§11.2 (Series)
§11.3 (Series with positive terms)
July 24, 2018

Name: Solutions

Convergence Tests for Series

- **The divergence test:** If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.
- A series that looks like $a_n = cr^n$ is called **geometric.**
  - (a) If $|r| \geq 1$, then it diverges.
  - (b) If $|r| < 1$, then $\sum_{n=K}^{\infty} cr^n = \frac{cr^K}{1-r}$
- **The integral test:** Assume that $a_n = f(n)$ for $n \geq M$.
  - (a) If $\int_M^\infty f(x)\,dx$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.
  - (b) If $\int_M^\infty f(x)\,dx$ diverges, then $\sum_{n=0}^{\infty} a_n$ diverges.
- **The comparison test:**
  - (a) If $a_n \leq b_n$, and $\sum_{n=0}^{\infty} b_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.
  - (b) If $\sum_{n=0}^{\infty} a_n$ diverges, then $\sum_{n=0}^{\infty} b_n$ diverges.
- **Limit comparison test:** Let $\{a_n\}$ and $\{b_n\}$ be sequences with positive terms. Let $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.
  - (a) If $L > 0$ (1), then $\sum a_n$ converges if and only if $\sum b_n$ converges.
  - (b) If $L = \infty$ (2) and $\sum a_n$ converges, then $\sum b_n$ converges.
  - (c) If $L = 0$ (3) and $\sum b_n$ converges, then $\sum a_n$ converges.

Problems

(1) Determine the limit of the series or show that the series diverges.

(a) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n$

SOLUTION: This is geometric, and converges to $\frac{1}{1-1/4} = \frac{4}{3}$.

(b) $\displaystyle\sum_{n=0}^{\infty} e^n$

SOLUTION: $\lim_{n\to\infty} e^n = \infty$, so this diverges.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

SOLUTION: This is the Harmonic series, which diverges.

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$

SOLUTION: This is a telescoping series. First perform partial fractions to see that

$$\frac{1}{n(n-1)} = \frac{-1}{n} + \frac{1}{n-1}$$

Then the sum is

$$\sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots = 1$$

(e) $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

SOLUTION: Since $\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = 1 \neq 0$, the series diverges by the Divergence Test.

(f) $\sum_{n=0}^{\infty} \frac{9^n + 2^n}{5^n}$

SOLUTION: We can write

$$\sum_{n=0}^{\infty} \frac{9^n + 2^n}{5^n} = \sum_{n=0}^{\infty} \left[\left(\frac{9}{5}\right)^n + \left(\frac{2}{5}\right)^n \right].$$

Since $\sum (9/5)^n$ diverges (as $9/5 > 1$), the entire series must diverge.

(g) $\sum_{n=1}^{\infty} \cos(\pi n)$

SOLUTION: Notice that $\cos(\pi n) = (-1)^n$, so this series diverges.

(h) $\sum_{n=1}^{\infty} \cos \frac{1}{n}$

SOLUTION: We have $\lim_{n\to\infty} = \cos 0 = 1$, so this series diverges by the divergence test.

(i) $\sum_{n=2}^{\infty} \frac{n^2}{n^4-1}$ (Limit Comparison Test)

SOLUTION: Use the limit comparison test. Let $a_n = \frac{n^2}{n^4-1}$. Since for n large, $\frac{n^2}{n^4-1} \approx \frac{n^2}{n^4} = \frac{1}{n^2}$, apply Limit comparison with $b_n = \frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^2}{n^4-1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^4}{n^4-1} = 1 \neq 0.$$

We know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges because it's a p-series, so $\sum_{n=2}^{\infty} a_n$ also converges.

(j) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+2^n}$ (Comparison Test)

SOLUTION: For $n \geq 1$, we have

$$\frac{1}{\sqrt{n}+2^n} \leq \frac{1}{2^n} = \left(\frac{1}{2}\right)^n.$$

The series $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ converges since it is geometric with $r = 1/2$. So the comparison test tells us that this series converges too.

(k) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ (Integral Test)

SOLUTION: Integrate

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\, dx.$$

Substitute $u = \ln x$, $du = \frac{1}{x}\, dx$. Then

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\, dx = \int_{\ln 2}^{\infty} \frac{1}{u^2}\, du = -\frac{1}{u}\Big|_{\ln 2}^{\infty} = -\frac{1}{\ln \infty} + \frac{1}{\ln 2} = \frac{1}{\ln 2}$$

The integral converges, so the series converges as well.

(2) Give a counterexample to show that each of the following statements is false.

(a) If the general term a_n tends to zero, then $\sum a_n$ converges.

SOLUTION: $\sum \frac{1}{n}$ diverges even through $\frac{1}{n} \to 0$ as $n \to \infty$.

(b) The Nth partial sum of the infinite series defined by $\{a_n\}$ is equal to a_N.

SOLUTION: Almost any nonzero series will work as a counterexample here. For instance, consider the series in 2(a), below.

(c) If $a_n \to L$, then $\sum_{n=0}^{\infty} a_n = L$.

SOLUTION: If a_n is a positive sequence for which $\sum a_n$ converges, then we must have $a_n \to 0$, but $\sum a_n > 0$. (Again, if you are looking for a concrete example, consider the series in Problem 2(a).)

(3) Determine a reduced fraction that is equal to 0.217217217217....

SOLUTION: The decimal can be regarded as a geometric series

$$0.217217217... = \frac{217}{10^3} + \frac{217}{10^6} + \frac{217}{10^9} + \cdots = \sum_{n=1}^{\infty} \frac{217}{10^{3n}} = \sum_{n=1}^{\infty} 217.\left(\frac{1}{10^3}\right)^n = \frac{217/10^3}{1-1/10^3} = \frac{217}{999}$$

(4) Let $b_n = \dfrac{\sqrt[n]{n!}}{n}$.

(a) Show that $\ln b_n = \dfrac{1}{n}\sum_{k=1}^{n} \ln \dfrac{k}{n}$.

SOLUTION: Start by taking logarithms:

$$\ln b_n = \ln \frac{\sqrt[n]{n!}}{n} = \ln \sqrt[n]{n!} - \ln n = \frac{1}{n}\ln n! - \ln n = \frac{1}{n}\left(\ln n! - n\ln n\right).$$

Next, notice that

$$\begin{aligned}\ln n! &= \ln[n(n-1)(n-2)\cdots(2)(1)] \\ &= \ln n + \ln(n-1) + \cdots + \ln 2 + \ln 1 = \sum_{k=1}^{n} \ln k,\end{aligned}$$

and so we have

$$\begin{aligned} b_n &= \frac{1}{n}\left(\ln n! - n\ln n\right) = \frac{1}{n}\left(\sum_{k=1}^{n}\ln k - n\ln n\right) \\ &= \frac{1}{n}\sum_{k=1}^{n}(\ln k - \ln n) = \frac{1}{n}\sum_{k=1}^{n}\ln\frac{k}{n},\end{aligned}$$

which was what we wanted.

(b) Show that $\ln b_n$ converges to $\displaystyle\int_0^1 \ln x\, dx$. Use this to compute $\lim b_n$.

SOLUTION: Notice that $\frac{1}{n}\sum_{k=1}^{n}\ln\frac{k}{n}$ is precisely the right hand approximation to $\int_0^1 \ln x\, dx$; since $\ln x$ is continuous, we will have

$$\ln b_n \to \int_0^1 \ln x\, dx = (x\ln x - x)\Big|_{x=0}^{1} = -1.$$

Hence $\ln b_n \to -1$ implies $b_n \to e^{-1}$.