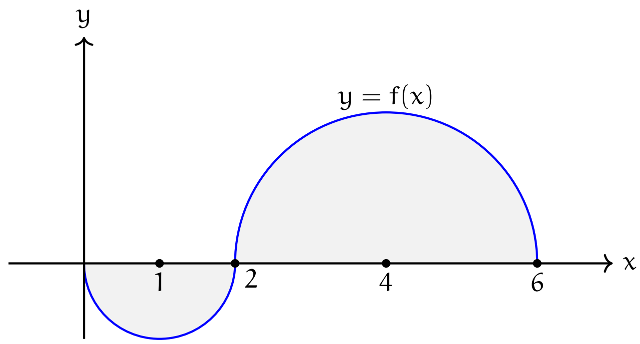
HOMEWORK 1

Math 1910, Summer 2018

NAME: SOLUTIONS

Due 3 July 2018

(1) Evaluate $\int_1^4 f(x)\,dx$ and $\int_1^6 |f(x)|\,dx$ using the graph below. The two parts of the graph are semicircles.



SOLUTION: The definite integral $\int_1^4 f(x)\,dx$ is the signed area of one-quarter of a circle of radius 1 which lies below the x-axis and one-quarter of a circle of radius 2 which lies above the x axis. Therefore,

$$\int_1^4 f(x)\,dx = \frac{1}{4}\pi(2)^2 - \frac{1}{4}\pi(1)^2 = \frac{3}{4}\pi.$$

The definite integral $\int_1^4 f(x)\,dx$ is the total area of the shaded region in the picture, which is one-half of a circle of radius 1 and one-half of a circle of radius 2. The total area is then

$$\int_1^6 |f(x)|\,dx = \frac{1}{2}\pi(1)^2 + \frac{1}{2}\pi(2)^2 = \frac{\pi}{2} + 2\pi = \frac{5}{2}\pi.$$

(2) Let A be the area under $f(x) = \sqrt{x}$ over the interval $[0, 1]$. Prove that $0.52 \leq A \leq 0.77$ without computing an integral. Explain your reasoning.

SOLUTION: To establish upper and lower bounds for the area under $f(x) = \sqrt{x}$, we estimate using either left-endpoint or right-endpoint rectangles. In either case, four rectangles suffice.

Let L_4 be the left endpoint approximation for $f(x)$ with four rectangles, and let R_4 be the right endpoint approximation with four rectangles.

For the left endpoint approximation,

$$L_4 = (0.25)\sqrt{0} + (0.25)\sqrt{0.25} + (0.25)\sqrt{0.5} + (0.25)\sqrt{0.75} \approx 0 + 0.125 + 0.177 + 0.216 = 0.518$$

For the right endpoint approximation,

$$R_4 = (0.25)\sqrt{0.25} + (0.25)\sqrt{0.5} + (0.25)\sqrt{0.75} + (0.25)\sqrt{1} \approx 0.125 + 0.177 + 0.216 + 0.25 = 0.768$$

Since $L_4 < A < R_4$, we know that $0.518 < A < 0.768$ with some rounding.

(3) Evaluate the indefinite integral.

(a) $\int \frac{1}{x^{4/3}}\, dx.$

SOLUTION: $\int \frac{1}{x^{4/3}}\, dx = \int x^{-4/3}\, dx = \frac{x^{-1/3}}{-1/3} + C = -\frac{3}{x^{1/3}} + C$

(b) $\int \left(\frac{4}{x} - e^x\right)\, dx$

SOLUTION: $\int \left(\frac{4}{x} - e^x\right)\, dx = \int \frac{4}{x}\, dx - \int e^x\, dx = 4\ln|x| + e^x + C$

(c) $\int (z^5 + 4z^2)(z^3+1)^{12}\, dz.$

SOLUTION: Let $u = z^3 + 1$. Then $du = 3z^2\, dz$ and $z^3 = u - 1$ and

$$\begin{aligned}\int (z^5 + 4z^2)(z^3+1)^{12} &= \frac{1}{3}\int (u+3)u^{12}\, du \\ &= \frac{1}{3}\int u^{13} + 3x^{12}\, du \\ &= \frac{1}{3}\left(\frac{1}{14}u^{14} + \frac{3}{13}u^{13}\right) + C \qquad = \frac{1}{42}(z^3+1)^{14} + \frac{1}{13}(z^3+1)^{14} + C\end{aligned}$$

(d) $\int x^2\sqrt{x+1}\, dx$

SOLUTION: Let $u = x + 1$. Then $x = u - 1$ and $du = dx$. Hence,

$$\begin{aligned}\int x^2\sqrt{x+1}\, dx = \int (u-1)^2 u^{1/2}\, du &= \int \left(u^{5/2} - 2u^{3/2} + u^{1/2}\right) du \\ &= \frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C \\ &= \frac{2}{7}(x+1)^{7/2} - \frac{4}{5}(x+1)^{5/2} + \frac{2}{3}(x+1)^{3/2} + C.\end{aligned}$$

(4) Evaluate the definite integral.

(a) $\int_1^{27} \frac{t+1}{\sqrt{t}}\, dt$

SOLUTION:

$$\begin{aligned}\int_1^{27} \frac{t+1}{\sqrt{t}}\, dt &= \int_1^{27} (t^{1/2} + t^{-1/2})\, dt \\ &= \left(\frac{2}{3}t^{3/2} + 2t^{1/2}\right)\Big|_1^{27} \\ &= \left(\frac{2}{3}(81\sqrt{3} + 6\sqrt{3})\right) - \left(\frac{2}{3} + 2\right) \\ &= 60\sqrt{3} - \frac{8}{3}.\end{aligned}$$

(b) $\int_0^5 |x^2 - 4x + 3|\, dx$

SOLUTION:

$$\begin{aligned}\int_0^5 |x^2 - 4x + 3|\, dx &= \int_0^5 (x-3)(x-1)\, dx \\ &= \int_0^1 (x^2 - 4x + 3)\, dx + \int_1^3 -(x^2 - 4x + 3)\, dx + \int 3^5 (x^2 - 4x + 3)\, dx \\ &= (\frac{1}{3}x^3 - 2x^2 + 3x)\Big|_0^1 - (\frac{1}{3}x^3 - 2x^2 + 3x)\Big|_1^3 + (\frac{1}{3}x^3 - 2x^2 + 3x)\Big|_3^5 \\ &= (\frac{1}{3} - 2 + 3) - 0 - (9 - 18 + 9) + (\frac{1}{3} - 2 + 3) + (\frac{125}{3} - 50 + 15) - (9 - 18 + 9) \\ &= \frac{28}{3}.\end{aligned}$$

(c) $\int_{\pi/4}^{5\pi/8} \cos 2x\, dx$

SOLUTION:

$$\begin{aligned}\int_{\pi/4}^{5\pi/8} \cos 2x\, dx &= \frac{1}{2}\sin 2x\Big|_{\pi/4}^{5\pi/8} \\ &= \frac{1}{2}\sin\frac{5\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2} \\ &= -\frac{\sqrt{2}}{4} - \frac{1}{2}\end{aligned}$$

(d) $\displaystyle\int_0^{\sqrt{e-1}} \frac{x^3}{x^2+1}\, dx$

SOLUTION: Let $u = x^2 + 1$, so $du = 2x\, dx$. The bounds change from $x = 0$ to $u = 1$ and from $x = \sqrt{e-1}$ to $u = e$.

$$\int_0^{\sqrt{e-1}} \frac{x^3}{x^2+1}\, dx = \int_1^e \frac{1}{2}\frac{x^2}{u}\, du.$$

To get rid of the x^2 in this integral, use the equation $u = x^2 + 1 \implies x^2 = u - 1$. The integral becomes

$$\begin{aligned}
\frac{1}{2}\int_1^e \frac{u-1}{u}\, du &= \frac{1}{2}\int_1^e \left(1 - \frac{1}{u}\right)\, du \\
&= \frac{1}{2}\int_1^e 1\, du - \frac{1}{2}\int_1^e \frac{1}{u}\, du \\
&= \frac{1}{2}u\Big|_1^e - \frac{1}{2}\ln|u|\Big|_1^e \\
&= \frac{1}{2}(e-1) - \frac{1}{2}(1-0) \\
&= \frac{e}{2} - 1
\end{aligned}$$

(5) Show that $f(x) = \tan^2(x)$ and $g(x) = \sec^2(x)$ have the same derivative. What can you conclude about the relationship between f and g?

SOLUTION:

$$f'(x) = \frac{d}{dx}\tan^2(x) = 2\tan(x)\cdot\sec^2(x) = 2\sec^2(x)\tan(x)$$
$$g'(x) = \frac{d}{dx}\sec^2(x) = 2\sec(x)\cdot\sec(x)\tan(x) = 2\sec^2(x)\tan(x)$$

This means that both f and g are antiderivatives of the function $2\sec^2(x)\tan(x)$; they therefore differ by a constant. To figure out what this constant is, plug in a particular value of x, say $x = 0$.

$$f(0) = \tan^2(0) = 0$$
$$g(0) = \sec^2(0) = 1$$

So $f(x) = g(x) + 1$.

(6) Calculate the derivative. $\frac{d}{dx}\int_0^{x^2}\frac{t\,dt}{t+1}$

SOLUTION: By the chain rule and the fundamental theorem of calculus,

$$\frac{d}{dx}\int_0^{x^2}\frac{t\,dt}{t+1} = \frac{x^2}{x^2+1}\cdot 2x = \frac{2x^3}{x^2+1}.$$

(7) Let $N(d)$ be the number of asteroids of diameter d kilometers. Data suggest that the diameters are distributed according to a piecewise power law:

$$N'(d) = \begin{cases} 1.9 \times 10^9 d^{-2.3}, \text{ for } d < 70 \\ 2.6 \times 10^{12} d^{-4}, \text{ for } d \geq 70 \end{cases}$$

(a) Compute the number of asteroids with a diameter between 0.1 km and 100 km.

SOLUTION: The number of asteroids with diameter between 0.1 and 100 km

$$\begin{aligned} \int_{0.1}^{100} N'(d)\, dd &= \int_{0.1}^{70} 1.9 \times 10^9 d^{-2.3}\, dd + \int_{70}^{100} 2.6 \times 10^{12} d^{-4}\, dd \\ &= -\frac{1.9 \times 10^9}{1.3} d^{-1.3}\Big|_{0.1}^{70} - \frac{2.6 \times 10^{12}}{3} d^{-3}\Big|_{70}^{100} \\ &= 2.916 \times 10^{10} + 1.66 \times 10^6 \approx 2.916 \times 10^{10}. \end{aligned}$$

(b) Using the approximation $N(d+1)N(d) \approx N'(d)$, estimate the number of asteroids of diameter 50km.

SOLUTION: Taking $d = 49.5$,

$$N(50.5) - N(49.5) \approx N'(49.5) = 1.9 \times 10^9 49.5^{-2.3} = 240,525.70.$$

Thus, there are approximately 240,526 asteroids of diameter 50 km.