§11.7 (TAYLOR SERIES) NAME: ________________________

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TAYLOR SERIES

(1) The power series

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the **Taylor Series** for $f(x)$ centered at $x = c$. If $c = 0$, this is called a **Maclaurin series**.

(2) The N-th partial sum

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(N)}(c)}{N!}(x-c)^N$$

of the Taylor series $T(x)$ is called the N-th **Taylor Polynomial** for $f(x)$ centered at $x = c$.

(3) **Taylor's Theorem.** The n-th Taylor polynomial $T_n(x)$ centered at $x = a$ approximates the function $f(x)$ with a remainder

$$f(x) - T_n(x) = \frac{1}{n!}\int_a^x (x-u)^n f^{(n+1)}(u)\, du.$$

Corollary. The n-th Taylor polynomial $T_n(x)$ centered at $x = a$ approximates $f(x)$ with error at most

$$|f(x) - T_n(x)| \leq K\frac{|x-a|^{n+1}}{(n+1)!},$$

where K is a number such that $|f^{(n+1)}(u)| \leq K$ for all $u \in (a, x)$.

(4) **Where functions agree with their Taylor series:** Suppose that $T(x)$ is the Taylor series for $f(x)$ centered at c, with radius of convergence R. If there is a number K such that $|f^{(n)}(x)| \leq K$ for all $x \in (c-R, c+R)$ for all n, then $f(x) = T(x)$ for all $x \in (c-R, c+R)$.

(5) $(1+x)^a = 1 + \sum_{n=1}^{\infty} \binom{a}{n} x^n$ for $|x| < 1$, where $\binom{a}{n} = \frac{a(a-1)(a-2)\cdots(a-n+1)}{n!}$

(6) Some Taylor series:

Function	**Series**	**Interval of Convergence**
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$(-\infty, \infty)$
$\sin(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$(-\infty, \infty)$
$\cos(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	$(-\infty, \infty)$
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$(-1, 1)$
$\ln(1+x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$	$(-1, 1]$

PROBLEMS

(1) Find the Taylor polynomial $T_3(x)$ for $f(x)$ centered at $c = 3$ if $f(3) = 1$, $f'(3) = 2$, $f''(3) = 12$, $f'''(3) = 3$.

(2) Find the Taylor polynomials $T_2(x)$ and $T_3(x)$ for $f(x) = \frac{1}{1+x}$ centered at $a = 1$.

(3) Find n such that $|T_n(1.3) - \sqrt{1.3}| \leq 10^{-6}$, where $T_n(x)$ is the Taylor polynomial for $\sqrt{x}$ at $a = 1$.

(4) (a) Use the fact that $\arctan(x)$ is an antiderivative of $\frac{1}{1+x^2}$ to find a Maclaurin series for $\arctan(x)$, and find the interval of convergence.

(b) Use the fact that $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$ and your answer to the previous part to find a series that converges to π.

(5) Find the interval of convergence of the following power series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n^4+2}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{3n}(x+3)^n$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(x+4)^n}{(n \ln n)^2}$

(6) Find the Taylor series of the following functions and determine the radius of convergence.

(a) $f(x) = \sin(2x)$, centered at $x = 0$.

(b) $f(x) = e^{4x}$, centered at $x = 0$.

(c) $f(x) = x^2 e^{x^2}$, centered at $x = 0$.

(d) $f(x) = \dfrac{1}{3x-2}$, centered at $c = -1$.

(e) $f(x) = (1+x)^{1/3}$, centered at $c = 0$.

(f) $f(x) = \sqrt{x}$, centered at $c = 4$.