Math 4410 HW 7 - Due Nov. 1 in class

1. Let P be a poset with 66 elements. Prove that P has an antichain with more than 5 elements or chain with more than 13 elements.

2. Suppose you have a deck of nm cards which has n suits and m different ranks, one card for each possible composition of rank and suit. For instance, in a normal deck of cards $n = 4$ and $m = 13$. Now place the cards in m distinct piles, each pile containing n cards. Show that it is possible to take exactly one card from each pile and end up with exactly one card of each rank.

3. Let $(P, \leq)$ be a poset and $p, q \in P$. An *upper bound* for p and q is any $u \in P$ such that $p \leq u$ and $q \leq u$. A *least upper bound* for p and q is an upper bound u such that if v is any upper bound for p and q, then $u \leq v$. For instance, if P is $B_5, p = \{1, 5\}$, and $q = \{1, 3, 4\}$, then $\{1, 2, 3, 4, 5\}$ is an upper bound for p and q, but not a least upper bound. A least upper bound for this p and q is $\{1, 3, 4, 5\}$. For an arbitrary P and $p, q \in P$ there may or may not be an upper bound for p and q. In addition, if there are upper bounds for p and q there may or may not be a least upper bound.

   Now assume that P is finite and let $\mathcal{A}$ be the antichains of P. For $A, B \in \mathcal{A}$ define $A \leq B$ if for all $a \in A$ there exists $b \in B$ such that $a \leq b$.

   (a) Prove that if c and d in P are both a least upper bound for a and b in P, then $c = d$.

   (b) Prove that $(\mathcal{A}, \leq)$ is a poset.

   (c) Prove that $\mathcal{A}$ has a maximum element C. That is, there exists C in $\mathcal{A}$ such that $A \leq C$ for all $A \in \mathcal{A}$.

   (d) Suppose the maximum size of an antichain in P is m. Let A and B be two antichains of P of size m. Prove that there is a least upper bound for A and B in $(\mathcal{A}, \leq)$.