§11.5 (Ratio and Root Tests)

26 July 2018

Name: Solutions

The tests

Ratio Test: Assume that $\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists. Then the series $\sum_{n=1}^{\infty} a_n$

(a) converges absolutely when $\rho < 1$ (1)

(b) diverges when $\rho > 1$ (2)

(c) inconclusive if $\rho = 1$ (3)

Root Test: Assume that $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists. Then the series $\sum_{n=1}^{\infty} a_n$

(a) converges absolutely if $L < 1$ (4)

(b) diverges if $L > 1$ (5)

(c) inconclusive if $L = 1$ (6)

Problems

(1) Apply the ratio test or the root test to determine the convergence or divergence of the following series, or state that the test is inconclusive. If the test is inconclusive, apply another test to determine convergence or divergence, if possible.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{5^n}$

Solution: Use the ratio test. Then $a_n = \frac{(-1)^{n-1} n}{5^n}$. Then $|a_n| = \frac{n}{5^n}$, so compute

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} n + 15n = \lim_{n\to\infty} \frac{1 + \frac{1}{n}}{5} = \frac{1}{5}.$$

Since $\rho < 1$, the series converges absolutely.

(b) $\sum_{n=1}^{\infty} \frac{3n+2}{5n^3+1}$

SOLUTION: Use the ratio test.

$$\rho = \lim_{n\to\infty} \frac{\frac{3(n+1)+2}{5(n+1)^3+1}}{\frac{3n+2}{5n^3+1}} = \lim_{n\to\infty} \frac{3n+5}{3n+2} \cdot \frac{5n^3+1}{5(n+1)^3+1} = \lim_{n\to\infty} \frac{15n^4+\dots}{15n^4+\dots} = 1.$$

So $\rho = 1$ and the test is inconclusive.

However, we can use the limit comparison test to compare the series with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and see that the series converges.

(c) $\sum_{n=1}^{\infty} \frac{2^n}{n}$

SOLUTION: Use the ratio test.

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{2^{n+1}}{n+1}}{\frac{2^n}{n}} = \lim_{n\to\infty} \frac{2n}{n+1} = 2$$

Since $\rho > 1$, the series diverges.

(d) $\sum_{n=0}^{\infty} \frac{1}{10^n}$

SOLUTION: Use the root test.

$$L = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{10^n}} = \lim_{n\to\infty} \frac{1}{10} = \frac{1}{10}.$$

Since $L < 1$, this series converges by the root test. Note that this is a geometric series, so we already knew that it converged.

(e) $\sum_{k=0}^{\infty} \left(\frac{k}{k+10}\right)^k$

SOLUTION: Use the root test.

$$L = \lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} \sqrt[k]{\left(\frac{k}{k+10}\right)} = \lim_{k\to\infty} \frac{k}{k+10} = 1$$

So the test is inconclusive.

(f) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

SOLUTION: Ratio test, with a bunch of algebraic manipulation:

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} = \frac{(n+1)n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)^n} \\ &= \left(\frac{n}{n+1}\right)^n = \left[\left(\frac{n+1}{n}\right)^n\right]^{-1} = \left[\left(1+\frac{1}{n}\right)^n\right]^{-1} \to e^{-1}.\end{aligned}$$

Since $e^{-1} < 1$, the series converges.

(g) $\sum_{n=1}^{\infty} a_n$ where $a_0 = 1$, $a_{n+1} = \frac{a_n}{n}$

SOLUTION: Apply the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{a_n}{n}}{a_n} = \frac{1}{n} \to 0,$$

so the series converges. (By the way, this series can be expressed more concretely as $\sum_{n=1}^{\infty} \frac{1}{n!}$.)