Math 4410 HW 2 - Due Sept. 23, 2019 in class

1. Let X be a nonempty finite set and $\mathcal{I}$ a nonempty set of subsets of X which is closed under subsets. Equivalently, if $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$. Furthermore, suppose there exist $I, J \in \mathcal{I}$ such that $|I| < |J|$ and for all $x \in J - I$, $I \cup \{x\} \notin \mathcal{I}$. Prove there exists a weight function $w : X \to \mathbb{R}$ such that the greedy algorithm defined in class does NOT work.

2. Let X be a nonempty finite set and $\mathcal{I}$ a nonempty set of subsets of X. Assume that $\mathcal{I}$ satisfies the following two properties.

   - If $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$.
   - If $I, J \in \mathcal{I}$ and $|I| < |J|$, then there exists $x \in J - I$ such that $I \cup \{x\} \in \mathcal{I}$.

   Prove that $\mathcal{I}$ satisfies the greedy algorithm defined in class.

3. A graph G is *bipartite* if there exist two disjoint subsets A and B of the vertices of G such that every edge has one endpoint in A and another in B. Prove that G is bipartite if and only if it has no odd polygons.

4. How many trees with vertex set $[n]$ do not contain the edge $\{1, 2\}$? (Hint: What is the probability that a random tree with vertex set $[n]$ does contain a given edge?)