Math 4410 HW 1 - Due Feb. 5, 2019 IN CLASS

1. A graph which satisfies the three conditions in discussion problem 1 is called a tree.

   (a) Prove that a tree has at least two vertices of valence one.
   (b) Prove that a tree with exactly two vertices of valence one is a path.

2. Problem 1E from the text.

3. Let $G$ be a simple graph with $n$ vertices. Suppose that for every vertex $x$ of $G$, $\deg(x) \geq \frac{n-1}{2}$. Prove that $G$ is connected.

4. A spanning circuit of $G$ is called a Hamiltonian circuit, and if $G$ has a Hamiltonian circuit it is called Hamiltonian. Any cyclic ordering of the vertices of $K_n$ produces a Hamiltonian circuit of $K_n$. Determining whether or not an arbitrary graph is Hamiltonian turns out to be a computationally hard problem.

   Let $Q$ be a nonempty finite set, $n \geq 1$. Set $Q^n = \{(q_1, \ldots, q_n) : q_i \in Q\}$ to be the ordered $n$-tuples of $Q$. Define a graph $G(n, Q)$ as follows. The vertices of $G(n, Q)$ are the elements of $Q^n$. Two $n$-tuples have an edge between them if and only if they differ in exactly one coordinate. For instance, $G(1, Q)$ is $K_{|Q|}$. For another example, if $Q = \{a, b\}$, then $G(2, Q)$ is a square. Notice that if we fix $q \in Q$, then the induced graph $H$ on the vertices $\{q, q_2, \ldots, q_n : q_2, \ldots, q_n \in Q\}$ is isomorphic to $G(n-1, Q)$. Prove that $G(n, Q)$ is Hamiltonian for all $n \geq 1$ and $Q$. 