REVIEW
1 August 2018

NAME: SOLUTIONS

The final exam is on Monday Aug 6 from 08:30 AM to 11:00 AM in Malott Hall 224.

Warning: *These problems are by no means a comprehensive representation of the material that might appear on the exam. That is, there may be topics not covered by these problems that you are still responsible for knowing. Let these problems be a supplement to your preparation for the exam, but be sure to review other sources (e.g. your notes, homework assignments, reading assignments, and the textbook) as well.*

(1) Compute the following integrals.

(a) $\displaystyle\int \frac{2-\cos x+\sin x}{\sin^2 x}\,dx$

ANSWER: $-2\cot x-\ln|\csc x+\cot x|+\csc x+C$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{2\,dx}{e^x+e^{-x}}$

ANSWER: π

(c) $\displaystyle\int \frac{dr}{1+\sqrt{r}}$

ANSWER: $2\sqrt{r}-2\ln(1+\sqrt{r})+C$

(d) $\displaystyle\int_{\pi/4}^{\pi/2} \sqrt{1+\cos 4x}\,dx$

ANSWER: $\dfrac{\sqrt{2}}{2}$

(e) $\int x^3 \sin x \, dx$

ANSWER: $3(x^2-2)\sin x - x(x^2-6)\cos x + C$

(f) $\int (27)^{3\theta+1} \, d\theta$

ANSWER: $\frac{1}{3}\left(\frac{27^{3\theta+1}}{\ln 27}\right) + C$

(g) $\int_{-1}^{1} \frac{dy}{y^{2/3}}$

ANSWER: 6

(h) $\int_{2}^{\infty} \frac{dx}{\sqrt{x} - \sqrt[4]{x}}$

ANSWER: diverges

(i) $\int_{2}^{4} \frac{dx}{x\sqrt{x^2-4}}$

ANSWER: $\pi/6$

(2) Find the following derivatives.

(a) $F'(x)$, where $F(x) = \int_{\tan^{-1} x}^{\pi/4} e^{\sqrt{t}}\, dt$

ANSWER: $-\dfrac{e^{\sqrt{\tan^{-1} x}}}{1+x^2}$

(b) $G'(x)$, where $G(x) = \int_{\ln x}^{\sin^2 x} e^{t}\, dt$

ANSWER: $2e^{\sin^2 x} \sin x \cos x - 1$

(3) (a) Suppose that $\int_{0}^{x^2} f(t)\, dt = x\cos(\pi x)$. Find $f(4)$.

ANSWER: $\dfrac{1}{4}$

(b) Suppose that $\int_{0}^{f(x)} t^2\, dt = x\cos(\pi x)$. Find $f(4)$.

ANSWER: $\sqrt[3]{12}$

(4) Let R be the "triangular" region in the first quadrant that is bounded above by the line $y = 1$, below by the curve $y = \ln x$, and on the left by $x = 1$.

(a) Find the area of the region R.

ANSWER: $e - 2$

(b) Find the volume of the solid obtained by rotating R around the x-axis.

ANSWER: π

(c) Find the volume of the solid obtained by rotating R around the line $x = 1$.

ANSWER: $\dfrac{\pi}{2}(5 - 4e + e^2)$

(5) The (infinite) region bounded by the coordinate axes and the curve $y = -\ln x$ in the first quadrant is revolved about the x-axis to generate a solid. Find the volume of the solid.

ANSWER: 2π

(6) (a) Find the length of the curve $y = \ln x$ from $x = 1$ to $x = e$.

ANSWER: $\sqrt{1+e^2} - \ln\left(\frac{\sqrt{1+e^2}}{e} + \frac{1}{e}\right) - \sqrt{2} + \ln(1+\sqrt{2})$

(b) Find the surface area of the surface generated by rotating $y = \ln x$ from $x = 1$ to $x = e$ around the y-axis.

ANSWER: $\pi\left(-\sqrt{2} + e\sqrt{1+e^2} - \sinh^{-1}(1) + \sinh^{-1}(e)\right)$

(7) A reservoir shaped like a right circular cone, point down, 20 ft across on the top and 8 feet deep, is full of water. How much work does it take to pump the water to a level of 6 feet above the top? (The density of water is approximately 62.4 lb/ft^3.)

ANSWER: $\approx 418,208.81$ (ft-lb)

(8) Find the following limits.

(a) $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n$

ANSWER: $\frac{1}{e}$

(b) $\lim_{n\to\infty}\frac{1}{n}\int_1^n\frac{1}{x}\,dx$

ANSWER: 0

(c) $\lim_{n\to\infty}\sum_{k=1}^{n}\ln\sqrt[n]{1+\frac{k}{n}}$.

ANSWER: $\ln 4-1$

(9) Determine if the following series converge absolutely, converge conditionally, or diverge.

(a) $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

ANSWER: Converges absolutely. (Direct comparison to $\sum(1/n^2)$.)

(b) $\sum_{n=3}^{\infty} \frac{\ln n}{\ln(\ln n)}$

ANSWER: Diverges. (Divergence test.)

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n(n^2+1)}{2n^2+n-1}$

ANSWER: Diverges. (Divergence test.)

(d) $\sum_{n=1}^{\infty} \frac{(\arctan n)^2}{n^2+1}$

ANSWER: Converges. (Limit comparison to $\sum(1/n^2)$.)

(e) $\sum_{n=2}^{\infty} \frac{\log_n(n!)}{n^3}$

ANSWER: Converges. (Direct comparison to $\sum(1/n^2)$.)

(f) $\sum_{n=1}^{\infty} a_n$, where $a_1 = 2$, $a_{n+1} = \frac{6n+1}{5n+3} a_n$

ANSWER: Diverges (Ratio Test.)

(10) Find the interval of convergence of the following power series.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(3x-1)^n}{n^2}$

ANSWER: $[0, 2/3]$

(b) $\sum_{n=1}^{\infty} (\operatorname{csch} n) x^n$

ANSWER: $(-e, e)$

(11) Find the Maclaurin series for the given functions.

(a) $f(x) = \cos(x^{5/2})$

ANSWER: $\sum_{n=0}^{\infty} \frac{x^{5n}}{(2n)!}$

(b) $f(x) = \frac{1}{(x-1)^2}$

ANSWER: $\sum_{n=1}^{\infty} nx^{n-1}$

(12) (a) Find the Maclaurin series of $\arctan x$. What is the interval of convergence of the series? (Hint: first find a series expansion for $(d/dx)(\arctan x)$.)

SOLUTION: We have

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

for $|x| < 1$. Since $\arctan 0 = 0$, it follows that

$$\arctan x = \int_0^x \frac{1}{1+t^2}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

on the interval $(-1, 1)$.

(b) Use the fact that $\tan(\pi/6) = 1/\sqrt{3}$ to express π as the sum of an infinite series.

SOLUTION: We have $\arctan(1/\sqrt{3}) = \pi/6$. Since $1/\sqrt{3}$ is in the interval $(-1, 1)$, we can use the series above.

$$\frac{\pi}{6} = \arctan(1/\sqrt{3}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{(2n+1)/2}(2n+1)}$$

Hence

$$\pi = \sum_{n=0}^{\infty} \frac{6 \cdot (-1)^n}{3^{(2n+1)/2}(2n+1)}.$$

(13) Let $f(x) = \sqrt{1+x}$.

(a) Find $f''(x)$ and $f''(0)$.

ANSWER: $f''(x) = -\frac{1}{4}(1+x)^{-3/2}$, and $f''(0) = -1/4$.

(b) Find the first three terms of the Maclaurin series of $f(x)$.

ANSWER: $1 + \frac{1}{2}x - \frac{1}{8}x^2$

(c) Find the first three *nonzero* terms of the Maclaurin series of $\int_0^x \sqrt{1+t^3}\, dt$.

ANSWER: $x + \frac{1}{8}x^4 - \frac{1}{56}x^7$