§11.7 (Taylor Series) — Name: Solutions

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Taylor Series

(1) The power series

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the **Taylor Series** for $f(x)$ centered at $x = c$. If $c = 0$, this is called a **Maclaurin series**.

(2) The N-th partial sum

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(N)}(c)}{N!}(x-c)^N$$

of the Taylor series $T(x)$ is called the N-th **Taylor Polynomial** for $f(x)$ centered at $x = c$.

(3) **Taylor's Theorem.** The n-th Taylor polynomial $T_n(x)$ centered at $x = a$ approximates the function $f(x)$ with a remainder

$$f(x) - T_n(x) = \frac{1}{n!}\int_a^x (x-u)^n f^{(n+1)}(u)\, du.$$

Corollary. The n-th Taylor polynomial $T_n(x)$ centered at $x = a$ approximates $f(x)$ with error at most

$$|f(x) - T_n(x)| \le K\frac{|x-a|^{n+1}}{(n+1)!},$$

where K is a number such that $|f^{(n+1)}(u)| \le K$ for all $u \in (a, x)$.

(4) **Where functions agree with their Taylor series:** Suppose that $T(x)$ is the Taylor series for $f(x)$ centered at c, with radius of convergence R. If there is a number K such that $|f^{(n)}(x)| \le K$ for all $x \in (c-R, c+R)$ for all n, then $f(x) = T(x)$ for all $x \in (c-R, c+R)$.

(5) $(1+x)^a = 1 + \sum_{n=1}^{\infty} \binom{a}{n} x^n$ for $|x| < 1$, where $\binom{a}{n} = \frac{a(a-1)(a-2)\cdots(a-n+1)}{n!}$

(6) Some Taylor series:

Function	**Series**	**Interval of Convergence**
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$(-\infty, \infty)$
$\sin(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$(-\infty, \infty)$
$\cos(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	$(-\infty, \infty)$
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$(-1, 1)$
$\ln(1+x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$	$(-1, 1]$

PROBLEMS

(1) Find the Taylor polynomial $T_3(x)$ for $f(x)$ centered at $c = 3$ if $f(3) = 1$, $f'(3) = 2$, $f''(3) = 12$, $f'''(3) = 3$.

SOLUTION:

$$\begin{aligned} T_3(x) &= f(3) + f'(3)(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \frac{f'''(3)}{3!}(x-3)^3 \\ &= 1 + 2(x-3) + \frac{12}{2!}(x-3)^2 + \frac{3}{3!}(x-3)^3 \\ &= 1 + 2(x-3) + 6(x-3)^2 + \frac{1}{2}(x-3)^3 \end{aligned}$$

(2) Find the Taylor polynomials $T_2(x)$ and $T_3(x)$ for $f(x) = \frac{1}{1+x}$ centered at $a = 1$.

SOLUTION: We need to take a few derivatives, and then plug in $a = 1$ to each one.

n	n-th derivative $f^{(n)}(x)$	$f^{(n)}(a)$
0	$f(x) = \dfrac{1}{1+x}$	$f(1) = 1/2$
1	$f'(x) = \dfrac{-1}{(1+x)^2}$	$f'(1) = -1/4$
2	$f''(x) = \dfrac{2}{(1+x)^3}$	$f''(1) = 1/4$
3	$f'''(x) = \dfrac{-6}{(1+x)^4}$	$f'''(1) = -3/8$

Then plug these values into the formula for the Taylor polynomial.

$$T_2(x) = \frac{1}{2} - \frac{(x-1)}{4} + \frac{(x-1)^2}{8}$$

$$T_3(x) = \frac{1}{2} - \frac{(x-1)}{4} + \frac{(x-1)^2}{8} - \frac{(x-1)^3}{16}$$

(3) Find n such that $|T_n(1.3) - \sqrt{1.3}| \leq 10^{-6}$, where $T_n(x)$ is the Taylor polynomial for $\sqrt{x}$ at $a = 1$.

SOLUTION: By the error formula, we have that

$$|T_n(1.3) - \sqrt{1.3}| \leq \frac{K_{n+1}(1.3-1)^{n+1}}{(n+1)!}$$

So we just need to find n such that

$$\frac{K_{n+1}(0.3)^{n+1}}{(n+1)!} < 10^{-6},$$

where K_{n+1} is the maximum value of the $(n+1)$-st derivative of $f(x) = \sqrt{x}$ between 1 and 1.3. Since $f^{(n+1)}(x)$ is the $(n+1)$-st derivative of $\sqrt{x}$, and this always has x in the denominator for any $n \geq 0$, this maximum will always occur at $x = 1$. Therefore, in this case,

$$K_{n+1} = |f^{(n+1)}(1)|.$$

So we just need to find n such that

$$\frac{|f^{(n+1)}(1)|(0.3)^{n+1}}{(n+1)!} < 10^{-6}.$$

The hard part is finding a pattern for the n-th derivative of $\sqrt{x}$, but that's not strictly necessary, although possible. If you keep taking derivatives of $\sqrt{x}$ and plugging into the formula, you find that this is valid for $\boxed{n \geq 7}$.

Alternatively, the general formula for the n-th derivative of $\sqrt{x}$ is

$$f^{(n)}(x) = (-1)^{n+1}\frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n}x^{\frac{-(2n-1)}{2}}$$

Then you can plug this in to the previous formula.

(4) (a) Use the fact that $\arctan(x)$ is an antiderivative of $\dfrac{1}{1+x^2}$ to find a Maclaurin series for $\arctan(x)$, and find the interval of convergence.

SOLUTION: Recall that $\arctan(x)$ is an antiderivative of $(1+x^2)^{-1}$. We can get a power series expansion for $\frac{1}{1+x^2}$ by substituting $-x^2$ into the geometric series formula:

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$$

This expansion is valid for $|x^2| < 1$, or equivalently, $|x| < 1$. Now integrate term-by-term:

$$\tan^{-1} x = \int \frac{dx}{1+x^2} = \int \left(1 - x^2 + x^4 - x^6 + \cdots\right) dx = A + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

We're not done yet! We need to find the constant of integration. To do this, plug in $x = 0$, so $A = \arctan(0) = 0$. Therefore,

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}.$$

Integrating term-by-term doesn't change the radius of convergence, so it still converges for $|x| < 1$. But we do need to check the endpoints of this interval: $x = \pm 1$.

For $x = 1$, we have the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1},$$

which converges by the alternating series test.

For $x = -1$, notice that $(-1)^{2n+1} = -1$, so we have the series

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1},$$

which again converges by the alternating series test.

Therefore, the interval of convergence is $[-1, 1]$.

(b) Use the fact that $\tan\left(\frac{\pi}{6}\right) = \dfrac{1}{\sqrt{3}}$ and your answer to the previous part to find a series that converges to π.

SOLUTION: We have $\arctan(1/\sqrt{3}) = \pi/6$. Since $x = 1/\sqrt{3}$ is inside the radius of convergence, so we can plug in $1/\sqrt{3}$ into the series from the previous part:

$$\begin{aligned}\frac{\pi}{6} &= \arctan\left(\frac{1}{\sqrt{3}}\right) \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{\left(1/\sqrt{3}\right)^{2n+1}}{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1/2}(2n+1)}.\end{aligned}$$

Thus we have

$$\pi = \sum_{n=0}^{\infty} \frac{6\,(-1)^n}{3^{n+1/2}(2n+1)}.$$

(5) Find the interval of convergence of the following power series.

(a) $\sum_{n=0}^{\infty} \frac{x^n}{n^4+2}$

SOLUTION: Start with the ratio test:

$$\left|\frac{x^{n+1}}{(n+1)^4+2}\frac{n^4+2}{x^n}\right| = \left|\frac{n^4+2}{(n+1)^4+2}x\right| \to |x|$$

So the series converges when $|x| < 1$ and diverges when $|x| > 1$. We must now check the cases when $|x| = 1$ manually: when $x = 1$ and $x = -1$, the resulting series converges by limit comparison to $\sum(1/n^4)$. Hence the interval of convergence is $[-1, 1]$.

(b) $\sum_{n=0}^{\infty} \frac{2^n}{3n}(x+3)^n$

SOLUTION: Ratio test:

$$\left|\frac{2^{n+1}(x+3)^{n+1}}{3(n+1)}\frac{3n}{2^n(x+3)^n}\right| = \left|\frac{3n}{3(n+1)}\cdot 2(x+3)\right| \to |2(x+3)|.$$

Thus the series converges for $|x+3| < 1/2$. Check the endpoints: when $x+3 = 1/2$ then the series is

$$\sum_{n=0}^{\infty} \frac{2^n}{3n}\frac{1}{2^n} = \sum_{n=0}^{\infty} \frac{1}{3n},$$

which diverges, and when $x+3 = -1/2$ the series is the alternating version of the above, which converges. Hence the interval of convergence is $[-3-1/2, -3+1/2) = [-7/2, -5/2)$.

(c) $\sum_{n=0}^{\infty} \frac{(x+4)^n}{(n\ln n)^2}$

SOLUTION: Ratio test:

$$\left|\frac{(x+4)^{n+1}}{((n+1)\ln(n+1))^2}\frac{(n\ln n)^2}{(x+4)^n}\right| = \left|\left(\frac{n}{n+1}\frac{\ln n}{\ln(n+1)}\right)^2(x+4)\right| \to |x+4|.$$

(Use L'Hôpital's rule if you are not confident with the limit.) So the series converges when $|x+4| < 1$, that is for $x \in (-5, -3)$. Checking the endpoints, we find that when $x = -5$, we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n\ln n)^2},$$

which converges by the Alternating Series Test, and when $x = -3$ we have

$$\sum_{n=0}^{\infty} \frac{1}{(n\ln n)^2},$$

which converges by limit comparison to $\sum 1/n^2$. Therefore the interval of convergence is $[-5, -3]$.

(6) Find the Taylor series of the following functions and determine the radius of convergence.

(a) $f(x) = \sin(2x)$, centered at $x = 0$.

SOLUTION:

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$
$$\sin(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!}$$

Since the formula for $\sin(x)$ is valid for all x, the formula for $\sin(2x)$ is also valid for all x.

(b) $f(x) = e^{4x}$, centered at $x = 0$.

SOLUTION:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$e^{4x} = \sum_{n=0}^{\infty} \frac{(4x)^n}{n!} = \sum_{n=0}^{\infty} \frac{4^n x^n}{n!}$$

Since the formula for e^x is valid for all x, so is the formula for e^{4x}.

(c) $f(x) = x^2 e^{x^2}$, centered at $x = 0$.

SOLUTION:

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n+2}}{n!}$$
$$x^2 e^{x^2} = x^2 \left(\sum_{n=0}^{\infty} \frac{x^{2n+2}}{n!} \right) = \sum_{n=0}^{\infty} \frac{x^{2n+2}}{n!}$$

Since the formula for x^2 is valid for all x, so is the formula for $x^2 e^{x^2}$.

(d) $f(x) = \frac{1}{3x-2}$, centered at $c = -1$.

SOLUTION: Rewrite the function as follows:

$$\frac{1}{3x-2} = \frac{1}{-5+3(x+1)} = \frac{-1}{5}\frac{1}{1-\frac{3(x+1)}{5}}$$

Now use the geometric series formula, valid for $|x| < 1$.

$$\frac{1}{3x-2} = -\frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{3(x+1)}{5}\right)^n = -\frac{1}{5}\sum_{n=0}^{\infty} 3^n 5^n (x+1)^n = -\sum_{n=0}^{\infty}\frac{3^n}{5^{n+1}}(x+1)^n$$

This formula is now valid for $\left|\frac{3(x+1)}{5}\right| < 1$, or $|x+1| < \frac{5}{3}$. So the radius of convergence is $\frac{5}{3}$.

(e) $f(x) = (1+x)^{1/3}$, centered at $c = 0$.

SOLUTION: Use the binomial series formula with $a = \frac{1}{3}$.

$$(1+x)^{\frac{1}{3}} = 1 + \sum_{n=1}^{\infty}\binom{\frac{1}{3}}{n}x^n$$

The radius of convergence is 1, since the formula is valid for $|x| < 1$.

(f) $f(x) = \sqrt{x}$, centered at $c = 4$.

SOLUTION: First rewrite the function

$$\sqrt{x} = \sqrt{4+(x-4)} = \sqrt{4\left(1+\frac{x-4}{4}\right)} = 2\sqrt{1+\frac{x-4}{4}}$$

Now find the MacLaurin series of $\sqrt{1+u}$ by setting $a = \frac{1}{2}$ in the binomial series formula.

$$(1+u)^{\frac{1}{2}} = \sqrt{1+u} = 1 + \sum_{n=1}^{\infty}\binom{\frac{1}{2}}{n}u^n.$$

This is valid for $|u| < 1$. Now replace u by $\frac{x-4}{4}$ to get

$$\sqrt{1+\frac{x-4}{4}} = 1 + \sum_{n=1}^{\infty}\binom{\frac{1}{2}}{n}\left(\frac{x-4}{4}\right)^n = 1 + \sum_{n=1}^{\infty}\binom{\frac{1}{2}}{n}\frac{1}{4^n}(x-4)^n$$

This is valid for $\left|\frac{x-4}{4}\right| < 1$ or $|x-4| < 4$. So the radius of convergence is 4.

The final answer is:

$$\sqrt{x} = 2 + \sum_{n=1}^{\infty}\binom{\frac{1}{2}}{n}\frac{2}{4^n}(x-4)^n$$

If you're willing to do a lot of simplifying, you can eventually get to:

$$\sqrt{x} = 2 + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}n(2n-2)!}{2^{4n-2}(n!)^2}(x-4)^n$$