§11.6 (Power Series) — Name: Solutions

26 July 2018

Power Series

(1) An infinite series of the form $F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ is called a **power series** and c is called the **center**.

(2) The **radius of convergence** of $F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ is a constant R such that $F(x)$ converges absolutely for $|x-c| < R$ and diverges for $|x-c| > R$. If $F(x)$ converges for all x, then $R = \infty$.

(3) To determine R, use the ratio test. (1)

(4) $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, with radius of convergence $R =$ 1 (2).

(5) If $R > 0$, then a power series $F(x)$ is differentiable on $(c-R, c+R)$, and

$$F'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1}.$$

$$\int F(x)\, dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-c)^{n+1}.$$

Problems

(1) Show that all three of the following power series have the same radius of convergence, but different behavior at the endpoints.

(a) $\sum_{n=1}^{\infty} \frac{(x-5)^n}{9^n}$

Solution: Use the ratio test to determine the radius of convergence.

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{|x-5|^{n+1} 9^n}{|x-5|^n 9^{n+1}} = \frac{|x-5|}{9}.$$

So this series converges if $\frac{1}{9}|x-5| < 1$, and has radius of convergence $R = 9$.
But now we need to check the endpoints, which are $x = -4$ and $x = 14$.

$x = 14:$	$\sum_{n=1}^{\infty} \frac{(14-5)^n}{9^n} = \sum_{n=1}^{\infty} 1$	diverges
$x = -4:$	$\sum_{n=1}^{\infty} \frac{(-4-5)^n}{9^n} = \sum_{n=1}^{\infty} (-1)^n$	diverges

So the interval of convergence is $(-4, 14)$.

(b) $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n 9^n}$

SOLUTION: Use the ratio test to determine the radius of convergence.

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x-5|^{n+1} n 9^n}{|x-5|^n (n+1) 9^{n+1}} = \frac{|x-5|}{9}.$$

So this series converges if $\frac{1}{9}|x-5| < 1$, and has radius of convergence $R = 9$.
But now we need to check the endpoints, which are $x = -4$ and $x = 14$.

$$x = 14: \quad \sum_{n=1}^{\infty} \frac{(14-5)^n}{n 9^n} = \sum_{n=1}^{\infty} \frac{1}{n} \qquad \text{diverges}$$

$$x = -4: \quad \sum_{n=1}^{\infty} \frac{(-4-5)^n}{9^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \qquad \text{converges}$$

The interval of convergence is $[-4, 14)$.

(c) $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2 9^n}$

SOLUTION: Use the ratio test to determine the radius of convergence.

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x-5|^{n+1} n^2 9^n}{|x-5|^n (n+1)^2 9^{n+1}} = \frac{|x-5|}{9}.$$

So this series converges if $\frac{1}{9}|x-5| < 1$, and has radius of convergence $R = 9$.
But now we need to check the endpoints, which are $x = -4$ and $x = 14$.

$$x = 14: \quad \sum_{n=1}^{\infty} \frac{(14-5)^n}{n^2 9^n} = \sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \text{converges}$$

$$x = -4: \quad \sum_{n=1}^{\infty} \frac{(-4-5)^n}{n^2 9^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \qquad \text{converges}$$

The interval of convergence is $[-4, 14]$.

(2) Use the geometric series formula to expand the function $\frac{1}{1+3x}$ in a power series with center $c = 0$ and determine radius of convergence.

SOLUTION: The formula for the geometric series implies that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

for $|x| < 1$. Replace x by $-3x$ in that formula to get

$$\frac{1}{1+3x} = \sum_{n=0}^{\infty} (-3x)^n = \sum_{n=0}^{\infty} (-1)^n 3^n x^n.$$

This formula is valid for $|-3x| < 1$, or $|x| < 1/3$. So the radius of convergence is $R = \frac{1}{3}$.

(3) Find a power series expansion for $\ln(1+x)$ and the interval on which this expansion is valid.

SOLUTION: We apply integration to the expansion

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$$

which is valid for $|x| < 1$, to see that

$$\ln(1+x) = \int \frac{1}{1+x}\, dx = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

which is also valid for $|x| < 1$.