HOMEWORK 3

Math 1910, Summer 2018

NAME: SOLUTIONS

Due 24 July 2018

(1) Evaluate the following integrals, or state that they diverge.

(a) $\int_0^\infty e^{-x}\cos(x)\,dx$

SOLUTION:

First evaluate the indefinite integral using integration by parts, with $u = e^{-x}$, $dv = \cos x\,dx$. Then $du = -e^{-x}$, $v = \sin x$, and

$$\int e^{-x}\cos x dx = e^{-x}\sin x - \int \sin x\left(-e^{-x}\right)dx = e^{-x}\sin x + \int e^{-x}\sin x dx$$

Now use integration by parts again, with $u = e^{-x}$, $dv = \sin x\,dx$. Then $du = -e^{-x}dx$, $v = -\cos x$, and

$$\int e^{-x}\cos x dx = e^{-x}\sin x + \left[-e^{-x}\cos x - \int e^{-x}\cos x dx\right].$$

Solving this equation for $\int e^{-x}\cos x dx$, we find

$$\int e^{-x}\cos x dx = \frac{1}{2}e^{-x}(\sin x - \cos x) + C.$$

Thus,

$$\int_0^R e^{-x}\cos x dx = \frac{1}{2}e^{-x}\left(\sin x - \cos x\right)\Big|_0^R = \frac{\sin R - \cos R}{2e^R} - \frac{\sin 0 - \cos 0}{2} = \frac{\sin R - \cos R}{2e^R} + \frac{1}{2},$$

and

$$\int_0^\infty e^{-x}\cos x dx = \lim_{R\to\infty}\left(\frac{\sin R - \cos R}{2e^R} + \frac{1}{2}\right) = 0 + \frac{1}{2} = \frac{1}{2}.$$

(b) $\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx$

SOLUTION:

We begin by letting $x = 3\sin\theta$, which gives $dx = 3\cos\theta d\theta$. Substituting into the integral, we obtain

$$\int_0^3 \frac{dx}{\sqrt{9-x^2}} = \int_0^{\frac{\pi}{2}} \frac{3\cos\theta d\theta}{\sqrt{9-9\sin^2\theta}} = \int_0^{\frac{\pi}{2}} \frac{\cos\theta d\theta}{\sqrt{1-\sin^2}} = \int_0^{\frac{\pi}{2}} \frac{\cos\theta d\theta}{\cos\theta} = \int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{2}$$

(c) $\int_4^\infty \frac{1}{(x-2)(x-3)}\,dx$

SOLUTION:

The partial fraction decomposition takes the form

$$\frac{1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}.$$

Clearing denominators gives us

$$1 = A(x-3) + B(x-2).$$

Setting $x = 2$ then yields $A = -1$, while setting $x = 3$ yields $B = 1$. Thus,

$$\int \frac{dx}{(x-2)(x-3)} = \int \frac{dx}{x-3} - \int \frac{dx}{x-2} = \ln|x-3| - \ln|x-2| + C = \ln\left|\frac{x-3}{x-2}\right| + C,$$

and, for $R > 4$,

$$\int_4^R \frac{dx}{(x-2)(x-3)} = \ln\left|\frac{x-3}{x-2}\right|_4^R = \ln\left|\frac{R-3}{R-2}\right| - \ln\frac{1}{2}.$$

Then

$$\int_4^\infty \frac{1}{(x-2)(x-3)}\,dx = \lim_{R\to\infty}\left(\ln\left|\frac{R-3}{R-2}\right| - \ln\frac{1}{2}\right) = \ln 1 - \ln\frac{1}{2} = \ln 2.$$

(d) $\int_0^1 \frac{1}{x^{1/3}+x^{2/3}}\,dx$

SOLUTION:

To begin, let $u = x^{1/3}$. Then $du = \frac{dx}{3x^{2/3}}$. Thus we have

$$\int \frac{1}{x^{1/3}+x^{2/3}}dx = \int \frac{1}{u+u^2}dx = \int \frac{3u}{(u+1)(3u^2)}dx = \int \frac{3u}{u+1}du$$

Since

$$3\frac{u}{u+1} = 3\frac{u+1-1}{u+1} = 3\frac{u+1}{u+1} - \frac{3}{u+1} = 3 - \frac{3}{u+1},$$

we have

$$\int 3\frac{u}{u+1} = \int 3 - \frac{3}{u+1} = u - 3\ln(u+1),$$

and thus

$$\int_0^1 \frac{1}{x^{1/3}+x^{2/3}} = 3x^{1/3} - 3\ln(x^{1/3}+1)|_0^1 = 3 - 3\ln(2).$$

(2) Find a constant C such that $p(x)$ is a probability density function on the given interval, and compute the probability indicated.

(a) $p(x) = \frac{C}{(x+1)^3}$ on $[0, \infty)$; $\quad P(0 \le X \le 1)$.

SOLUTION: Compute the indefinite integral using the substitution $u = x + 1$, $du = dx$:

$$\int p(x)dx = \int \frac{C}{(x+1)^3}dx = -\frac{1}{2}C(x+1)^{-2} + K$$

For p to be a probability density function, we must have

$$1 = \int_0^\infty p(x)dx = -\frac{1}{2}C \lim_{R\to\infty} (x+1)^{-2}\Big|_0^R = \frac{1}{2}C - \frac{1}{2}C \lim_{R\to\infty}(R+1)^{-2} = \frac{1}{2}C$$

so that $C = 2$, and $p(x) = \frac{2}{(x+1)^3}$. Then using the indefinite integral above,

$$P(0 \le X \le 1) = \int_0^1 \frac{2}{(x+1)^3} = -\frac{1}{2} \cdot 2 \cdot (x+1)^{-2}\Big|_0^1 = -\frac{1}{4} + 1 = \frac{3}{4}.$$

(b) $p(x) = \frac{Ce^{-x}}{1+e^{-2x}}$ on $(-\infty, \infty)$; $\quad P(X \le -4)$.

SOLUTION:

Compute the indefinite integral using the substitution $u = e^{-x}$; then $du = -e^{-x}dx$, and

$$\int p(x)dx = \int \frac{Ce^{-x}}{1+e^{-2x}}dx = -C\int \frac{du}{1+u^2} = -C\tan^{-1}u + K = -C\tan^{-1}\left(e^{-x}\right) + K.$$

Using the identity

$$\tan^{-1}y + \tan^{-1}\frac{1}{y} = \frac{\pi}{2}$$

the indefinite integral can be expressed as

$$\int p(x)dx = C\tan^{-1}(e^x) + K',$$

where $K' = K - \frac{\pi}{2}C$. For p to be a probability density function, we must have

$$\begin{aligned}1 &= \int_{-\infty}^\infty p(x)dx = \int_{-\infty}^0 p(x)dx + \int_0^\infty p(x)dx \\ &= -C\left(\lim_{R\to-\infty} \tan^{-1}(e^{-x})\big|_R^0 + \lim_{R\to\infty} \tan^{-1}(e^{-x})\big|_0^R\right) \\ &= -C\left(\frac{\pi}{4} - \lim_{R\to-\infty} \tan^{-1}\left(e^{-R}\right) + \lim_{R\to\infty} \tan^{-1}\left(e^{-R}\right) - \frac{\pi}{4}\right) \\ &= -C(-\frac{\pi}{2} + 0) = \frac{\pi}{2}C\end{aligned}$$

so that $C = \frac{2}{\pi}$ and $p(x) = \frac{2e^{-x}}{\pi(1+e^{-2x})}$. Then using the definite integral above,

$$P(X \le -4) = \int_{-\infty}^{-4} p(x)dx = \lim_{R\to-\infty} \frac{2}{\pi}\tan^{-1}(e^x)\Big|_R^{-4} = \frac{2}{\pi}\tan^{-1}\left(e^{-4}\right) - \frac{2}{\pi}\lim_{R\to-\infty}\tan^{-1}\left(e^R\right)$$
$$= \frac{2}{\pi}\tan^{-1}\left(e^{-4}\right) \approx 0.0117$$

(3) The distance r between the electron and the nucleus in a hydrogen atom is a random variable with probability density $p(r) = 4a_0^{-3}r^2e^{-2r/a_0}$ for $r \geq 0$, where a_0 is the Bohr radius, $a_0 \approx 5.29 \times 10^{-11}$ m.

(a) Calculate the probability P that the electron is within one Bohr radius of the nucleus.

SOLUTION:
The probability P is the integral of $p(x)$ from 0 to a_0. To calculate P, use the substitution $u = \frac{2r}{a_0}$:

$$P = \int_0^{a_0} p(r)dr = \frac{4}{a_0^3}\int_0^{a_0} r^2e^{-2r/a_0}dr = \left(\frac{4}{a_0^3}\right)\left(\frac{a_0^3}{8}\right)\int_0^2 u^2e^{-u}du.$$

The constant in front simplifies to $\frac{1}{2}$, and the formula in the margin gives us

$$P = \frac{1}{2}\int_0^2 u^2e^{-u}du = \frac{1}{2}\left(-\left(u^2+2u+2\right)e^{-u}\right)\Big|_0^2 = \frac{1}{2}\left(2-10e^{-2}\right) \approx 0.32.$$

Thus, the electron is within a distance a_0 of the nucleus with probability 0.32.

(b) Calculate the average distance between the electron and the nucleus.

SOLUTION:
The mean of the distribution is

$$\mu = \int_0^\infty rp(r)dr = \int_0^\infty r\cdot 4a_0^{-3}r^2e^{-2r/a_0}dr = \frac{4}{a_0^3}\int_0^\infty r^3e^{-2r/a_0}dr.$$

To calculate this integral, use as before the substitution $x = \frac{2r}{a_0}$ to get

$$\mu = \frac{4}{a_0^3}\cdot\frac{a_0^3}{8}\cdot\frac{a_0}{2}\int_0^\infty x^3e^{-x}dx = \frac{a_0}{4}\int_0^\infty x^3e^{-x}dx.$$

To calculate this integral, we use integration by parts, with $u = x^3$, and $dv = e^{-x}dx$, so that $du = 3x^2dx$, and $v = -e^{-x}$; then

$$\mu = \frac{a_0}{4}\left(-x^3\,e^{-x}\big|_0^\infty + 3\int_0^\infty x^2e^{-x}dx\right)$$

The first term is evaluated as follows, using L'Hopital's Rule multiple times:

$$-x^3\,e^{-x}\big|_0^\infty = \lim_{R\to\infty}\left(-x^3e^{-x}\right)\Big|_0^R = \lim_{R\to\infty}\left(-\frac{R^3}{e^R}\right)$$
$$= \lim_{R\to\infty}\left(-\frac{3R^2}{e^R}\right) = \lim_{R\to\infty}\left(-\frac{6R}{e^R}\right) = \lim_{R\to\infty}\left(-\frac{6}{e^R}\right) = 0$$

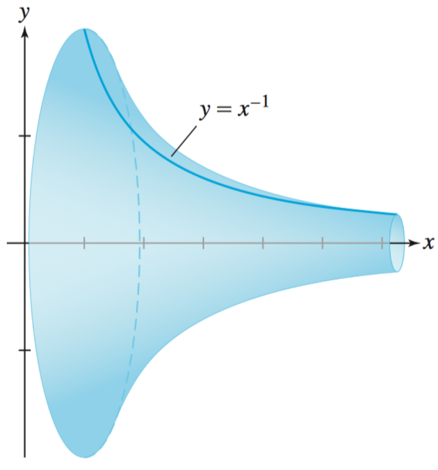
The second term, by part (a), is

$$\int_0^\infty x^2 e^{-x} dx = \lim_{R\to\infty} \left(\left(-u^2 + 2u + 2 \right) e^{-u} \right) \Big|_0^R = \lim_{R\to\infty} \left(2 - \frac{-R^2 + 2R + 2}{e^R} \right) = 2$$

using L'Hopital's Rule as in the previous formulas. Thus, finally,

$$\mu = \frac{a_0}{4}(0 + 3 \cdot 2) = \frac{3}{2} a_0.$$

(4) The solid S obtained by rotating the region below the graph of $y = x^{-1}$ around the x axis for $1 \leq x < \infty$ is called *Gabriel's Horn*.



(a) Compute the volume of S.

SOLUTION:

The volume is given by

$$V = \int_1^\infty \pi \left(\frac{1}{x}\right)^2 dx.$$

First compute the volume over a finite interval

$$\int_1^R \pi \left(\frac{1}{x}\right)^2 dx = \pi \int_1^R x^{-2} dx = \pi \left.\frac{x^{-1}}{-1}\right|_1^R = \pi \left(\frac{-1}{R} - \frac{-1}{1}\right) = \pi \left(1 - \frac{1}{R}\right).$$

Thus,

$$V = \lim_{R\to\infty} \int_1^\infty \pi x^{-2} dx = \lim_{R\to\infty} \pi \left(1 - \frac{1}{R}\right) = \pi$$

Question (4), continued.

(b) Compute the surface area of S.

SOLUTION:

For $x > 1$, we have

$$\frac{1}{x}\sqrt{1+\frac{1}{x^4}} = \frac{1}{x}\sqrt{\frac{x^4+1}{x^4}} = \frac{\sqrt{x^4+1}}{x^3} \geq \frac{\sqrt{x^4}}{x^3} = \frac{x^2}{x^3} = \frac{1}{x}$$

The integral $\int_1^\infty \frac{1}{x}dx$ diverges, since $p = 1 \geq 1$. Therefore, by the comparison test,

$$\int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}dx \text{ also diverges.}$$

Finally,

$$A = 2\pi\int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}dx$$

diverges, and thus the surface area of the solid is infinite.

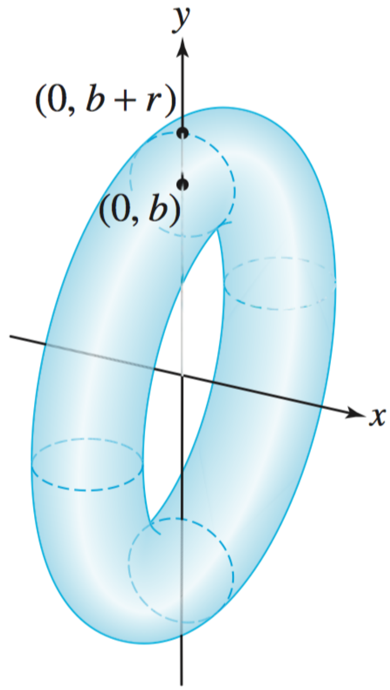
(c) What is surprising about this? Would you rather use one of these as a cup or cut it up and use the pieces as paper?

SOLUTION:

The obvious answer here is that we would typically expect a surface to have finite surface area if and only if it has finite volume. This is just another example of how weird infinity can be.

The choice is tricky. A cup with infinite surface area would be pretty awesome to have. On the other hand, if you had infinite paper, you could sell paper as cheaply as you wanted and become rich as the world's number one paper supplier. Cool cup or infinite riches? Kind of a toss up, I suppose.

(5) Find the surface area of the torus obtained by rotating the circle $x^2 + (y - b)^2 = r^2$ around the x-axis.



SOLUTION: $y = b + \sqrt{a^2 - x^2}$ gives the top half of the circle, and $y = b - \sqrt{a^2 - x^2}$ gives the bottom half. Note that in each case,

$$1 + (y')^2 = 1 + \frac{x^2}{a^2 - x^2} = \frac{a^2}{a^2 - x^2}.$$

Rotating the two halves of the circle around the x-axis then yields

$$\begin{aligned} SA &= 2\pi \int_{-a}^{a} \left(b + \sqrt{a^2 - x^2}\right) \frac{a}{\sqrt{a^2 - x^2}} dx + 2\pi \int_{-a}^{a} \left(b - \sqrt{a^2 - x^2}\right) \frac{a}{\sqrt{a^2 - x^2}} dx \\ &= 2\pi \int_{-a}^{a} 2b \frac{a}{\sqrt{a^2 - x^2}} dx = 4\pi ba \int_{-a}^{a} \frac{1}{\sqrt{a^2 - x^2}} dx \\ &= 4\pi ba \cdot \sin^{-1}\left(\frac{x}{a}\right)\Big|_{-a}^{a} = 4\pi ba \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 4\pi^2 ba \end{aligned}$$