Math 4410 Discussion questions, Jan. 29, 2019

Throughout these problems $G$ is a graph with a finite number of vertices and edges.

1. Let $n = |V(G)|$. Consider the following three possible properties of $G$.
   - $|E(G)| = n - 1$.
   - $G$ is connected.
   - $G$ has no circuits.
   Prove that any two of these properties implies the third.

2. Prove that if $|E(G)| \geq |V(G)|$, then $G$ has at least one circuit.

3. Let $G$ be a simple graph with at least two vertices. We are going to prove the following: There are two distinct vertices $x, y$ of $G$ such that $\deg(x) = \deg(y)$.

   Proof. Proof by contradiction: Suppose the statement is not true. Then there is a simple graph $G'$ all of whose vertices have different degrees such that any other simple graph with fewer vertices does contain at least two distinct vertices with the same degree. So for each vertex $v_i$ of $G'$ there exists vertices $x_i, y_i$ of $G'$ such that the degrees of $x_i$ and $y_i$ are the same in the induced subgraph of $G'$ whose vertex set is the vertices of $G'$ without $v_i$. Now define a third graph $G''$ as follows. The vertices of $G''$ are the same as the vertices of $G'$. The edges of $G''$ are all of the edges $(x_i, y_i)$. This leads to a contradiction since .... □

   (Remark - we are not looking alternative proofs of the original statement, just how to finish off this proof.)

4. Let $G$ be a simple graph and let $V(G) = \{v_1, \ldots , v_n\}$. Define an $n \times n$ matrix $A(G)$ by setting the $i, j$ coefficient of $A(G)$ to be the number of edges whose ends are $v_i$ and $v_j$. For all $k \geq 1$, the $i, j$ entries of $A(G)^k$ count something. What do they count? Let $A'(G) = A(G) + I_n$, where $I_n$ is the $n \times n$ identity matrix. Once again the entries of $A'(G)^k$ count something. What do they count?

5. Suppose the girth of $G$ is at least 5 and the degree of every vertex of $G$ is at least $d$. Prove that $|V(G)| \geq d^2 + 1$. Now suppose that the girth of $G$ is at least 7 and the degree of every vertex is at least $d$. Prove the best lower bound you can for $|V(G)|$.

---Added on 1/29---

6. A sequence $(d_1, \ldots , d_n)$ of positive integers is a degree sequence if there exists a graph with vertices \{v_1, \ldots, v_n\} such that $\deg(v_i) = d_i$. We saw in class that if $(d_1, \ldots , d_n)$ is a degree sequence, then $\sum_{i=1}^{n} d_i$ is even. Prove that this is also a sufficient condition for $(d_1, \ldots , d_n)$ to be a degree sequence.

7. Is $(5, 5, 5, 4, 3, 2)$ the degree sequence of a simple graph? Either construct a simple graph with this degree sequence or prove that there is no such graph.

8. Suppose $(d_1, \ldots , d_n)$ is a sequence of positive integers such that $d_1 \geq d_2 \geq \cdots \geq d_n$. Prove that if $(d_1, \ldots , d_n)$ is the degree sequence of a simple graph, then for all $1 \leq j \leq n$,

$$d_1 + \cdots + d_j \leq j(j-1) + \sum_{k=j+1}^{n} \min(d_k, j).$$