Math 4410 HW 2 - Due Sept. 16, 2019 IN CLASS

1. Prove that a tree with exactly two vertices of degree one is a path

2. Let G be a graph and define $\mathcal{T}(G)$ to be the number of subgraphs of G which are trees. For instance, $\mathcal{T}(K_n) = n^{n-2}, \mathcal{T}(P_n) = n$ (recall P_n is the polygon of length n) and if G is not connected, then $\mathcal{T}(G) = 0$.

   Now let e be an edge of G. The *deletion* of e is the graph which is the same as G except the edge e has been removed. It is denoted $G - e$. The *contraction* of e is the graph denoted G/e and is defined as follows: If e is a loop, then $G/e = G - e$. If e is not a loop, let x and y be the endpoints of e. Then G/e is the graph whose vertex set is $V(G) - \{y\}$ and edges are:

   - The edge e is gone.
   - All edges of G which were not incident to y are still edges in G/e.
   - All edges other than e of the form (y, z) are replaced by an edge (x, z) in G/e.

   Geometrically, the edge e is shrunk to a single vertex which identifies x and y,

   Prove that if e is not a loop, then $\mathcal{T}(G) = \mathcal{T}(G - e) + \mathcal{T}(G/e)$.

3. (a) Let T be a tree. Prove that for every edge $e \in E(T),\ T - e$ is the disjoint union of two trees.

   (b) Let T_n be the number of trees on n vertices. Without using Cayley's theorem prove that

   $$T_n = \sum_{k=1}^{n-1} k \binom{n-2}{k-1} T_k T_{n-k}.$$

4. For this question $\widehat{G}$ is a directed graph and G is the underlying undirected graph. Given $\widehat{G}$ we construct a matrix M whose columns are indexed by the edges of G and whose rows are indexed by the vertices of $\widehat{G}$. The entry of M corresponding to edge e and vertex v is determined as follows:

   - If v is not an endpoint of e, then the entry is zero.
   - If v is the endpoint of e corresponding to the head of the orientation, then the entry is 1.
   - If v is the endpoint of e corresponding to the tail of the orientation, then the entry is -1.
   - If e is a loop, then all entries in the corresponding column are zero.

   (a) The column dependencies of M do not depend on the orientations chosen. They only depend on G.

   (b) Specifically, a set of columns of M is linearly dependent if and only if the corresponding edges contain a polygon of G.

   (c) The rank of M is the number of vertices of G minus the number of components of G.