§8.9 (NUMERICAL INTEGRATION) NAME: SOLUTIONS

20 July 2018

(1) Find the T_4 approximation for $\int_0^4 \sqrt{x}\, dx$.

SOLUTION: Let $f(x) = \sqrt{x}$. We divide $[0, 4]$ into 4 subintervals of width

$$\Delta x = \frac{4-0}{4} = 1,$$

with endpoints $0, 1, 2, 3, 4$. With this data, we get

$$T_4 = \frac{1}{2}\Delta x \left(\sqrt{0} + 2\sqrt{1} + 2\sqrt{2} + 2\sqrt{3} + \sqrt{4}\right) \boxed{\approx 5.14626.}$$

(2) State whether M_{10} underestimates or overestimates $\int_1^4 \ln(x)\, dx$.

SOLUTION: Let $f(x) = \ln(x)$. Then $f'(x) = \frac{1}{x}$ and

$$f''(x) = -\frac{1}{x^2} < 0$$

on the interval $[1, 4]$, so $f(x)$ is concave down. Therefore, the midpoint rule overestimates the integral.

(3) Approximate the arc length of the curve $y = \sin(x)$ over the interval $[0, \pi/2]$ using the midpoint approximation M_8.

SOLUTION: Since $y = \sin(x)$, we have

$$1 + (y')^2 = 1 + \cos^2(x)$$

Therefore, $\sqrt{1 + (y')^2} = \sqrt{1 + \cos^2(x)}$, and the arc length over $[0, \pi/2]$ is

$$\int_0^{\pi/2} \sqrt{1 + \cos^2(x)}\, dx.$$

Let $f(x) = \sqrt{1 + \cos^2(x)}$. M_8 is the midpoint approximation with eight subdivisions. So

$$\begin{aligned}
\Delta x &= \frac{\pi/2 - 0}{8} = \frac{\pi}{16} \\
x_i &= 0 + (i - \tfrac{1}{2})\Delta x \qquad \text{for } i = 1, 2, \ldots, 8 \\
y_i &= f\left((i - \tfrac{1}{2})\Delta x\right) \\
M_8 &= \sum_{i=1}^{8} y_i \Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_8)\Delta x
\end{aligned}$$

i	x_i	$f(x_i) = y_i$
1	0.5	1.41081
2	1.5	1.3841
3	2.5	1.3333
4	3.5	1.26394
5	4.5	1.18425
6	5.5	1.10554
7	6.5	1.04128
8	7.5	1.00479

The final answer is that the arc length is approximately $\boxed{1.9101}$.

(4) Find a number N for which $\text{Error}(T_N) \leq 10^{-6}$ for $\int_0^3 e^{-x}\, dx$.

SOLUTION: Let $f(x) = e^{-x}$, so that $f''(x) = e^{-x}$, which has a maximum value of 1 (at $x = 0$) on $[0, 3]$. Hence we can take $K_2 = 1$, and so

$$\text{Error}(T_N) \leq \frac{K_2(b-a)^3}{12N^2} = \frac{(1)(3)^3}{12N^2} = \frac{27}{12N^2}.$$

We want to find N for which

$$\frac{27}{12N^2} \leq 10^{-6},$$

which means that we need

$$N \geq \sqrt{\frac{27 \cdot 10^6}{12}} = 1500.$$

Hence taking N greater than or equal to 1500 will do the trick.

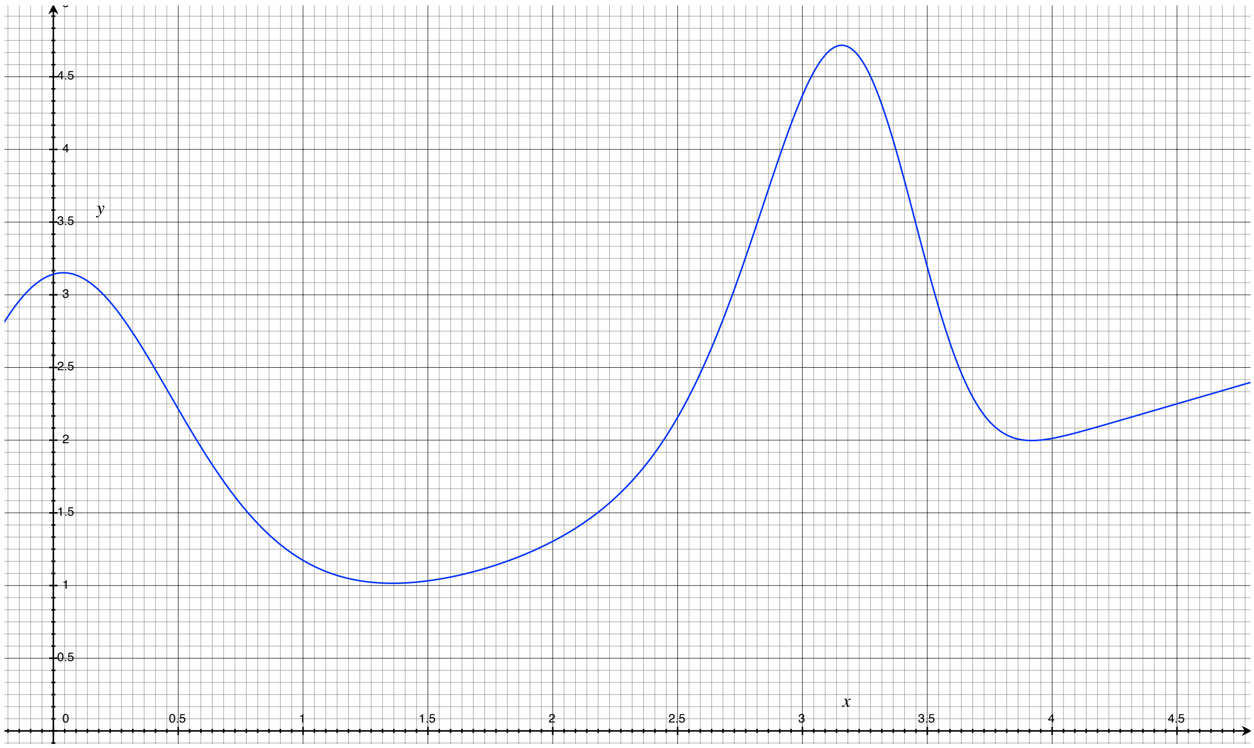
(5) Since Simpson's Rule can be derived by using quadratic polynomials (parabolas) to approximate a function, it makes sense that Simpson's rule gives the exact value for integrals of quadratic polynomials.

(a) Prove the statement above. In other words, show that the integral of a quadratic polynomial $f(x) = A + Bx + Cx^2$ over an interval $[a, b]$ exactly coincides with the Simpson's Rule approximation S_2.

(b) Perhaps unexpectedly, Simpson's Rule also gives the exact result for integrals of cubic polynomials. Show this as well: the integral of $g(x) = A + Bx + Cx^2 + Dx^3$ over $[a, b]$ is equal to the Simpson's Rule approximation S_2.

(c) Take another look at the error bound for Simpson's Rule. Is there a quicker way to prove the previous two results without calculating the integrals?

SOLUTION: The error bound formula has a term K_4, which is a bound on the fourth derivative of $f(x)$ over the interval $[a, b]$. Since quadratic and cubic polynomials both have fourth derivatives equal to zero, we can take $K_4 = 0$, and so according to the error formula the Simpson's Rule error should be zero. In other words, the approximation gives the exact value.



This is the graph of

$$y = (\pi)\, e^{-x^2(1+\cos(x))} + \frac{1}{2}x$$

The integral from $x = 0$ to $x = 4.5$ is approximately
$10.08816325863157 \pm 1.52581389892617 \cdot 10^{-8}$