HOMEWORK 4

Math 1910, Summer 2018

NAME: SOLUTIONS

Due 2 August 2018

(1) Determine the limit of the sequence $x_n = \frac{e^n + (-3)^n}{5^n}$ or show that it diverges.

SOLUTION:

$$\lim_{n\to\infty} \frac{e^n + (-3)^n}{5^n} = \lim_{n\to\infty} \left(\frac{e}{5}\right)^n + \lim_{n\to\infty} \left(\frac{-3}{5}\right)^n$$

assuming both limits on the right-hand side exist. But by the limit of Geometric Sequences, since

$$-1 < \frac{-3}{5} < 0 < \frac{e}{5} < 1$$

both limits on the right-hand side are 0, so that x_n converges to 0.

(2) Give an example of a divergent sequence $\{a_n\}$ such that $\lim_{n\to\infty} |a_n|$ converges.

SOLUTION: Let $a_n = (-1)^n$. Then $\{a_n\}$ diverges, and $\{|a_n|\}$ is the constant sequence which is always 1.

(3) Find the sum: $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots$

SOLUTION: This series can be written

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}.$$

Use partial fraction decomposition to find that

$$\frac{1}{(2n-1)(2n+1)}=\frac{1}{2}\frac{1}{2n-1}-\frac{1}{2}\frac{1}{2n+1},$$

Thus the series telescopes: it has partial sums

$$\begin{aligned}S_N &= \frac{1}{2}\left(\frac{1}{1}-\frac{1}{3}\right)+\frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}\right)+\frac{1}{2}\left(\frac{1}{5}-\frac{1}{7}\right)\\ &\quad+\cdots+\frac{1}{2}\left(\frac{1}{2N-1}-\frac{1}{2N+1}\right)\\ &=\frac{1}{2}\left(1+\frac{1}{2N+1}\right).\end{aligned}$$

Hence we have

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}=\lim_{N\to\infty}S_N=\frac{1}{2}.$$

(4) Express the integral $\int_0^1 \arctan(x^2)\,dx$ as an infinite series and find its value to within 10^{-4}.

SOLUTION: Substituting x^2 for x in the Maclaurin series for $\tan^{-1}(x)$ yields

$$\tan^{-1}\left(x^2\right)=\sum_{n=0}^{\infty}(-1)^n\frac{\left(x^2\right)^{2n+1}}{2n+1}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{2n+1};$$

therefore,

$$\int_0^1 \tan^{-1}\left(x^2\right)dx=\sum_{n=0}^{\infty}(-1)^n\left.\frac{x^{4n+3}}{(2n+1)(4n+3)}\right|_0^1=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(4n+3)}.$$

This is an alternating series with $a_n=\frac{1}{(2n+1)(4n+3)}$; therefore, the error incurred by using S_N to approximate the value of the definite integral is bounded by

$$\left|\int_0^1 \tan^{-1}\left(x^2\right)dx-S_N\right|\le a_{N+1}=\frac{1}{(2N+3)(4N+7)}.$$

To guarantee the error is at most 0.0001, we must choose N so that

$$\frac{1}{(2N+3)(4N+7)}<0.0001 \quad \text{or} \quad (2N+3)(4N+7)>10000.$$

For $N=33$, $(2N+3)(4N+7)=(69)(139)=9591<10000$ and for $N=34$, $(2N+3)(4N+7)>10000$; thus, the smallest acceptable value for N is $N=34$. The corresponding approximation is

$$S_{34}=\sum_{n=0}^{34}\frac{(-1)^n}{(2n)!(4n+1)}=0.297953297$$

(5) Determine convergence or divergence of the series.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

SOLUTION:
Let $f(x) = \frac{1}{x(\ln x)^2}$. This function is positive and continuous for $x \geq 2$. Moreover,

$$f'(x) = -\frac{1}{x^2(\ln x)^4}\left(1 \cdot (\ln x)^2 + x \cdot 2(\ln x) \cdot \frac{1}{x}\right) = -\frac{1}{x^2(\ln x)^4}\left((\ln x)^2 + 2\ln x\right).$$

Since $\ln x > 0$ for $x > 1$, $f'(x)$ is negative for $x > 1$; hence, f is decreasing for $x \geq 2$. To compute the improper integral, we make the substitution $u = \ln(x)$, $du = \frac{1}{x}dx$. We obtain:

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{R\to\infty} \int_2^R \frac{1}{x(\ln x)^2}\,dx = \lim_{R\to\infty} \int_{\ln 2}^{\ln R} \frac{du}{u^2}$$
$$= -\lim_{R\to\infty}\left(\frac{1}{\ln R} - \frac{1}{\ln 2}\right) = \frac{1}{\ln 2}.$$

The integral converges; hence, the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ also converges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n + n}{e^{2n} - n^2}$

SOLUTION: Let

$$a_n = \frac{e^n + n}{e^{2n} - n^2} = \frac{e^n + n}{(e^n - n)(e^n + n)} = \frac{1}{e^n - n}$$

For large n,

$$\frac{1}{e^n - n} \approx \frac{1}{e^n} = e^{-n},$$

so we apply the Limit Comparison Test with $b_n = e^{-n}$. We find

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{e^n - n}}{e^{-n}} = \lim_{n\to\infty} \frac{e^n}{e^n - n} = 1.$$

The series $\sum_{n=1}^{\infty} e^{-n} = \sum_{n=1}^{\infty}\left(\frac{1}{e}\right)^n$ is a geometric series with $r = \frac{1}{e} < 1$, so it converges. Because L exists, by the Limit Comparison Test we can conclude that the series $\sum_{n=1}^{\infty} \frac{e^n+n}{e^{2n}-n^2}$ also converges.

(c) $\displaystyle\sum_{n=4}^{\infty} \left(1 + \frac{1}{n}\right)^{-n^2}$

SOLUTION: With $a_k = \left(1 + \frac{1}{n}\right)^{-n^2}$,

$$\sqrt[n]{a_n} = \sqrt[n]{\left(1 + \frac{1}{n}\right)^{-n^2}} = \left(1 + \frac{1}{n}\right)^{-n} \text{ and } \lim_{n\to\infty} \sqrt[n]{a_n} = e^{-1} < 1$$

Therefore, the series $\sum_{k=0}^{\infty} \left(1 + \frac{1}{n}\right)^{-n^2}$ converges by the Root Test.

Question (5), continued.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + \sin n}$

SOLUTION: Apply the Limit Comparison Test with $a_n = \frac{1}{n^2+\sin n}$ and $b_n = \frac{1}{n^2}$:

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^2+\sin n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{1+\frac{\sin n}{n^2}} = 1.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series. Because L exists, by the Limit Comparison Test we can conclude that the series $\sum_{n=1}^{\infty} \frac{1}{n^2+\sin n}$ also converges.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$

SOLUTION: Apply the Limit Comparison Test with $a_n = \frac{\sin(1/n)}{\sqrt{n}}$ and $b_n = \frac{1/n}{\sqrt{n}}$:

$$L = \lim_{n\to\infty} \frac{\sin(1/n)}{\sqrt{n}} \cdot \frac{\sqrt{n}}{1/n} = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{u\to 0} \frac{\sin u}{u} = 1$$

so that Σa_n and Σb_n either both converge or both diverge. But

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1/n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

is a convergent p-series, and so our original series converges as well.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n!}$

SOLUTION: We use the ratio test:

$$\lim_{n\to\infty} \frac{\frac{e^{n+1}}{(n+1)!}}{\frac{e^n}{n!}} = \lim_{n\to\infty} \frac{e}{n+1} = 0$$

and thus the series converges.

(6) Find the Taylor series centered at 0 and the interval on which the expansion is valid.

(a) $x^4 + 3x - 1$

SOLUTION: To determine the Taylor series for $x^4 + 3x - 1$, we must evaluate the function and its first four derivatives of the function at 0. This yields $f(0) = -1$, $f'(0) = 3$, $f''(0) = 0$, $f'''(0) = 0$, and $f''''(0) = 24$.

From this, and using the Taylor Series formula $T = \Sigma_{n=0}^{\infty} \frac{f^{(n)}}{n!}(x-c)^n$, we obtain $T = x^4 + 3x - 1$, which converges everywhere.

(b) $(x^2 + 2x)e^x$

SOLUTION:
Using the Maclaurin series for e^x, we find

$$\left(x^2+2x\right)e^x = x^2\sum_{n=0}^{\infty}\frac{x^n}{n!} + 2x\sum_{n=0}^{\infty}\frac{x^{n+2}}{n!} + \sum_{n=0}^{\infty}\frac{2x^{n+1}}{n!} = 2x + \sum_{n=1}^{\infty}\left(\frac{1}{(n-1)!}+\frac{2}{n!}\right)x^{n+1}$$
$$= 2x + \sum_{n=1}^{\infty}\frac{n+2}{n!}x^{n+1} = \sum_{n=0}^{\infty}\frac{n+2}{n!}x^{n+1}$$

This series is valid for all x by the ratio test.

(c) $\dfrac{1}{3x-2}$

SOLUTION: Write

$$\frac{1}{3x-2} = \frac{1}{-2+3(x)} = -\frac{1}{2}\frac{1}{1-\frac{3(x)}{2}}$$

and then substitute $\frac{3(x)}{2}$ for x in the Maclaurin series for $\frac{1}{1-x}$ to obtain

$$\frac{1}{1-\frac{3(x)}{2}} = \sum_{n=0}^{\infty}\left(\frac{3(x)}{2}\right)^n = \sum_{n=0}^{\infty}\frac{3^n}{2^n}x^n$$

Thus

$$\frac{1}{3x-2} = -\sum_{n=0}^{\infty}\frac{3^n}{2^{n+1}}x^n$$

This series is valid for $|x| < \frac{3}{2}$ by the ratio test.

Question (6), continued.

(a) $\cos^2(x)$

SOLUTION: We will use the identity $\cos^2(x) = \frac{1}{2}(1 + \cos 2x)$.

The Maclaurin series for $\cos(2x)$ is

$$\sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} x^{2n}}{(2n)!}$$

so the Maclaurin series for $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ is

$$\frac{1 + \left(1 + \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n} x^{2n}}{(2n)!}\right)}{2} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n-1} x^{2n}}{(2n)!},$$

which is convergent everywhere.

(b) $\int_0^x e^{t^2}\, dt$

SOLUTION: Substituting t^2 for t in the Maclaurin series for e^t yields

$$e^{t^2} = \sum_{n=0}^{\infty} \frac{(t^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{t^{2n}}{n!};$$

therefore,

$$\int_0^x e^{t^2}\, dx = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{n!(2n+1)}\Bigg|_0^x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!(2n+1)}$$

This series converges everywhere by the ratio test.