Math 4410 Discussion questions, Feb. 5, 2019

Let $T_1, \ldots, T_r$ be trees on disjoint vertex sets $V(T_1), \ldots, V(T_r)$, and let $V = \bigcup_{i=1}^{r} V(T_i)$. We want to try to count the number of trees on $V$ which contain $T_1, \ldots, T_r$.

For this purpose we introduce $W = \{w_1, \ldots, w_r\}$. Now let $T$ be a tree with vertices $V$ which contains $T_1, \ldots, T_r$. Using $T$ we define a graph $W_T$ whose vertex set is $W$. A pair $(w_i, w_j)$ is an edge of $W_T$ if and only if there is an edge in $T$ whose ends are in $V(T_i)$ and $V(T_j)$.

(1) The function $f : T \rightarrow W_T$ is a surjective function from trees on $V$ which contain $T_1, \ldots, T_r$ to trees on $W$.

(2) Let $S$ be a tree on $W$. Find a formula for $|f^{-1}(S)|$. Your formula should be in terms of $|V(T_1)|, |V(T_2)|, \ldots, |V(T_r)|$ and $\deg_S(w_i), 1 \leq i \leq r$.

(3) Prove that the number of trees on $V$ which contain $T_1, \ldots, T_r$ is

$$|V(T_1)| \cdot |V(T_2)| \cdots |V(T_r)| \cdot |V|^{r-2}.$$

(4) Using the above formula compute the probability that a random tree with vertex set $[n]$ has $(1,2)$ as an edge.

(5) There is a sense in which the answer to the previous problem is ‘obvious’. How?

(6) Now assume that you are given a random tree $T$ with vertex set $[n]$ which has the edge $(1,2)$. What is the probability that $T$ also has the edge $(a,b)$. Does your answer depend on $a$ and $b$?