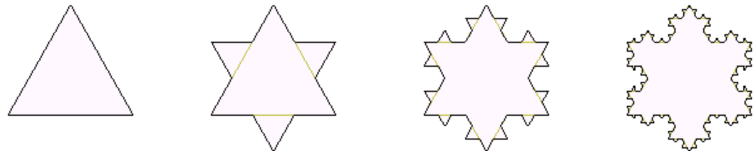
Introduction: *Fractals* are natural phenomena or mathematical sets which exhibit (among other properties) *self–similarity*: no matter how much we zoom in, the structure remains the same. The Koch snowflake, created by the infinite–step process whose first four iterations are shown below, is one example of a fractal:



The process of constructing a mathematical fractal is always infinite. In practice, though, self–similarity is limited by our perception and the actual construction or calculation limits of the system or object. Even the naturally–occurring fractal properties of an ocean coastline, for instance, can't continue down to the molecular level! Despite this, self–similarity and fractals have many applications ranging from computer graphics (generating realistic landscapes for games, etc.) to signal compression, to soil mechanics, to highly-efficient antenna designs (a good antenna often needs to have large surface area while remaining very compact).

Goals:

- Practice working with infinite series.
- Understand the relationship between volume and surface area in 3D fractals.

Problems: We'll consider a 3–dimensional version of the Koch snowflake: a sphereflake! This fractal is created as follows: start with a sphere of radius 1. To this large sphere, attach 9 smaller spheres of radius 1/3. To each of these nine spheres, attach nine spheres of radius 1/9, and so on. To each sphere of radius r, attach nine spheres of radius $r/3$, for infinite iterations.

a) What is the total volume of the sphereflake?
Hint: You may use the fact that the volume of a sphere is $V = \frac{4}{3}\pi r^3$.

b) What is the total surface area of the sphereflake?
Hint: You may use the fact that the surface area of a sphere is $A = 4\pi r^2$.

c) To generalize this example, suppose that the initial sphere is of radius $r_0 = R$, each next level consists of spheres of radius $r_{n+1} = \alpha r_n$ for some positive $\alpha < 1$, and there are $m \geq 1$ balls of radius r_{n+1} attached to each ball of radius r_n. We will ignore the possibility of spheres intersecting ("attached" does not have to mean "touching"). What relation between α and m guarantees that the sphereflake will have a finite volume? finite surface area?

d) Without resorting to "the interwebs," brainstorm with your group to come up with a list of objects or concepts which exhibit self-similarity.

§11.1 (SEQUENCES) NAME: ______________________

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REVIEW

- A [ ] (1) is a list of numbers $a_0, a_1, a_2, \ldots$. It doesn't have to start with zero.
- A [ ] (2) is the sum of the terms in a sequence:

$$\sum_{i=0}^{\infty} a_i$$

- A sequences is called:
  - (a) [ ] (3) if there exists M such that $|a_n| \leq M$ for all n.
  - (b) [ ] (4) if either $a_n < a_{n+1}$ or $a_n > a_{n+1}$ for all n.

  If a sequence is both of the above, then it converges.
- If f is [ ] (5) and $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} f(a_n) = f(L)$.
- A sequence that looks like $a_n = cr^n$ is called [ ] (6).

PROBLEMS

(1) Determine the limit of the sequence or show that the sequence diverges.

(a) $a_n = \dfrac{e^n}{2^n}$

(b) $b_n = \dfrac{3n+1}{2n+4}$

(c) $c_n = \dfrac{\sqrt{n}}{\sqrt{n}+4}$

(d) $c_n = \dfrac{(\ln n)^2}{n}$

(2) Show that the sequence given by $a_n = \dfrac{3n^2}{n^2+2}$ is strictly increasing, and find an upper bound.

(3) Let $\{a_n\}$ be the sequence defined recursively by

$$a_0 = 0, \quad a_{n+1} = \sqrt{2 + a_n}$$

(a) Write the first four terms of the sequence.

(b) Show that the sequence $\{a_n\}$ is increasing.

(c) Show that the sequence is bounded above by $M = 2$.

(d) Prove that $\lim_{n \to \infty} a_n$ exists and compute it.

(4) Consider the sequence $\{a_n\}$ where $a_n = \dfrac{1}{2n+1}$.

(a) Show that $\{a_n\}$ is decreasing.

(b) Find bounds M_l and M_u such that $M_l \leq a_n \leq M_u$ for every n.

(c) Show that $\lim_{n\to\infty} a_n$ exists without computing it. Then compute it.

(5) Use the fact that $\dfrac{\sin\frac{1}{n}}{\frac{1}{n}} \to 1$ as $n \to \infty$ to find the limit of

$$a_n = n\left(1 - \sqrt{1 - \sin\frac{1}{n}}\right).$$