Math 4410 Discussion questions, Sept. 16, 2019

(1) Let $X = \{x_1, \ldots, x_m\}$ be nonzero vectors in a vector space and let U be the span of X. We set d to be the dimension of U. Let $w : X \to \mathbb{R}$ be a weight function. Since X spans U the following set is not empty:

$$\mathcal{B} = \{B \subseteq X : B \text{ is a basis of } U.\}.$$

For any basis of U in $\mathcal{B}$ we define the weight of B as

$$w(B) = \sum_{x \in B} w(x).$$

We are going to try to find the basis in $\mathcal{B}$ of minimum weight by using a 'greedy alogorithm' as follows. We are going to choose $e_1, e_2, \ldots$ until we reach e_d so that for all i, $\{e_1, e_2, \ldots, e_i\} \subset X$ and is a linearly independent set. We start by setting e_1 to be a vector in X of minimum weight. lf there is more than one vector with minimum weight choose one at random. Once you have picked $e_1, \ldots, e_i$ choose e_{i+1} as follows. If $i = d$ stop since $\{e_1, \ldots, e_i\}$ is a basis of U. (Do you know why?) $B = \{e_1, \ldots, e_d\}$ is your basis. If not, then let $Z = \{x \in X : x \text{ is not in the span of } \{e_1, \ldots, e_i\}\}$ and set e_{i+1} to be a vector in Z of minimum weight (of vectors in Z). Then $\{e_1, \ldots, e_i, e_{i+1}\}$ is linearly independent. (Do you know why?) We repeat this procedure until we get a basis $B = \{e_1, \ldots, e_d\}$ of U which is a subset of X. (Why do we always end up with a basis?)

Example: $X = \{x_1, x_2, x_3, x_4, x_5\}$ with

$$x_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, x_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, x_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}, x_4 = \begin{bmatrix}1\\1\\0\end{bmatrix}, x_5 = \begin{bmatrix}1\\1\\1\end{bmatrix},$$

and $w(x_1) = 2, w(x_2) = 1, w(x_3) = 8, w(x_4) = 3$ and $w(x_5) = 6$. In this case your basis would be $B = \{x_2, x_1, x_5\}$.

Prove that if you choose B using this greedy algorithm, then B is a basis of U contained in X with the following strong minimizing property. Let $C = \{a_1, \ldots, a_d\}$ be any basis of U contained in X ordered so that $w(a_1) \le w(a_2) \le \cdots \le w(a_d)$. Then for all i, $w(e_i) \le w(a_i)$.

(2) Use the above question and hw 1 question # 4 to prove Theorem 2.2 of the text.

(3) Let E be a finite set and $\mathcal{I}$ be a set of subsets of E. A *maximal subset* of $\mathcal{I}$ is a set $A \subseteq E$ such that

- $A \in \mathcal{I}$.
- If $B \in \mathcal{I}$ and $A \subseteq B$, then $A = B$.

For example, if E is a spanning subset of vectors in a vector space V and $\mathcal{I}$ consists of the subsets of E which are linearly independent, then the maximal subsets of $\mathcal{I}$ are the subsets of E which are also bases of V.

Let $w : E \to \mathbb{R}$ be a weight function. For $A \in \mathcal{I}$ define

$$w(A) = \sum_{e \in A} w(e).$$

Define a "greedy algorithm" to find a maximal subset of $\mathcal{I}$ which has minimal weight so that your definition agrees with the first problem when E is a spanning subset of vectors and $\mathcal{I}$ are the linearly independent subsets of $\mathcal{I}$.

Do not worry (yet!) about when your algorithm works and when it doesn't.

(4) Let $X = \{x_1, \ldots, x_m\}$ be vectors in a vector space and let $\mathcal{I} = \{I \subseteq X : I \text{ is linearly independent .}\}$ Prove that if $I, J \in \mathcal{I}$ and $|I| < |J|$, then there exists $x \in J - I$ such that $I \cup \{x\} \in \mathcal{I}$.