Math 4410 Discussion questions, Nov. 8, 2019

(1) Let G be a simple graph with n vertices and m edges. Order the edges $E = \{e_1, \ldots, e_m\}$. A *broken circuit* is a circuit with its least element removed. (Note: A circuit is a polygon.) So if a circuit was $\{e_5, e_7, e_1, e_3\}$ then $\{e_3, e_5, e_7\}$ is a broken circuit. The order in which the elements of the circuit (and broken circuit) are listed does not matter. Only the order determined by listing them e_1, e_2, etc. Now define f_i to be the number of subsets of edges of cardinality i which do **not** contain a broken circuit.

Theorem (Whitney, '32)

$$\chi_G(t) = f_0 t^n - f_1 t^{n-1} + \cdots + (-1)^{n-1} f_{n-1} t + (-1)^n f_n.$$

We prove this in several steps. (Remark - remember, $\chi_G(t)$ counts the number of colorings using t or fewer colors.)

(a) Let A be a nonempty subset of $[m]$. Define E_A to be the subgraph of G consisting of all of the vertices of G and all edges $\{e_i : i \in A\}$. Now let $N(A)$ be the number of colorings (not proper!) using t or fewer colors such that for every edge $e \in E_A$, the endpoints of the edge e have the same color.

Example: $G = K_4$ and we order the edges $\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}$. If $A = \{1,6\}$, then $N(A) = t^2$. If $A = \{5\}$, then $N(A) = t^3$. Write down a formula for $N(A)$ in terms of the number of components of E_A. Show that inclusion-exclusion immediately implies that $\chi_G(t)$ is a polynomial of degree n whose initial term is t^n.

(b) Let $\mathcal{B}$ be the broken circuits of G. Order them $\mathcal{B} = \{B_1, B_2, \ldots, B_s\}$ so that if $i < j$, then B_j is not contained in B_i. For instance, you could start with the broken circuits of cardinality three, then cardinality four, etc. Now define P_1 to be all $A \subseteq [m]$ such that $B_1 \subseteq E_A$. Prove that

$$\sum_{A \in P_1} (-1)^{|A|} N(A) = 0.$$

(c) Now, for $2 \le i \le s$ define P_j to be all subsets A of $[m]$ such that $B_j \subseteq E_A$ and $B_i \not\subseteq E_A$ for $i < j$. Some of the P_j may, of course, be empty. By definition, the set of all $A \subseteq [m]$ such that E_A contains a broken circuit is the disjoint union of $P_1, P_2, \ldots, P_s$. Prove that

$$\sum_{j=1}^{s} \sum_{A \in P_j} (-1)^{|A|} N(A) = 0.$$

(d) Prove the theorem. (If E_A has no broken circuits, then E_A has no circuits - so the number of components in E_A is ....)

(2) Let

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

be the Riemann zeta function. Prove that

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$