Math 4410 Discussion questions, Sept. 23, 2019

(1) Let G be a simple graph with vertices $V(G) = \{x_1, \dots, x_n\}$ Construct a new simple graph H as follows:
- $V(H) = V(G) \cup \{y_1, \dots, y_n, z\}$
- G is an induced subgraph of H.
- The neighbors of z are $\{y_1, \dots, y_n\}$.
- The neighbors of y_i are z and the neighbors of x_i in G.

(a) Prove that if G has no triangles, then H has no triangles.
(b) Prove that $\chi(H) = \chi(G) + 1$.
(c) Prove that for all $n \geq 3$ there exists a simple graph G with no triangles and $\chi(G) = n$.

(2) Let G be a graph. For $t \geq 1$ define $\chi_G(t)$ to be the number of proper t-colorings of G.
(a) Prove that $\chi_G(t) = \chi_{G-e}(t) - \chi_{G/e}(t)$.
(b) Prove that if G has no loops, then $G(t)$ is a polynomial of degree $|V|$ with integer coefficients whose leading term is $t^{|V|}$ and whose nonzero coefficients alternate in sign.
(c) Prove that if G is simple, then the first two terms are $\chi_G(t) = t^{|V|} - |E|t^{|V|-1} + \dots$.