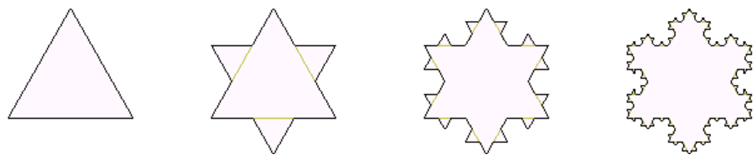
Introduction: *Fractals* are natural phenomena or mathematical sets which exhibit (among other properties) *self–similarity*: no matter how much we zoom in, the structure remains the same. The Koch snowflake, created by the infinite–step process whose first four iterations are shown below, is one example of a fractal:



The process of constructing a mathematical fractal is always infinite. In practice, though, self–similarity is limited by our perception and the actual construction or calculation limits of the system or object. Even the naturally–occurring fractal properties of an ocean coastline, for instance, can't continue down to the molecular level! Despite this, self–similarity and fractals have many applications ranging from computer graphics (generating realistic landscapes for games, etc.) to signal compression, to soil mechanics, to highly-efficient antenna designs (a good antenna often needs to have large surface area while remaining very compact).

Goals:

- Practice working with infinite series.
- Understand the relationship between volume and surface area in 3D fractals.

Problems: We'll consider a 3–dimensional version of the Koch snowflake: a sphereflake! This fractal is created as follows: start with a sphere of radius 1. To this large sphere, attach 9 smaller spheres of radius 1/3. To each of these nine spheres, attach nine spheres of radius 1/9, and so on. To each sphere of radius r, attach nine spheres of radius $r/3$, for infinite iterations.

a) What is the total volume of the sphereflake?
Hint: You may use the fact that the volume of a sphere is $V = \frac{4}{3}\pi r^3$.
Solution. Since the sphereflake is constructed in an infinite process, we expect to represent its volume with an infinite series.

There are 9^n spheres of radius $\dfrac{1}{3^n}$ in the sphereflake.

The volume of a sphere of radius r is $\dfrac{4}{3}\pi r^3$, so the volume of the sphereflake is

$$\sum_{n=0}^{\infty} 9^n \cdot \frac{4\pi}{3}\left(\frac{1}{3^n}\right)^3 = \sum_{n=0}^{\infty} \frac{4\pi}{3} \cdot \frac{3^{2n}}{3^{3n}} = \sum_{n=0}^{\infty} \frac{4\pi}{3} \cdot \frac{1}{3^n} = \frac{4\pi}{3} \sum_{n=0}^{\infty} \frac{1}{3^n}.$$

We note that this infinite series is a geometric series of the form $\sum_{n=0}^{\infty} Cr^n$ with $r = 1/3$. Since $|r| < 1$, we know that it converges to

$$\sum_{n=0}^{\infty} Cr^n = \frac{C}{1-r}.$$

For $r = 1/3$ and $C = 1$, the geometric series converges to $3/2$ and the volume of the sphereflake is

$$\boxed{V = \frac{4\pi}{3} \cdot \frac{3}{2} = 2\pi.}$$

As a reminder, here is how we find the value of a geometric series when $|r| < 1$:

$$\sum_{n=0}^{N} Cr^n - r\sum_{n=0}^{N} Cr^n = C + Cr + \cdots + Cr^N - (Cr + Cr^2 + \cdots + Cr^{N+1}),$$

$$(1-r)\sum_{n=0}^{N} Cr^n = C - Cr^{N+1},$$

$$\sum_{n=0}^{\infty} Cr^n = \lim_{N\to\infty}\sum_{n=0}^{N} Cr^n = \lim_{N\to\infty}\frac{C(1-r^{N+1})}{1-r} = \frac{C}{1-r}, \quad |r| < 1.$$

b) What is the total surface area of the sphereflake?
Hint: You may use the fact that the surface area of a sphere is $A = 4\pi r^2$.
Solution. The surface area of a sphere of radius r is $4\pi r^2$, so the area of the sphereflake is

$$\sum_{n=0}^{\infty} 9^n 4\pi \left(\frac{1}{3^n}\right)^2 = \sum_{n=0}^{\infty} 4\pi = 4\pi\sum_{n=0}^{\infty} 1 \longrightarrow \infty.$$

So, a sphereflake has an "infinite surface area" but a finite volume!

Remark: Although this result may seem surprising, there is no paradox here. Remember that the sphereflake is obtained by an infinite process, and its volume and area are defined to be the limits of growing partial sums, and those limits can either converge (as in the first case) or diverge (as in the second case). At the same time, an unbounded geometric body does not have to be fractal-like to have a finite volume and infinite surface area; by now you should have seen an example of that in the homework.

c) To generalize this example, suppose that the initial sphere is of radius $r_0 = R$, each next level consists of spheres of radius $r_{n+1} = \alpha r_n$ for some positive $\alpha < 1$, and there are $m \geq 1$ balls of radius r_{n+1} attached to each ball of radius r_n. We will ignore the possibility of spheres intersecting ("attached" does not have to mean "touching"). What relation between α and m guarantees that the sphereflake will have a finite volume? finite surface area?

Solution. The total number of spheres of radius r_n is m^n. Generalizing the above, the volume is

$$V = \sum_{n=0}^{\infty} m^n \cdot \frac{4\pi}{3} (\alpha^n R)^3 = \frac{4\pi R^3}{3} \sum_{n=0}^{\infty} (m\alpha^3)^n;$$

so, $m < \alpha^{-3}$ will guarantee convergence. For the surface area,

$$A = \sum_{n=0}^{\infty} m^n \cdot 4\pi (\alpha^n R)^2 = 4\pi R^2 \sum_{n=0}^{\infty} (m\alpha^2)^n;$$

so, $m < \alpha^{-2}$ will guarantee convergence. Since $\alpha < 1$, whenever the surface area is finite, the volume is finite as well.

d) Without resorting to "the interwebs," brainstorm with your group to come up with a list of objects or concepts which exhibit or approximate self-similarity.

Solution. A (by no means exhaustive) list:

- trees
- feathers
- blood vessels
- river systems
- ocean waves

All of these examples have the same "shape" on many length scales.

Workshop takeaways:

- 3D fractals can have infinite surface area with finite volume.
- Mathematical series can be used to describe natural phenomena.

§11.1 (SEQUENCES) NAME: SOLUTIONS

July 23, 2018

REVIEW

- A sequence (1) is a list of numbers $a_0, a_1, a_2, \ldots$. It doesn't have to start with zero.
- A series (2) is the sum of the terms in a sequence:

$$\sum_{i=0}^{\infty} a_i$$

- A sequences is called:

  (a) bounded (3) if there exists M such that $|a_n| \leq M$ for all n.

  (b) monotone (4) if either $a_n < a_{n+1}$ or $a_n > a_{n+1}$ for all n.

  If a sequence is both of the above, then it converges.
- If f is continuous (5) and $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} f(a_n) = f(L)$.
- A sequence that looks like $a_n = cr^n$ is called geometric (6).

PROBLEMS

(1) Determine the limit of the sequence or show that the sequence diverges.

(a) $a_n = \dfrac{e^n}{2^n}$

SOLUTION:

$$a_n = \frac{e^n}{2^n} = \left(\frac{e}{2}\right)^n$$

Note that $e > 2$, so $e/2 > 1$. Hence,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{e}{2}\right)^n = \infty.$$

(b) $b_n = \dfrac{3n+1}{2n+4}$

SOLUTION: As $n \to \infty$, the top and the bottom are both polynomial of the same degree, so only the leading coefficients matter. Hence,

$$\lim_{n\to\infty} \frac{3n+1}{2n+4} = \frac{3}{2}.$$

(c) $c_n = \dfrac{\sqrt{n}}{\sqrt{n}+4}$

SOLUTION:

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+4} = \lim_{n\to\infty} \frac{\frac{\sqrt{n}}{\sqrt{n}}}{\frac{\sqrt{n}}{\sqrt{n}} + \frac{4}{\sqrt{n}}} = \lim_{n\to\infty} \frac{1}{1+\frac{4}{\sqrt{n}}} = \frac{1}{1+0} = 1.$$

(d) $c_n = \dfrac{(\ln n)^2}{n}$

SOLUTION: Use L'Hôpital's Rule twice:

$$\lim_{n\to\infty} \frac{(\ln n)^2}{n} = \lim_{n\to\infty} \frac{2(\ln n)\frac{1}{n}}{1} = 2\lim_{n\to\infty} \frac{\ln n}{n} = 2\lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0.$$

(2) Show that the sequence given by $a_n = \dfrac{3n^2}{n^2+2}$ is strictly increasing, and find an upper bound.

SOLUTION: Consider the function $f(x) = \frac{3x^2}{x^2+2}$. The derivative of f is

$$f'(x) = \frac{12x}{(x^2+2)^2}.$$

For $x > 0$, $f'(x) > 0$, so the function is strictly increasing. Therefore, the sequence $a_n = f(n)$ is strictly increasing.

To find an upper bound, observe that

$$a_n = \frac{3n^2}{n^2+2} \leq \frac{3n^2+6}{n^2+2} = \frac{3(n^2+2)}{n^2+2} = 3.$$

Therefore, $M = 3$ is an upper bound.

(3) Let $\{a_n\}$ be the sequence defined recursively by

$$a_0 = 0, \quad a_{n+1} = \sqrt{2 + a_n}$$

(a) Write the first four terms of the sequence.

SOLUTION: $a_0 = 0$, $a_1 = \sqrt{2}$, $a_2 = \sqrt{2 + \sqrt{2}}$, $a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}}$

(b) Show that the sequence $\{a_n\}$ is increasing.

SOLUTION: We have $a_0 = 0 < \sqrt{2} = a_1$. Now assume that it's true for n, that is, $a_n \leq a_{n+1}$, and show it for $n+1$; that is, show that $a_{n+1} \leq a_{n+2}$.

$$a_{n+1} = \sqrt{2 + a_n} \leq \sqrt{2 + a_{n+1}} = a_{n+2}$$

(c) Show that the sequence is bounded above by $M = 2$.

SOLUTION: Again, the first case is simple since $a_0 = 0 \leq 2$. Now assume that it's true for n, that is, $a_n \leq 2$, and show it's true for $n+1$, that is, $a_{n+1} \leq 2$.

$$a_{n+1} = \sqrt{2 + a_n} \leq \sqrt{2 + 2} = \sqrt{4} = 2$$

(d) Prove that $\lim_{n \to \infty} a_n$ exists and compute it.

SOLUTION: Since the sequence is increasing and bounded above, the limit exists. To compute it, we do

$$L = \lim_{n \to \infty} a_n = \lim_{n \to \infty} \sqrt{2 + a_{n-1}} = \sqrt{2 + \lim_{n \to \infty} a_{n-1}} = \sqrt{2 + L}$$

Then $L = \sqrt{2 + L}$, from which we get that either $L = 2$ or $L = -1$. But all terms of the sequence are positive, so it must be $L = 2$.

(4) Consider the sequence $\{a_n\}$ where $a_n = \dfrac{1}{2n+1}$.

(a) Show that $\{a_n\}$ is decreasing.

SOLUTION: $a_{n+1} = \dfrac{1}{2(n+1)+1} = \dfrac{1}{2n+3} < \dfrac{1}{2n+1} = a_n$, so $\{a_n\}$ is decreasing. You can also use derivatives to show this.

(b) Find bounds M_l and M_u such that $M_l \leq a_n \leq M_u$ for every n.

SOLUTION: Note that $\dfrac{1}{2n+1}$ is always positive, so we can take $M_l = 0$. Also, since $\{a_n\}$ is decreasing, we know that the biggest term will be the first, $a_0 = 1$, so we can take $M_u = 1$.

(c) Show that $\lim_{n\to\infty} a_n$ exists without computing it. Then compute it.

SOLUTION: From parts (a) and (b) we know that $\{a_n\}$ is a decreasing sequence that's also bounded below, so it must converge.

To compute it, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{1}{2n+1} = 0$.

(5) Use the fact that $\frac{\sin\frac{1}{n}}{\frac{1}{n}} \to 1$ as $n \to \infty$ to find the limit of

$$a_n = n\left(1 - \sqrt{1 - \sin\frac{1}{n}}\right).$$

SOLUTION: Multiply and divide by $1 + \sqrt{1 - \sin(1/n)}$:

$$\begin{aligned}
a_n &= n\left(1 - \sqrt{1 - \sin\frac{1}{n}}\right)\frac{1 + \sqrt{1 - \sin\frac{1}{n}}}{1 + \sqrt{1 - \sin\frac{1}{n}}} \\
&= n\left(1 - \left(1 - \sin\frac{1}{n}\right)\right)\frac{1}{1 + \sqrt{1 - \sin\frac{1}{n}}} \\
&= \frac{\sin\frac{1}{n}}{\frac{1}{n}}\frac{1}{1 + \sqrt{1 - \sin\frac{1}{n}}} \\
&= 1 \cdot \frac{1}{1 + \sqrt{1 - 0}} \\
&= \frac{1}{2}
\end{aligned}$$