From polytopes to enumeration

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Preface

Chapter 1

Affine and convex geometry

1.1 Affine subspaces

The geometry of polytopes begins with the study of affine spaces and convexity. An affine subspace of Euclidean space is any translation of a linear subspace. Since most geometric concepts such as distance, angle and colinearity are invariant under translation we should expect them to behave reasonably in affine spaces. The translation of a subset A of $\mathbb{R}^d$ by $\mathbf{v} \in \mathbb{R}^d$ is defined by

$$A + \mathbf{v} = \{\mathbf{x} + \mathbf{v} : \mathbf{x} \in A\}.$$

An *affine subspace* of $\mathbb{R}^d$ is a subset of $\mathbb{R}^d$ which is either empty or of the form $W + \mathbf{v}$, where W is a linear subspace. Familiar examples include points, lines and planes.

Problem 1 *Suppose $A = W + \mathbf{v} = W' + \mathbf{v}'$ is an affine subspace of $\mathbb{R}^d$. Prove that $W = W'$. Also, $A + (-\mathbf{y}) = W$ for every $\mathbf{y} \in A$.*

As is the case with linear subspaces, the notions of dimension, span and independence are critical to understanding affine subspaces. The *dimension* of the empty set is defined to be -1, otherwise the dimension of an affine subspace $W + \mathbf{v}$ is defined to be the dimension of the linear subspace W. The previous problem shows that this is well defined and that the dimensions of points, lines and planes are what we expect: $0, 1$ and 2.

Problem 2 *The intersection of a set of affine subspaces of $\mathbb{R}^d$ is an affine subspace of $\mathbb{R}^d$.*

Let $A \subseteq \mathbb{R}^d$. Consider the intersection of all affine subspaces which contain A. This is a nonempty intersection since $\mathbb{R}^d$ is such a set. By the above problem this intersection is the smallest affine subspace which contains A. It is called the *affine span* of A and we denote it by $aspan(A)$.

The affine analogues of linear combinations are called affine combinations. A linear combination of the form

$$a_1\mathbf{x}_1 + \cdots + a_n\mathbf{x}_n$$

is an *affine combination* if $a_1 + \cdots + a_n = 1$.

Problem 3 *The affine span of A equals the set of all affine combinations of elements of A.*

What is the affine notion corresponding to linear independence? If $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is a finite set of distinct point of $\mathbb{R}^d$, then we say they are *affinely independent* if their affine span has dimension $n - 1$.

Problem 4 *A subset A of $\mathbb{R}^d$ is affinely independent if and only if no element of A can be written as an affine combination of the other elements of A.*

The definition implies that the empty set is affinely independent as is any set of cardinality one.

Just as linear maps are the natural morphisms for linear spaces, affine maps are the natural morphsims for affine spaces. A function $f : \mathbb{R}^d \to \mathbb{R}^e$ is an *affine map* if $f(\mathbf{x}) = T(\mathbf{x}) + \mathbf{v}$ for some linear map $T : \mathbb{R}^d \to \mathbb{R}^e$ and fixed vector $\mathbf{v} \in \mathbb{R}^e$.

Problem 5 *If $f : \mathbb{R}^d \to R^e$ is an affine map and A is an affine subspace of $\mathbb{R}^d$, then $f(A)$ is an affine subspace of $\mathbb{R}^e$. Suppose B is an affine subspace of $\mathbb{R}^e$. Is $f^{-1}(B)$ an affine subspace of $\mathbb{R}^d$?*

Exercise 1.2

1. *Prove that $A \subseteq \mathbb{R}^d$ is an affine subspace if and only if A is the solution to a set of simultaneous linear equations in d variables.*

2. *Recall that for two distinct elements $\mathbf{x}$ and $\mathbf{y}$, the line determined by them consists of all points of the form $t\mathbf{x}+(1-t)\mathbf{y},\ t \in \mathbb{R}$. Prove that A is an affine subspace if and only if for all $\mathbf{x} \neq \mathbf{y}$ in A, the line determined by $\mathbf{x}$ and $\mathbf{y}$ is contained in A.*

3. *Let A and B be two affinely independent subsets of $\mathbb{R}^d$. Prove that if $|B| > |A|$, then there exists $\mathbf{x} \in B - A$ such that $A \cup \{\mathbf{x}\}$ is affinely independent.*

4. *An affine hyperplane of $\mathbb{R}^d$ is an affine subspace of dimension $d-1$. If H is an affine hyperplane, then there exists constants $a_1, \ldots a_d$, and c such that $H = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : a_1x_1 + \cdots + a_nx_n = c\}$. Let A be an affine subspace and H an affine hyerplane which does not contain A. Prove that if $A \cap H \neq \emptyset$, then $\dim(A \cap H) = \dim A - 1$.*

5. *Prove that if A is an i-dimensional affine subspace of $\mathbb{R}^d$, then A is the intersection of $d-i$ affine hyperplanes.*

6. *Let*

$$\mathbf{x}_1 = \begin{bmatrix} x_{1,1} \\ \vdots \\ x_{d,1} \end{bmatrix}, \ldots, \mathbf{x}_n = \begin{bmatrix} x_{1,n} \\ \vdots \\ x_{d,n} \end{bmatrix}$$

be n elements of $\mathbb{R}^d$ written as column vectors. Prove that $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are affinely independent if and only if

$$\hat{\mathbf{x}}_1 = \begin{bmatrix} 1 \\ x_{1,1} \\ \vdots \\ x_{d,1} \end{bmatrix}, \ldots, \hat{\mathbf{x}}_n = \begin{bmatrix} 1 \\ x_{1,n} \\ \vdots \\ x_{d,n} \end{bmatrix}$$

are linearly independent in $\mathbb{R}^{d+1}$.

7. *Let $T : \mathbb{R}^d \to \mathbb{R}^e$ be a linear map and $\mathbf{v}$ a fixed vector in $\mathbb{R}^d$. Then $S : \mathbb{R}^d \to \mathbb{R}^e$ defined by $S(\mathbf{x}) = T(\mathbf{x} + \mathbf{v})$ is an affine map. Show that not all affine maps are of this form. Characterize which ones are.*

1.3 Convexity

One way to view affine subspaces is as those sets which are closed under forming lines (Exerc. 2). Convex sets are those that are closed under forming line segments. For two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, the line segment between them, which we denote by $[\mathbf{x}, \mathbf{y}]$, is the set of all points of the form $t\,\mathbf{x} + (1-t)\mathbf{y},\ 0 \leq t \leq 1$.

Definition 1.3.1 *A subset A of $\mathbb{R}^d$ is* **convex** *if for all $\mathbf{x}, \mathbf{y} \in A$ the line segment $[\mathbf{x}, \mathbf{y}] \subseteq A$.*

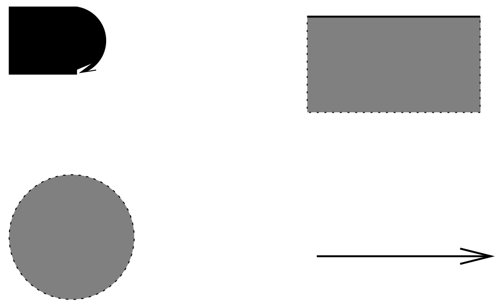

Figure 1.1: Some convex subsets of $\mathbb{R}^2$

Problem 6 *The intersection of a collection of convex sets is convex.*

As was the case with affine spaces, the above problem implies that for any subset A of $\mathbb{R}^d$ there is a smallest convex set which contains A. It is called the *convex hull* of A and we denote it by $ch(A)$.

As can be seen from the above examples convex sets can be empty, open, closed, neither open nor closed, bounded or unbounded. In a fashion similar to using affine combinations to form the affine span, convex combinations can be used to form the convex hull of a set. A *convex combination* is a linear combination

$$a_1\mathbf{x}_1 + \cdots + a_n\mathbf{x}_n$$

such that $0 \leq a_i \leq 1$ and $a_1 + \cdots + a_n = 1$. Equivalently, a convex combination is an affine combination with all of the scalars nonnegative.

Problem 7 *The convex hull of A consists of all convex combinations of elements of A.*

At this point we know that if $\mathbf{y} \in ch(A)$, then $\mathbf{y}$ is a convex combination of points of A. However, we do not yet have a bound on the number of elements needed.

Problem 8 *(Carathéodory) Let $\mathbf{y} \in ch(A)$, where $A \subseteq \mathbb{R}^d$. Then there exists $\mathbf{x}_1, \ldots, \mathbf{x}_{d+1}$ (not necessarily distinct) in A such that $\mathbf{y}$ is a convex combination of the $\mathbf{x}_i$.*

Exercise 1.4

1. *If K is convex and T is an affine map, then $T(K)$ is convex.*

2. *Let K be a convex set and $\mathbf{x}, \mathbf{y}$ be distinct points not in K. If $\mathbf{x} \in ch(K \cup \{\mathbf{y}\})$, then $\mathbf{y} \notin ch(K \cup \{\mathbf{x}\})$.*

3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function with $f(x) \geq 0$ and $f''(x) \leq 0$ for all $x \in \mathbb{R}$. Prove that for any $a < b$ the points under the curve $y = f(x)$, i.e.*
$$\{(x, y) : a \leq x \leq b, y \leq f(x)\}$$
is a convex subset of $\mathbb{R}^2$.

1.5 Simplices

Simplices are one of the fundamental examples and building blocks of polytopes. They are the higher dimensional analogues of line segments, triangles and tetrahedra. The d-simplex, which we denote by Δ^d, is the convex hull of the standard unit coordinate vectors $\mathbf{e}_1, \ldots, \mathbf{e}_{d+1}$ in $\mathbb{R}^{d+1}$. Equivalently,

$$\Delta^d = \{(x_1, \ldots, x_{d+1}) : 0 \leq x_i \leq 1, \text{ and } x_1 + \cdots + x_{d+1} = 1.\}$$

The dimension of a convex set is the dimension of its affine span. Since the affine span of Δ^d is the hyperplane $x_1 + \cdots + x_{d+1} = 1$ it is d-dimensional.

A d-simplex (as opposed to **the** d-simplex) is the convex hull of $d + 1$ affinely independent points. A d-simplex can live in any $\mathbb{R}^e$ as long as $e \geq d$. For instance, any triangle is a 2-simplex.

Exercise 1.6

1. *A d-simplex is d-dimensional.*

2. *Some texts define* **the** *d-simplex as the convex hull of $\{\vec{0}, \mathbf{e}_1, \ldots, \mathbf{e}_d$ in $\mathbb{R}^d.\}$ Denote this version of the simplex by $\tilde{\Delta}^d$. Prove that if S is a d-simplex in $\mathbb{R}^e$, then there exists an injective affine map $T : \mathbb{R}^d \to \mathbb{R}^e$ such that $T(\tilde{\Delta}) = S$. To what extent is T uniquely defined?*

3. *Let $S = ch(\mathbf{x}_0, \ldots, \mathbf{x}_d\}$ be a d-simplex. A face of S is the convex hull of any subset of $\{\mathbf{x}_0, \ldots, \mathbf{x}_d\}$. Prove that if F is a face of S other than S itself, then there exists a hyperplane H such that $F = S \cap H$.*

4. *Is the closure of a convex set convex?*

5. *Is the convex hull of a closed set closed?*

Chapter 2

What is a polytope?

Some of the first geometric objects that a child encounters are convex polygons and polyhedra. Squares, hexagons, cubes and pyramids are familiar to almost anyone over the age of five. What are the higher dimensional analogues of these shapes? There are several approaches to this question. Here we will consider two apparently different possibilities. The first idea is that a polyhedron is the smallest convex subset which contains its vertices. The second approach is to examine the problem from the point of view of linear optimization.

2.1 $\mathcal{V}$-polytopes

Definition 2.1.1 *A subset P of $\mathbb{R}^d$ is a* $\mathcal{V}$**-polytope** *if it is the convex hull of a finite set of points.*

By definition the empty set is a $\mathcal{V}$-polytope. What are some examples of $\mathcal{V}$-polytopes? Perhaps the easiest to write down are the d-simplex, the d-cube and the d-crosspolytope. We have already seen the d-simplex. The d-cube is defined by

$$\Box^d = \{\mathbf{x} = (x_1, \ldots, x_d) : |x_i| \leq 1, i = 1, \ldots, d\}.$$

So, the 2-cube is a square and the 3-cube is a cube. The d-cube is a d-dimensional $\mathcal{V}$-polytope. The d-crosspolytope is the convex hull of $\{\pm\mathbf{e}_i : 1 \leq i \leq d\}$ in $\mathbb{R}^d$. By definition it is a $\mathcal{V}$-polytope and it is easy to see that it is d-dimensional. We denote it by $\diamond^d$. The 3-crosspolytope is the octahedron.

Problem 9 *Let $f : \mathbb{R}^d \to \mathbb{R}^e$ be an affine map and P a $\mathcal{V}$-polytope. Is $f(P)$ necessarily a $\mathcal{V}$-polytope? What about $f^{-1}(P)$?*

Problem 10 *Let P be a $\mathcal{V}$-polytope in $\mathbb{R}^d$. If A is an affine subspace of $\mathbb{R}^d$, is $P \cap A$ a $\mathcal{V}$-polytope?*

An alternative way to generalize convex polygons is from the point of view of linear optimization. The general idea of linear optimization is that you are asked to find the maximum (or minimum) of a linear objective function when restricted to a number of linear inequalities. Here is an example to illustrate the point. Suppose that you are responsible for purchasing the gold and platinum your company needs during the coming year. The company has contracts with two mines, one for each metal. In order to insure continued business with the mines you must purchase at least 2 tons of platinum and 3 tons of gold. Your budget, plus the fact that platinum is approximately 50 percent more expensive than gold, implies that if you purchase x tons of platinum and y tons of gold, then $2x + 3y \leq 24$. Your last constraint is that due to political pressure from the owner of the platinum mine you must be sure to purchase at least 2 more tons of platinum than gold. As usual, your objective is to maximize the company's profit. Currently the items with more platinum have a greater profit margin, so you can estimate the profit by the expression $6x + y$. We can sum up your problem as trying to maximize $6x + y$ given that

$$\begin{array}{rcrcl} & & x & \geq & 2 \\ & & y & \geq & 3 \\ 2x & + & 3y & \leq & 24 \\ x & - & y & \geq & 2. \end{array}$$

The situation is depicted in Figure 2.1. The shaded quadrilateral is the intersection of the four inequalities above and represents the *feasible region*, those points (x, y) which satisfy all the constraints. The dotted lines represent the objective function. From the diagram the optimal answer is 6 tons of platinum and 4 tons of gold. Notice that no matter what the objective function is, as long as it is linear and your goal is to maximize (or minimize) it, then an optimal solution is at one of the vertices of the polygon. (Can you explain why?) In practice the number of variables and constraints can easily be several thousand.

With the above model in mind we now introduce $\mathcal{H}$-polytopes. The *dual* of $\mathbb{R}^d$, which we denote by $(\mathbb{R}^d)^\star$, is the real vector space of all linear functions $\mathbf{a} : \mathbb{R}^d \to \mathbb{R}$. We will denote the value of $\mathbf{a}$ at $\mathbf{x} \in \mathbb{R}^d$ by $\mathbf{a} \cdot \mathbf{x}$. This notation suggests the usual dot product on $\mathbb{R}^d$, and if we coordinatise $(\mathbb{R}^d)^\star$ in the usual way it is the 'usual' dot product. A *closed half-space* of is any subset of the form

$$H_{\mathbf{a},b} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} \geq b\}$$

for some fixed $\mathbf{a} \neq 0$ in $(\mathbb{R}^d)^\star$ and $b \in \mathbb{R}$. While $H_{\mathbf{0},b}$ is not a closed half-space, we will use this notation where appropriate. Obviously $H_{\mathbf{0},b}$ is either all of $\mathbb{R}^d$ or empty depending on the sign of b.

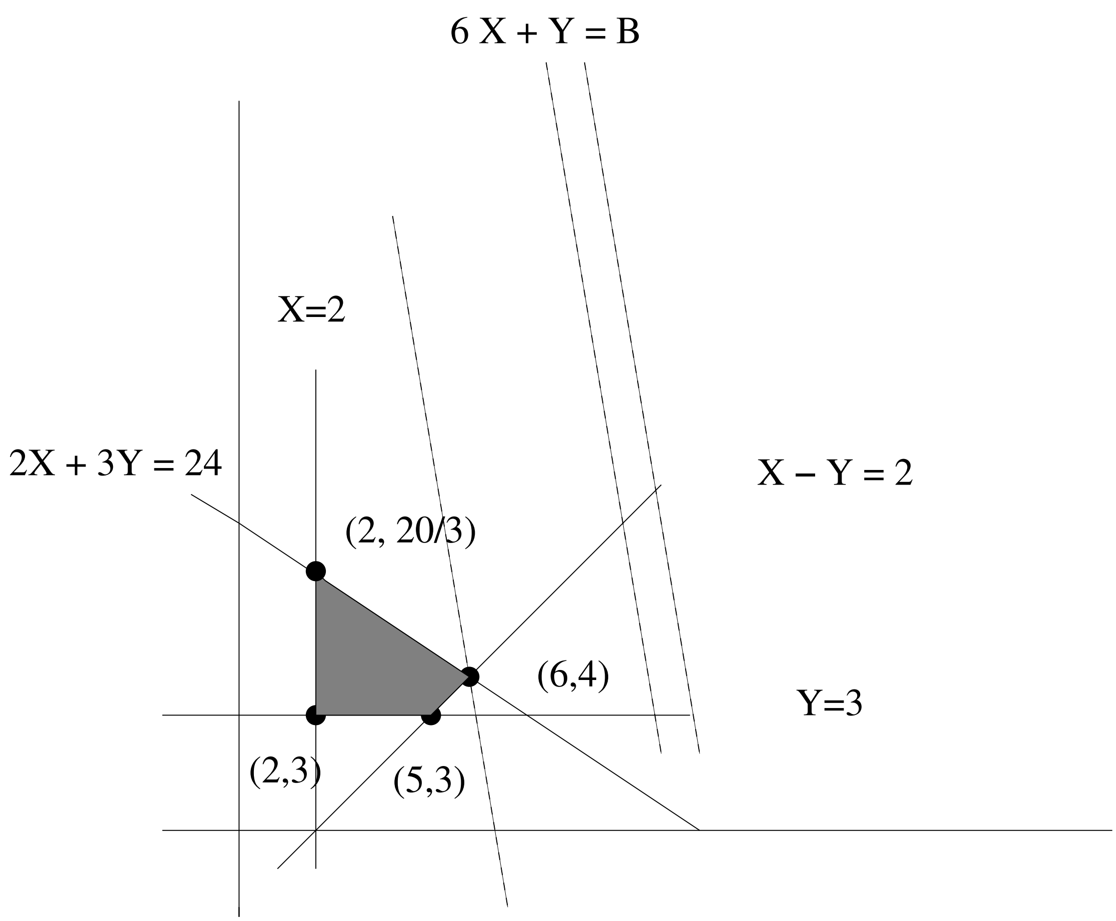

Figure 2.1: An $\mathcal{H}$-polytope

Definition 2.1.2 *An $\mathcal{H}$-polytope is a bounded set which is the intersection of finitely many closed half-spaces.*

Since closed half-spaces are convex, any $\mathcal{H}$-polytope is convex.

Problem 11 $\square^d, \diamond^d$, *and any simplex are $\mathcal{H}$-polytopes.*

As a hyperplane is the intersection of two closed half-spaces we can see that if P is an $\mathcal{H}$-polytope and H a hyperplane, then $P \cap H$ is an $\mathcal{H}$-polytope. By (Exerc. 1.2.5) any affine subspace can be written as the intersection of hyperplanes. So, Problem 10 is very easy when applied to $\mathcal{H}$-polytopes.

Problem 12 *Is the affine image of an $\mathcal{H}$-polytope an $\mathcal{H}$-polytope?*

2.2 Examples of Polytopes

Here we give some examples of polytopes. In each case it will be obvious from the definition that it is either an $\mathcal{H}$ or $\mathcal{V}$-polytope, but no so obvious if it is the other.

The *permutahedron* in $\mathbb{R}^d$ is the convex hull of all possible permutations of the coordinates of the vector

$$\begin{bmatrix} 1 \\ 2 \\ \vdots \\ d \end{bmatrix}.$$

For instance, the permutahedron in $\mathbb{R}^2$ is the line segment between $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Let P be a convex subset of $\mathbb{R}^d$. The prism of P is defined by

$$\text{prism}(P) = P \times [0,1] = \{(x_1, \ldots, x_d, x_{d+1}) \in \mathbb{R}^{d+1} : (x_1, \ldots, x_d) \in P \text{ and } 0 \le x_{d+1} \le 1.\}$$

Figure 2.2 is a simple example. As we will see in Exercise 3, if P is an $\mathcal{H}$ or $\mathcal{V}$-polytope, then so is prism (P).

In order to define our next example, order polytopes, we recall the definition of a partially ordered set. A partially ordered set, which usually called a *poset*, is a pair $(\Pi, \le)$ such that for all $x, y, z \in \Pi$

- $x \le x$. (reflexivity)

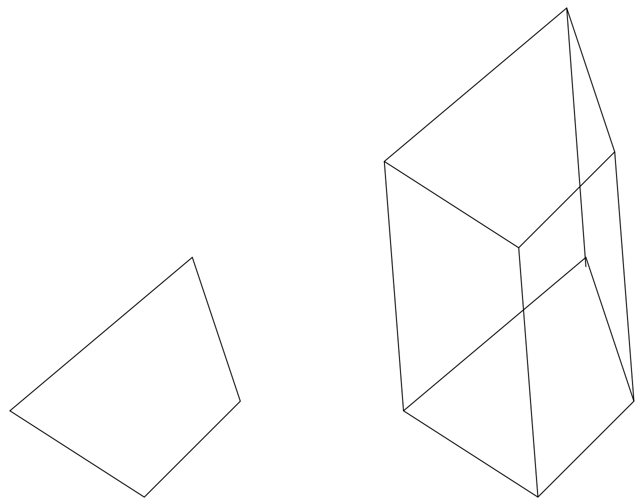

Figure 2.2: Prism (P)

- If $x \le y$ and $y \le x$, then $x = y$ (antisymmetry).
- If $x \le y$ and $y \le z$, then $x \le z$ (transitivity).

When the binary relation is clear we will frequently suppress $\le$. If $x \le y$ or $y \le x$, then x and y are **comparable**. Otherwise they are **incomparable**. It may well happen that the binary relation is empty, in which case all pairs of distinct elements of the poset are incomparable and Π is called an **antichain**.

Examples of posets are everywhere in mathematics. Familiar ones include the rationals or reals with the usual ordering. For any set X, we use B_X to represent the poset of all subsets of X with $\subseteq$ as the binary relation. In particular, the poset of all subsets of $[n]$ is denoted B_n. Another interesting poset is $(\mathbb{Z}^+, |)$ where the binary relation is "divides". For instance, in this poset $3|6$ but $3 \not| 5$. We will use D_n to denote $(\mathbb{Z}^+, |)$ restricted to the finite poset consisting of those elements which divide n. For instance, D_4 has three elements, $1 < 2 < 4$.

An **isomorphism** between posets $(\Pi, \le)$ and $(\Pi', \le')$ is a bijection $\phi : \Pi \to \Pi'$ such that $x \le y$ if and only if $\phi(x) \le' \phi(y)$. The **face poset** of a convex set K, denoted $\mathcal{F}(K)$, is the partially ordered set of faces of K with the subset relation. (Faces of convex sets are defined below, Definition 2.4.1)

Let $[n] = \{1, \ldots, n\}$. A *natural* poset on $[n]$ is a poset structure $([n], \le_{[n]})$ such that if $i \le_{[n]} j$, then $i \le j$ (the usual $\le$ on the integers). The *order polytope* for $([n], \le_{[n]})$ is

$$\{(x_1, \ldots, x_n) \in [0,1]^n : \text{ If } i \le_{[n]} j, \text{ then } x_i \le x_j.$$

By definition, order polytopes are $\mathcal{H}$-polytopes.

Exercise 2.3

1. *What is the dimension of the permutahedron in* $\mathbb{R}^d$*?*

2. *What is the dimension of prism* (P)?

3. *Show that if* P *is an* $\mathcal{H}$ *or* $\mathcal{V}$*-polytope, then so is prism* (P).

4. *Let* $(L, \leq)$ *be a finite poset with* $|L| = n$*. Show that there exists a natural poset on* $[n]$ *isomorphic to* $(L, \leq)$.

5. *Let* $(L, \leq)$ *be a poset with* L *countably infinite. Is there a natural poset on* $\mathbb{Z}$ *isomorphic to* $(L, \leq)$?

6. *Let* $([n], \leq_{[n]})$ *be a natural poset on* $[n]$*. What is the dimension of its order polytope?*

7. *Show that the order polytope of the usual linear order on* $[n]$ *is a simplex. What are its vertices?*

As you can see, sometimes $\mathcal{V}$-polytopes appear to be better while other times $\mathcal{H}$-polytopes look more useful. Our analysis of the exact nature of the relationship between these two definitions will occupy the next two chapters. In the mean time we will use 'polytope' to refer to both types. Central to the study of any polytope are its faces.

2.4 Faces of polytopes

What are the higher dimensional generalizations of the vertices, edges and faces of a three-dimensional polyhedron? A brief examination of the cube shows that each of these is the intersection of the cube and a hyperplane where the cube is completely contained on one side of the hyperplane.

An inequality of the form $\mathbf{a} \cdot \mathbf{x} \leq b, \mathbf{a} \in (\mathbb{R}^d)^\star, \mathbf{x} \in \mathbb{R}^d, b \in \mathbb{R}$ is *valid* for a convex set K if it holds for all $\mathbf{x} \in K$. An important special case is $\vec{0} \cdot \mathbf{x} \leq 0$.

Definition 2.4.1 *A face of a convex set* K *in* $\mathbb{R}^d$ *is any subset of the form* $H_{\mathbf{a},b} \cap K$*, where* $\mathbf{a} \cdot \mathbf{x} \leq b$ *is a valid inequality for* K.

Since $\vec{0} \cdot \mathbf{x} \leq 0$ is always valid, K is always a face of K. It is called the improper face of K. The maximal proper faces of K are called the *facets* of K. If K is bounded, then it is easy to see that the empty set is a face of K. The *vertices* of K are its zero-dimensional faces. Similarly, the *edges* of K are its one-dimensional faces. Since the intersection of convex sets is convex, faces of convex sets are convex.

Problem 13 *Let $P = ch(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ be a $\mathcal{V}$-polytope. Then the vertices of P are a subset of $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$.*

We can easily see that if F is a face of a 3-dimensional polyhedron and G is a face of F, then G is also a face of the polyheron. We will see later that this holds for any polytope. However, it is not true for arbitrary convex sets.

Problem 14 *Construct an example in $\mathbb{R}^2$ of a convex set K with a face F such that F contains a face G which is not a face of K.*

When are two polytopes 'the same'? Two possible approaches to this question are affine and combinatorial equivalence.

Definition 2.4.2 *Two convex subsets $K \subseteq \mathbb{R}^d$ and $K' \subseteq \mathbb{R}^e$ are* **affinely equivalent** *if there exists an affine map $f : \mathbb{R}^d \to \mathbb{R}^e$ such that f restricted to K is a bijection onto K'.*

Affine equivalence allows us to assume that whatever convex set we are investigating has full dimension.

Problem 15 *Let K be a convex set. Prove that K is affinely equivalent to a convex set $K' \subseteq \mathbb{R}^d$ where d is the dimension of K.*

The notion of affine equivalence is very restrictive. For instance, there are convex quadrilaterals in the plane that are not affinely equivalent (Exercise 5). In order to make precise the idea that any two polygons with the same number of sides should be considered equivalent we introduce the face poset of a convex set.

Definition 2.4.3 *Two convex sets K and K' are* **combinatorially equivalent** *if their face posets $\mathcal{F}(K)$ and $\mathcal{F}(K')$ are isomorphic.*

It is now easy to see that any two polygons with the same number of sides are combinatorially isomorphic.

Problem 16 *Prove that if P and Q are two affinely equivalent convex sets, then P and Q are combinatorially equivalent.*

One class of polytopes which have received a great deal of attention are simplicial polytopes. A polytope P is *simplicial* if all of its proper faces are simplices. For instance, any polygon is simplicial as is the octahedron and the icosahedron. However, the cube and the dodecahedron are not simplicial.

One of the most fundamental algorithms in linear optimization is Dantzig's simplex algorithm [5]. The idea is that the maximum (or minimum) of an objective function will always be at the vertices of the feasible region. Combined with an intelligent method for searching the vertices yields an algorithm which is very effective in practice. This leads to the natural question of what is the maximum number of vertices the simplex algorithm will have to examine given a fixed number of constraints? In Chapter 5 we will see that this is equivalent to asking what is the maximal number of facets in a simplicial polytope with a given number of vertices. It turns out that this is answered by the cyclic polytopes.

The moment curve in $\mathbb{R}^d$ is given by $\gamma : \mathbb{R} \to \mathbb{R}^d,\ \gamma(t) = (t, t^2, \ldots, t^d)$. For $n \geq d+1$ the *cyclic polytope* $C(n, d)$ is defined to be the convex hull of $\{\gamma(t_1), \ldots, \gamma(t_n)\}$, where $t_1 < \cdots < t_n$. Evidently $C(n, d)$ depends on the choices for t_i. However, as we are about to see, the face poset, and in particular the number facets, does not depend on these choices!

Problem 17 *$C(n, d)$ is a simplicial d-dimensional $\mathcal{V}$-polytope.*

We will eventually show that the vertices of a face F of a polytope P are the vertices of P contained in F. Furthermore, any face (including P) is the convex hull of its vertices. Hence, we can determine all of the faces of a simplicial polytope P by specifying which vertices are the vertices of facets. By Problem 13 this amounts to figuring out when the hyperplane determined by a d-subset of $\{\gamma(t_1), \ldots, \gamma(t_n)\}$ gives a valid inequality.

Problem 18 *(Gale evenness condition) A subset $V = \{\gamma(t_{i_1}), \ldots, \gamma(t_{i_d})\},\ t_{i_1} < \cdots < t_{i_d}$ is a set of vertices for a facet of $C(n, d)$ if and only if for all $t_j < t_m$ with t_j and t_m not in V, the cardinality of $\{l : j < l < m$ and $t_l \in V\}$ is even.*

Exercise 2.5

1. *Suppose K is a convex proper subset of $\mathbb{R}^d$. Is the empty face necessarily a face of K?*

2. *Let K be a convex set. If F is a proper face of K, then $\dim F < \dim K$. If the dimension of a face of K is exactly one dimension less than the dimension of K, then F is a facet of K. Is the converse true?*

3. *What are the faces of $\square^d$?*

4. *What are the faces of $\diamond^d$?*

5. *Give an example of two convex quadrilaterals in $\mathbb{R}^2$ which are not affinely equivalent.*

6. *Give an example of two compact convex sets of different dimensions whose face posets are isomorphic. Can this happen if the two convex sets are polytopes?*

7. *Assuming that any face of a face of $C(n, d)$ is a face of $C(n, d)$, show that cyclic polytopes of dimension four or higher are two-neighborly. That is, the line segment between any two vertices of $C(n, d)$ is a face whenever $d \geq 4$.*

8. *Let $f_i(P)$ be the number of i-dimensional faces a d-polytope P. Compute*

$$\sum_{i=0}^{d} (-1)^i f_i$$

for the d-cube, d-simplex and d-crosspolytope.

Chapter 3

Convexity

In this chapter we develop the tools necessary to understand the relationship between $\mathcal{V}$ and $\mathcal{H}$-polytopes. Our presentation is based on Grünbaum's classic text [9].

3.1 Projection

Our initial investigation of the relationship between $\mathcal{H}$ and $\mathcal{V}$-polytopes revolves around convexity. Here we establish a number of important properties of convex sets. Perhaps the single most useful property is the existence of projections for closed convex sets.

Problem 19 *Let $K \subseteq \mathbb{R}^d, K \neq \emptyset$, be a closed and $\mathbf{y} \in \mathbb{R}^d$. Prove there exists $\mathbf{x} \in K$ which minimizes the distance to $\mathbf{y}$. Specifically, there exists $\mathbf{x} \in K$ such that for all $\mathbf{z} \in K, ||\mathbf{x} - \mathbf{y}|| \leq ||\mathbf{z} - \mathbf{y}||$. Prove that if K is also convex, then there exists a* **unique** *such* $\mathbf{x}$.

The point $\mathbf{x}$ whose existence and uniqueness is given by the above problem is called the *projection* of $\mathbf{y}$ onto K and is denoted $\text{proj}_K(\mathbf{y})$.

Any hyperplane $\{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} = b\}$ partitions $\mathbb{R}^d$ into three sets, $H^+_{\mathbf{a},b} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} > b\}, H^-_{\mathbf{a},b} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} < b\}$, and $H^=_{\mathbf{a},b}$ the hyperplane itself. The two half-spaces $H^+_{\mathbf{a},b}$ and $H^-_{\mathbf{a},b}$ are called the sides of the hyperplane and when no confusion is possible we will simply write H^+ and H^-. The hyperplane *separates* two sets A and B if $A \subseteq H^+$ and $B \subseteq H^-$ or vice versa.

Problem 20 *If K is a closed convex set and $\mathbf{x} \notin K$, then there exists a hyperplane which separates K and $\{\mathbf{x}\}$.*

Looking at polygons and three-dimensional polyhedra it is fairly obvious that any point on the boundary is contained in a proper face, and the only face a point in the relative interior is contained in is the improper face. This holds for any closed convex set.

Theorem 3.1.1 *Let K be a closed d-dimensional convex subset of $\mathbb{R}^d$. If $\mathbf{x}$ is on the boundary of K, then $\mathbf{x}$ is contained in a proper face of K.*

Proof: W.L.O.G. we assume that $\mathbf{0}$ is in the interior of K. Let $K_\epsilon = \{\epsilon\mathbf{x} : \mathbf{x} \in K\}$. Since $\mathbf{x} \in \partial K, \mathbf{x} \notin K_{(1-\epsilon)}$ for $0 < \epsilon < 1$. (Why?) By the previous proposition, for each such ϵ there exists a hyperplane H_ϵ which separates $\mathbf{x}$ and $K_{(1-\epsilon)}$. The collection of all the H_ϵ must have at least one limit hyperplane H which will satisfy the proposition. (What is a limit hyperplane? Why does one exist? Why does it work?) □

Problem 21 *Fill in the details left out in the above proof.*

Exercise 3.2

1. *Every closed convex set is the intersection of the closed half-spaces which contain it.*
2. *Is the interior of a convex set convex?*
3. *For any closed convex K, proj_K is a continuous idempotent ($\operatorname{proj}_K \circ \operatorname{proj}_K = \operatorname{proj}_K$.)*
4. *If $\mathbf{x} \in H^+_{\mathbf{a},b}$ and $\mathbf{y} \in H^-\mathbf{a}, b$, then any path from $\mathbf{x}$ to $\mathbf{y}$ passes through $H^=_{\mathbf{a},b}$.*
5. *Let K and K' be closed convex sets in $\mathbb{R}^d$. Prove that if one of them is bounded, then there exists a hyperplane which separates them. What if neither is bounded?*

3.3 Extreme points

Our intuition from two and three dimensional polyhedra suggests to us that a compact convex set is the convex hull of its vertices. But Probem 14 warns us that this is not the case. The next best thing involves the extreme points of a convex set.

Definition 3.3.1 $\mathbf{x}$ *is an extreme point of a convex set* K *if* $\mathbf{x}$ *is never in the relative interior of a line segment contained in* K*. We denote the extreme points of* K *by* $\operatorname{ext} K$.

If you look at any compact solution to Problem 14, you will notice that the convex set is the convex hull of its extreme points. Before we can prove this in general we need some elementary properties of extreme points.

Lemma 3.3.2 *Let* K *be convex.*

- *Every vertex of* K *is an extreme point of* K.
- *If* $K = ch(A)$*, then* $\operatorname{ext} K \subseteq A$.
- *If* F *is a face of* K*, then* $\operatorname{ext} F = F \cap \operatorname{ext} K$.

Problem 22 *Prove the above lemma.*

Theorem 3.3.3 *If* K *is a compact convex set, then* $ch(\operatorname{ext} K) = K$.

Proof: Evidently, $ch(\operatorname{ext} K) \subseteq K$. We prove the reverse inclusion by induction on the dimension of K. Dimension one is obvious.

Suppose that $\mathbf{x} \in K$. If $\mathbf{x}$ is an extreme point then obviously $\mathbf{x} \in ch(\operatorname{ext} K)$. So, let $[\mathbf{y}_0, \mathbf{z}_0]$ be a line segment in K containing $\mathbf{x}$ in its relative interior. Extend this segment in both directions. It will intersect the boundary of K at two points, $\mathbf{y}$ and $\mathbf{z}$. By the previous proposition there exist faces $F_\mathbf{y}$ and $F_\mathbf{z}$ which contain $\mathbf{y}$ and $\mathbf{z}$ respectively. Each of these faces has dimension less that K, so by the induction hypothesis, $F_\mathbf{y} = ch(\operatorname{ext} F_\mathbf{y})$ and $F_\mathbf{z} = ch(\operatorname{ext} F_\mathbf{z})$. As $\mathbf{x} \in ch(\operatorname{ext} F_\mathbf{y} \cup \operatorname{ext} F_\mathbf{z})$ and the lemma tells us that $\operatorname{ext} F_\mathbf{y} \cup \operatorname{ext} F_\mathbf{z} \subseteq \operatorname{ext} K$, we are done. □

3.4 $\mathcal{V}$ and $\mathcal{H}$-polytopes

As the reader may (or may not) of guessed by now, the main theorem of polytopes is that $\mathcal{V}$ and $\mathcal{H}$-polytopes are the same. We are finally ready to prove this.

Theorem 3.4.1 $P \subseteq \mathbb{R}^d$ *is a* $\mathcal{V}$*-polytope if and only if* P *is an* $\mathcal{H}$*-polytope.*

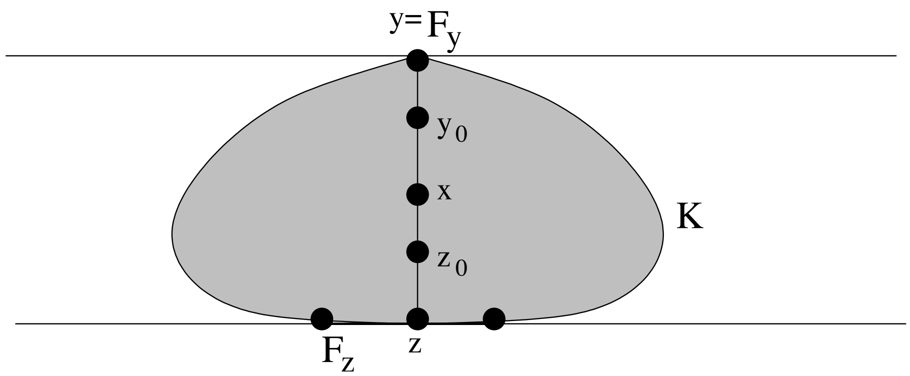

Figure 3.1: $ch(\text{ext}\, K) = K$

Proof: Using an affine equivalent polytope if necessary, we can assume that P is d-dimensional. Let P be an $\mathcal{H}$-polytope. We prove P is a $\mathcal{V}$-polytope by induction on dimension. As usual, dimension one is trivial.

Write P as a minimal intersection of closed half-spaces $P = \cap_{i=1}^{m} H_i$. By minimal we mean that if we set $P_i = \cap_{j \neq i} H_j$, then $P_i \neq P$ for any i. Minimality guarantees that the boundary of P is contained in the union of the $(d-1)$-dimensional faces of P and there are m of these.

Problem 23 *Why?*

By induction, each such face has a finite number of extreme points. All of the extreme points of P are on the boundary. Hence, the previous lemma and the last theorem imply that P has a finite number of extreme points and must be a $\mathcal{V}$-polytope.

Now suppose that P is a d-dimensional $\mathcal{V}$-polytope in $\mathbb{R}^d$. Write $P = ch(V)$, $|V| = n$. By the previous theorem, we might as well assume that $V = \text{ext}\, P$. Any k-face of P is determined by $(k+1)$ affinely independent points in the face. Hence, for every k the number of k-dimensional faces is less than or equal to $\binom{|V|}{k+1}$. In particular it is finite. Let F_i enumerate the facets of P and let H_i be the corresponding closed half-spaces. The proof is complete once we show the following.

Claim: $P = \cap H_i$.

Proof: (of claim) Certainly $P \subseteq \cap H_i$. Suppose $\mathbf{x} \notin P$. For each face G_j of P of dimension $d-2$ or less, let A_j be affine span of G_j and $\mathbf{x}$. The A_j form a finite collection of affine subspaces each of which has dimension at most $d-1$. Therefore their union does not cover the interior of P.

Problem 24 *Why?*

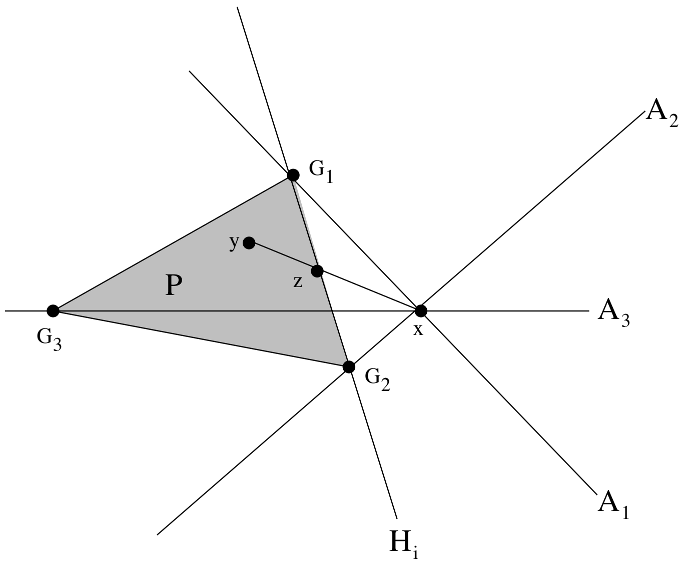

Figure 3.2: $P = \cap H_i$

So, there exists $\mathbf{y}$ in the interior of P so that $\mathbf{y}$ is not in the union of the A_j. Consider the line segment from $\mathbf{y}$ to $\mathbf{x}$. Since $\mathbf{y} \in P$ and $\mathbf{x} \notin P$ there exists $\mathbf{z}$ on the boundary of P and the line segment from $\mathbf{y}$ to $\mathbf{x}$. By Theorem 3.1.1 there is a proper face of P which contains $\mathbf{z}$. As $\mathbf{y}$ is not in any of the A_j, it must be the case that $\mathbf{z}$ is on the boundary of one of the H_i. Hence, $\mathbf{x} \notin \cap H_i$. □

Theorem 3.4.1 makes it clear the $\mathcal{V}$ and $\mathcal{H}$-polytopes are the same. As we have already seen, each point of view has its advantages and disadvantages. In chapter 5 we will see that this is only the beginning of the story.

Exercise 3.5

1. *Open convex sets have no extreme points.*
2. *A convex set K is* **centrally symmetric** *if whenever $\mathbf{x} \in K$, then $-\mathbf{x} \in K$. Prove that if $P \subseteq \mathbb{R}^e$ is a nonempty centrally symmetric polytope, then there exists $d > 0$ and an affine map $f : \mathbb{R}^d \to \mathbb{R}^e$ such that $f(\diamond^d) = P$.*
3. *Every face of a polytope is a face of a facet.*
4. *Let F be a face of a polytope P, and let G be a face of F. Prove that G is a face of P.*
5. *Let F be a k-dimensional face of a d-dimensional polytope P. Show that F is the intersection of $d - k$ facets of P.*

6. *(The basis of the simplex algorithm) Let* $\{v_1, \ldots, v_n\}$ *be the vertices of a polytope* $P \subseteq \mathbb{R}^d$. *Prove that for any* $\mathbf{a} \in (\mathbb{R}^d)^\star$

$$\sup_{\mathbf{x} \in P} \mathbf{a} \cdot \mathbf{x} = \max\{\mathbf{a} \cdot v_1, \ldots, \mathbf{a} \cdot v_n\}.$$

Chapter 4

Shelling

The boundary of a d-polytope is homeomorphic to the $(d-1)$-sphere. It turns out that this boundary can be built up by gluing the facets together in a 'nice' way. This will lead us to some remarkable enumerative results for the f-vector of the polytope, especially when P is simplicial. The f-vector of a d-polytope is $(f_{-1}, f_0, f_1, \ldots, f_d)$, where f_i is the number of i-dimensional faces of P. Until further notice, P is a simplicial d-polytope in $\mathbb{R}^d$.

A **shelling** of P is an ordering $F_1, \ldots, F_m$ of the facets of P such that for all $j \geq 2, F_j \cap (\cup_{i=1}^{j-1} F_i)$ is a nonempty union of facets of F_j. If P has a shelling order, then we say P is **shellable**. Figure 4.1 shows a shelling of the boundary of the octrahedron, while figure 4.2 shows an ordering of the four facets of the boundary of the octahedron which is not the beginning of any shelling.

Now view the ordering of the facets as a way of building up ∂P. Each facet adds new faces of ∂P as we take the union of the facet with the previous ones. The

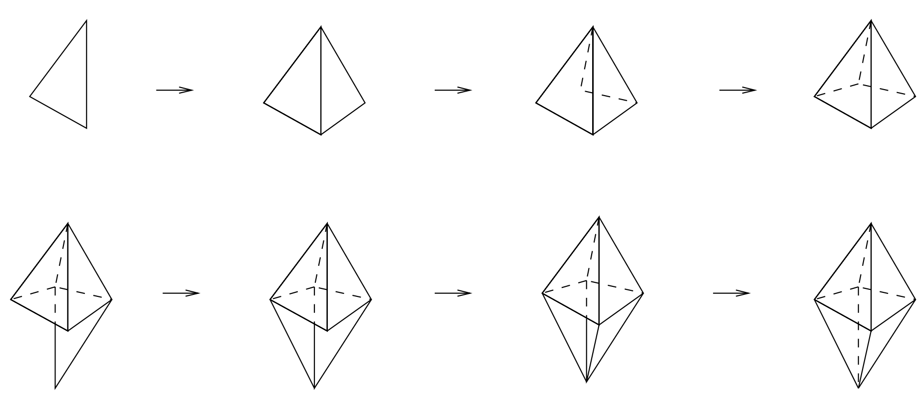

Figure 4.1: A shelling of the boundary of the octahedron

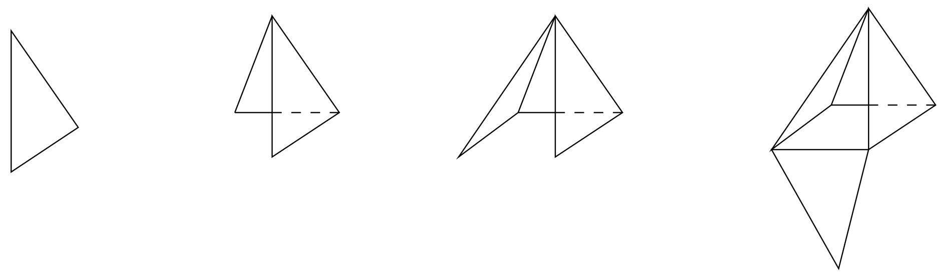

Figure 4.2: The beginning of a non-shelling of the boundary of the octohedron

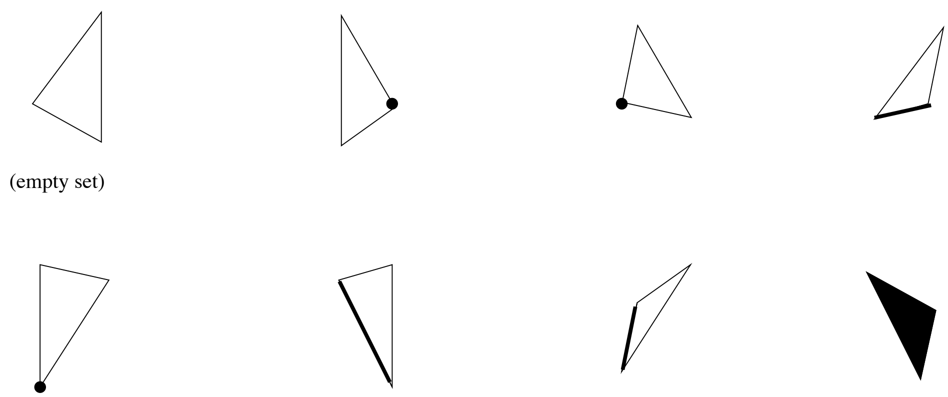

Figure 4.3: Minimal new faces of the shelling

shelling condition insures that there is a unique new minimal face added at each step. Indeed, at the j^{th} step $F_j \cap (\cup_{i=1}^{j-1} F_i)$ is a union of facets of F_j. The minimal face is $M_j = ch(\{v_{j_1}, \ldots, v_{j_m}\})$, where the v_{j_k} are the vertices opposite the facets of F_j in the intersection. Figure 4.3 shows the minimal new faces of the shelling in Figure 4.1.

Problem 25 *Prove that M_j is the minimal new face at the j^{th} step of the shelling.*

Exercise 4.1

1. *Any ordering of the facets of Δ^d is a shelling of its boundary.*
2. *Are there any other simplicial polytopes such that any ordering of their facets are shellable?*
3. *Is $\partial \diamond^d$ shellable?*
4. *Can a simplicial polytope with f-vector $(1, 7, 19, 20, 8, 1)$ be shellable?*

The concept of shellability is very important and extends to a number of situations including abstract simplicial complexes. An **abstract simplicial complex** consists of a set V, the vertices of the complex, and a set of **faces** $\Delta \subseteq 2^V$ The faces must be closed under subsets. If $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$. The maximal faces (with respect to inclusion) of Δ are the **facets** of the complex. The dimension of a face $G \in \Delta$ is $|G| - 1$. The dimension of Δ is the maximum of the dimensions of its faces.

Example 4.2 *Some examples of abstract simplicial complexes.*

1. *Boundaries of simplicial polytopes. The faces are the subsets of vertices whose convex hull is a proper face of the polytope.*
2. *Simple graphs. The faces are the pairs of vertices which have an edge between them.*
3. *Let V be a subset of vectors in a vector space and let Δ be the subsets of V which form independent subsets of vectors. What are the facets of this complex?*

The definition of shellable for a pure abstract simplicial complex is almost exactly the same as for simplicial polytopes; for $j \geq 2, F_j \cap (\cup_{i=1}^{j-1} F_i)$ must be a union of facets of the boundary of F_j. The boundary of a face of Δ is all subsets of F_j of cardinality less than or equal to $|F_j| - 1$. An abstract simplicial complex is **pure** if all facets have the same dimension. For instance, if $V = [4]$ and the facets of Δ are $\{1, 2, 3\}, \{1, 4\}, \{2, 4\}$, then Δ is not pure, but does satisfy the above conditions. There is a notion of nonpure shellability, but we will not discuss that here. Whenever we are considering shellability the complex will be pure.

Problem 26 *Show that connected graphs and the complexes described in Example 4.2 (3) are shellable.*

Example 4.3 *Let G be the graph in Figure 4.4. Let the vertices of Δ be the edges of the graph. The faces of Δ are those subsets of edges whose removal does not disconnect the graph. Specifically, $\emptyset$, all singleton, and all doubletons except $\{a, b\}$ and $\{c, d\}$ are the faces of Δ. We represent Δ in Figure 4.4. Singletons are represented by vertices, doubletons by edges. Ordering the edges (which are the facets of Δ) as shown, we see that $\emptyset$ is the minimal new face for* 1, *and the minimal new face is a single vertex for faces* $2, 3, 4$ *and an edge for faces* $5, 6, 7, 8$.

Suppose that G above represents a network and each edge has equal and independent probability of failing $p, 0 < p < 1$. What is the probability that that network will

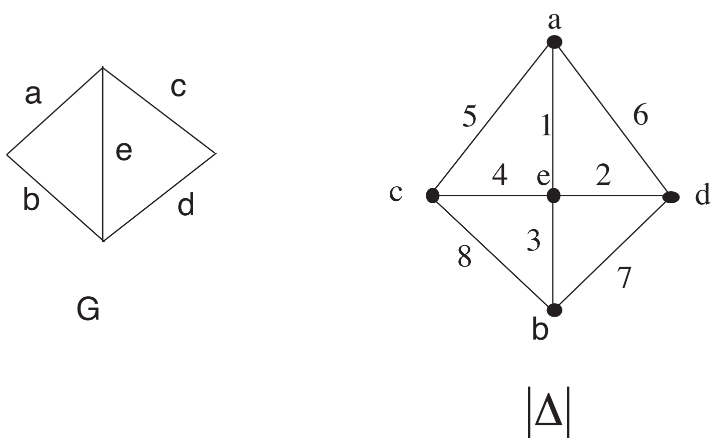

Figure 4.4: G and Δ

remain connected? Directly checking each possible subset of edges which keep the graph connected we see that this probability is

$$(1-p)^5 + 5p(1-p)^4 + 8p^2(1-p)^3 = (1-p)^3[1+3p+4p^2].$$

Notice that the coefficient in each degree i of the last factor is the number of steps in the shelling in which we added a minimal new face of cardinality i.

Suppose Δ is a shellable abstract simplicial complex and we are given a shelling. For each i, let h_i be the number of facets whose unique new minimal face has cardinality i. The *h-vector* of Δ is $(h_0, \ldots, h_d)$.

Example 4.4 *The shelling in Figure 4.1 gives h-vector* $(1,3,3,1)$.

The *shelling polynomial* of the shelling $F_1, \ldots, F_m$ is

$$h_\Delta(x) = h_0 x^d + h_1 x^{d-1} + \cdots + h_{d-1}x + h_d. \tag{4.1}$$

The shelling polynomial appears to depend on the shelling. We can also encode the f-vector of Δ in a polynomial. The f-vector of an abstract simplicial complex Δ is defined like the f-vector of a polytope, f_i is the number of i-dimensional faces and the $f(\Delta) = (f_{-1}, f_0, \ldots, f_{d-1})$, where $(d-1)$ is the dimension of Δ. Define the *face polynomial* of Δ to be

$$f_\Delta(x) = f_{-1}x^d + f_0 x^{d-1} + f_1 x^{d-2} + \cdots + f_{d-1}.$$

Theorem 4.4.1 $h_\Delta(x+1) = f_\Delta(x)$.

Problem 27 *Prove this theorem.*

Corollary 4.4.2 *The h-vector of Δ does not depend on the shelling order.*

Problem 28 *Write down formulas for h_i in terms of f_j and vice versa. Which f_j does h_i depend on (and vice versa)? Show that if $\mathcal{D}$ is a collection of abstract simplicial complexes and $\Delta \in \mathcal{D}$ has the property that $h_i(\Delta) \geq h_i(\Delta')$ for all $0 \leq i \leq d$ and $\Delta' \in \mathcal{D}$, then $f_i(\Delta)$ also maximizes all $f_i(\Delta')$ in $\mathcal{D}$. Is this still true if we reverse the role of the f- and h-vector?*

Aside from the formulas established in Problem 28 there is another method for computing h-vectors known as 'Stanley's trick'. We write the f-vector (including $f_{-1} = 1$) along the right-hand side of a triangle that looks like Pascal's triangle. Then put ones along the left-hand side. Fill in the rest of the triangle as you would Pascal's triangle, except subtract instead of add. Compute one extra row of subtractions and you get the h-vector. Here is an example which shows that the f-vector $(1, 7, 21, 14)$ becomes the h-vector $(1, 4, 10, -1)$.

$$\begin{array}{cccccccc}
 & & & & 1 & & & \\
 & & & 1 & & 7 & & \\
 & & 1 & & 6 & & 21 & \\
 & 1 & & 5 & & 15 & & 14 \\
\hline
1 & & 4 & & 10 & & -1 &
\end{array}$$

Problem 29 *Prove that Stanley's trick works.*

4.5 Geometric simplicial complexes

We saw in Example 4.2 (1) a geometric object, the boundary of a simplicial polytope, represent the combinatorial data of an abstract simplicial complex. Is this possible for all abstract simplicial complexes? The answer lies with geometric simplicial complexes.

Definition 4.5.1 *A* **geometric simplicial complex** Δ *in $\mathbb{R}^d$ is a finite set of simplices in $\mathbb{R}^d$ such that*

- $\Delta \neq \emptyset$.

- *If $F \in \Delta$ and G is a face of F, then $G \in \Delta$.*
- *If F_1 and F_2 are in Δ, then $F_1 \cap F_2$ is a face of both F_1 and F_2. (Recall that the empty set is always a face of both.)*

Examples illustrating the above definition are in Figures 4.5 and 4.6

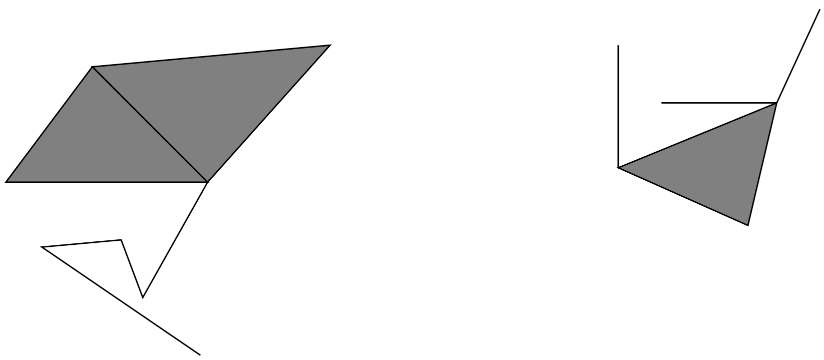

Figure 4.5: A geometric simplicial complex in $\mathbb{R}^2$

Figure 4.6: Two examples which are not geometric simplicial complexes in $\mathbb{R}^2$

Suppose that Δ is a geometric simplicial complex. Let V be the 0-dimensional faces in Δ. The abstract simplicial complex associated to Δ has as its vertices V, and as its faces those subsets of vertices whose convex hull is a simplex in Δ. For instance, the figure on the right hand side of Figure 4.4 is a geometric simplicial complex in $\mathbb{R}^2$ whose associated abstract simplicial complex is the one described in Example 4.3. Does every abstract simplicial complex come from a geometric simplicial complex?

Let Δ be an abstract simplicial complex with vertex set $[n]$. The **geometric realization** of Δ is the subset of $\mathbb{R}^n$ given by

$$|\Delta| = \bigcup_{F \in \Delta} ch(\cup_{i \in F}\ \mathbf{e}_i).$$

It is immediate from the definition that the abstract simplicial complex associated to $|\Delta|$ is Δ. A geometric realization of an abstract simplicial complex Δ (as opposed to **the** geometric realization) is any geometric simplicial complex whose associated abstract simplicial complex is Δ. Any two geometric realizations of the same abstract simplicial complex are homeomorphic. The idea is simple. Map corresponding vertices of the two complexes to each other and 'extend linearly'. Because it does not matter what geometric realization is used, we will simply say that Δ is homeomorphic to a given space if any of its geometric realizations is.

There are many geometric realizations for a given abstract simplicial complex Δ. One of the big disadvantages of $|\Delta|$ is that it lies in $\mathbb{R}^n$ for n much bigger than necessary. The geometric realization of the complex in Example 4.4 would be in $\mathbb{R}^5$, but obviously only $\mathbb{R}^2$ is needed. In general, what is the smallest $\mathbb{R}^d$ needed to realize a given complex?

Problem 30 *Let Δ be an abstract simplicial complex of dimension d. Prove that there is a geometric realization of Δ in $\mathbb{R}^{2d+1}$.*

Note that the bound in the above problem is optimal. Indeed, there are one-dimensional complexes, such as K_5, which can not be realized in $\mathbb{R}^2$. Because of the close connection between abstract and geometric simplicial complexes we will simply say simplicial complex from here on and hope that the context makes it clear which type we mean.

4.6 Line shellings

A little experimentation shows that not all simplicial complexes are shellable. Disconnected graphs and two triangles with exactly one vertex in common are two examples. What about the boundary of a simplicial polytope? For a long time it was simply assumed that any simplicial complex homeomorphic to a ball or sphere was shellable. However, this turned out not to be true! In 1958 Rudin demonstrated how to construct a simplicial complex homeomorphic to a ball which was not shellable [14]. Finally, in 1971 Brugessor and Mani proved that all polytopes were shellable [4]. What does it mean for a nonsimplicial polytope to be shellable? To make sense of this we need the notion of a polytopal complex.

Definition 4.6.1 *A* **polytopal complex** *in* $\mathbb{R}^d$ *is a nonempty finite set of polytopes* Δ *in* $\mathbb{R}^d$ *such that*

- *If* P *is in* Δ *and* F *is a face of* P*, then* F *is in* Δ*.*
- *If* P_1 *and* P_2 *are in* Δ*, then* $P_1 \cap P_2$ *is a face of* P_1 *and* P_2*.*

The prototypical example of a polytopal complex is the boundary of a polytope. Of course, any geometric simplicial complex is a polytopal complex. Just as in the case of simplicial complexes a polytopal complex is **pure** if all of its maximal polytopes (in the sense of inclusion) have the same dimension. As before, the maximal polytopes of Δ are called **facets**. The definition of a shelling of a polytopal complex is almost the same as for a simplicial complex.

Definition 4.6.2 *A* **shelling** *of a polytopal complex* Δ *is an ordering* $P_1, \ldots, P_t$ *of the facets of* Δ *such that for all* $j \geq 2$,

$$P_j \bigcap \left(\bigcup_{i=1}^{j-1} P_i \right)$$

is a union of a subset of the facets of P_j *which form an initial segment of a shelling order for* ∂P_j*.*

In view of Exercise 4.1 (1) the above definition reduces to our previous definition if Δ is a geometric simplicial complex. Note that the definition is very inductive!

The shellings that Bruggessor and Mani constructed for polytopes are called line shellings. Let $\mathbf{x}$ be a point in the interior of P, a d-polytope in $\mathbb{R}^d$, not necessarily simplicial. Now choose a line l through $\mathbf{x}$ with the following two properties:

1. l intersects all the hyperplanes of the facets of P. (These are the affine spans of the facets.)
2. l does not intersect any nontrivial intersection of the hyperplanes of the facets of P.

Problem 31 *Why is it obvious we can always choose such a line?*

Now imagine you are in a rocket ship inside the planet P heading along the line l in a chosen positive direction. Initially you can not see anything as you are inside

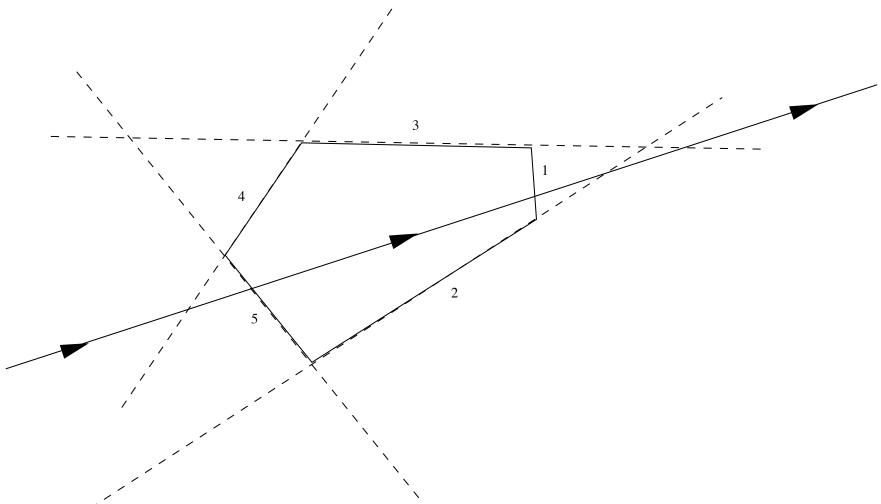

Figure 4.7: Line shelling of a polygon

the planet. By (2) you will emerge at a facet, F_1. As you travel away from P you will see one new facet each time you pass through its corresponding hyperplane. The shelling order for these facets is the same as the order you see them. By (2) you will never see two (or more) new facets simultaneously. Eventually you will go far enough towards "$+\infty$" so that you can see as many facets as possible in that direction. Now begin your return to P coming from the "$-\infty$" direction. At this point you can see all the facets you could not see from the "$+\infty$" direction. As you pass through each of their corresponding hyperplanes the corresponding facet will disappear from your vision. The shelling order continues from before in the order in which the facets disappear from your vision. By (1) each facet of P occurs exactly once in your shelling order.

Problem 32 *Show that this is a shelling of P. Where did you use convexity?*

The fact that every simplicial polytope is shellable already shows there are strong restrictions on their f-vectors. For instance, $(1, 6, 15, 18, 7, 1)$ is not the f-vector of any simplicial 4-polytope. Indeed, this would give $(1, 2, 3, 2, -1)$ as the h-vector of its boundary, and this is impossible. In fact, for the same reason, $(1, 6, 15, 18, 7)$ is not the f-vector of any shellable abstract simplicial complex.

What happens if we travel along l in the opposite direction? This simply reverses the shelling order. But now, if P is a simplicial polytope, then each facet which originally contributed to h_i, contributes to h_{d-i}. Since the h-vector is independent of the shelling order we obtain the *Dehn-Sommerville* equations.

Theorem 4.6.3 *[6], [15] If P is a simplicial d-polytope, then $h_i = h_{d-i}$.*

This is an even stronger restriction on the h-vector of P. Combining this theorem with Problem 28, we can see now that once we know $f_0, \ldots, f_{\lfloor d/2 \rfloor}$, we know the entire f-vector of P. Are there any other restrictions on the h-vectors of simplicial polytopes? In order to discuss this we must introduce some notation.

Let a, i be natural numbers. Then there is a unique way of writing

$$a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}, a_i > a_{i-1} > \cdots > a_j \geq j.$$

Problem 33 *Prove the above statement.*

Example 4.7 $a = 14, i = 3$. *Then* $14 = \binom{5}{3} + \binom{3}{2} + \binom{1}{1}$.

With a decomposed as above define

$$a^{<i>} = \binom{a_i + 1}{i+1} + \binom{a_{i-1} + 1}{i} + \cdots + \binom{a_j + 1}{j+1}.$$

Example 4.8 *So,* $14^{<3>} = \binom{6}{4} + \binom{4}{3} + \binom{2}{2} = 15 + 4 + 2 = 21$.

In 1971 P. McMullen conjectured that necessary and sufficient conditions for $(h_0, \ldots, h_d)$ to be the h-vector of the boundary of a simplicial d-polytope are

- $h_i \geq 0$.
- $h_i = h_{d-i}$.
- $h_0 \leq h_1 \leq \cdots \leq h_{\lfloor d/2 \rfloor}$.
- For $i \leq d/2$, define $g_i = h_i - h_{i-1}$. Then for $i \leq d/2$, $g_{i+1} \leq g_i^{<i>}$.

Stanley [17] proved necessity, while Billera and Lee [1] proved sufficiency. These results give a complete characterization of h-vectors (and hence f-vectors) of the boundaries of simplicial polytopes. For instance, here are two problems that are now fairly easy.

Problem 34 *Show that $C_d(n)$ maximizes every f_i among all possible simplicial d-polytopes with n vertices. What is fewest number faces in dimension i that a simplicial d-polytope with n vertices can have?*

Looking at many examples, it appears that the f-vector of a simplicial polytope is unimodal. A sequence $(f_0, f_1, \ldots, f_d)$ is *unimodal* if there exists i such that $f_0 \leq f_1 \leq \cdots \leq f_i \geq f_{i+1} \geq \cdots \geq f_d)$. By Stanley's proof of the necessity of McMullen's conditions, the h-vector of a simplicial polytope is symmetric and unimodal. It can be shown that the f-vector of a simplicial d-polytope is unimodal when $d \leq 19$ [7]. Shortly after Billera and Lee proved the sufficiency of McMullen's conditions, Björner [2] and Lee [12] gave examples of 20-dimensional simplicial polytopes with trillions of vertices with $f_{11} > f_{12} < f_{13}$.

What can we say about f-vectors of nonsimplicial polytopes? You may have guessed from Exercise 2.5 (8) that the alternating sum of the face numbers of a polytope only depended on the dimension. Analogously to our definition for simplicial complexes, for a polytopal complex Δ we define $f_i(\Delta)$ to be the number of i-dimensional polytopes in Δ.

Definition 4.8.1 *Let Δ be a d-dimensional polytopal complex. The* **Euler characteristic** *of Δ is*

$$\chi(\Delta) = \sum_{i=0}^{d} (-1)^i f_i(P).$$

Theorem 4.8.2 *Let P be a d-polytope. Then $\chi(P) = 1$. Equivalently, $\chi(\partial P)$ is zero if $\dim P$ is even ($\dim \partial P$ is odd) and two if $\dim P$ is odd ($\dim \partial P$ is even).*

Proof: The proof is by induction on d. In fact, we will prove something stronger. Let $F_1, \ldots, F_t$ be a line shelling of the facets of P. Define $\Delta_i = \cup_{j=1}^{i} F_j$. So $\Delta_1 = F_1$ and $\Delta_t = \partial P$. We will show that $\chi(\Delta_i) = 1$ for $i < t$ and $\chi(\Delta_t) = \chi(\partial P)$ is as claimed in the theorem. Our induction hypothesis is this stronger statement. It is easy to check that it holds for d equal to zero, one or two.

Now assume the induction hypothesis holds for $(d-1)$-polytopes. We proceed by induction on i. For $i = 1$, $\Delta_1 = F_1$, a $(d-1)$-polytope. So $\chi(\Delta_1) = 1$. What is $\Delta_{i-1} \cap F_i$? When $i < t$ the intersection is the initial segment of a partial shelling of the boundary of F_i. When $i = t$ the intersection is the entire boundary of F_i. In either case

$$\chi(\Delta_i) = \chi(\Delta_{i-1} \cup F_i) = \chi(\Delta_{i-1}) + \chi(F_i) - \chi(\Delta_{i-1} \cap F_i).$$

The theorem now follows by all of the inductive hypotheses. □

The fact that the Euler characteristic of a polytope is always one is just the tip of a remarkable iceberg. For instance, the Euler characteristic of any polytopal complex homeomorphic to a ball is one, and if Δ_1 and Δ_2 are two polytopal complexes which are homeomorphic, then $\chi(\Delta_1) = \chi(\Delta_2)$. The interested reader is invited to check out any of the many texts on algebraic topology where vast generalizations of these results are derived.

Chapter 5

Duality

Duality takes many shapes and sizes in mathematics. We have already met $(\mathbb{R}^d)^\star$, the linear dual of $\mathbb{R}^d$. In this chapter we take a closer look at polytope duality, and more generally, duality for closed convex sets. Anyone who has done Exercise 2.5.8 will (hopefully!) have noticed that $f_i(\square^d) = f_{d-i}(\diamond^d)$. This is one manifestation of the fact that the d-cube and the d-cross polytope are dual to one another. For a myriad of reasons, some of which we will see, duality in this context is an important idea with many applications.

5.1 $K^\star$

Definition 5.1.1 *Let K be a closed convex subset of $\mathbb{R}^d$. Then the* **dual** *of K is*

$$K^\star = \{\mathbf{a} \in (\mathbb{R}^d)^\star : \mathbf{a} \cdot \mathbf{x} \leq 1 \textit{ for all } \mathbf{x} \in K.\}$$

Example 5.2

- $K = B_r(\vec{0}) \subseteq \mathbb{R}^d$. $K^\star = B_{1/r}(\vec{0})$ *(in* $(\mathbb{R}^d)^\star$*).*
- $K = [1, 2] \subseteq \mathbb{R}$. $K^\star = (-\infty, 1/2] \subseteq \mathbb{R}^\star$.
- $K = \{(x, 0) \in \mathbb{R}^2 : -1 \leq x \leq 1\}$. $K^\star = \{(x, y) \in (\mathbb{R}^2)^\star : -1 \leq x \leq 1.\}$
- $(\square^d)^\star = \diamond^d$ *and* $(\diamond^d)^\star = \square^d$.

Without further restrictions the dual of a closed convex set may be of a different dimension, be unbounded when the set is bounded, or vice versa. However, with very mild restrictions it is very well behaved.

Lemma 5.2.1 *Let K be a compact convex subset of $\mathbb{R}^d$ with the origin in its interior. Then $K^\star$ is a also a compact convex subset with the origin in its interior.*

Problem 35 *Prove the above lemma.*

One of the most useful properties of duality for convex sets is that under mild conditions $K = K^{\star\star}$. In order to even make sense of this statement we need a method of identifying $\mathbb{R}^d$ and $(\mathbb{R}^d)^{\star\star} = ((\mathbb{R}^d)^\star)^\star$. So far we have frequently identified $\mathbb{R}^d$ with its dual $(\mathbb{R}^d)^\star$ using the dot product. This isomorphism between $\mathbb{R}^d$ and its dual depends on the choice of the unit coordinate vectors as a basis for $\mathbb{R}^d$. In a certain sense (which we will not follow up on here) this is unavoidable. So perhaps it is a little surprising to discover that there is a natural choice of isomorphism between $\mathbb{R}^d$ and its double dual which does not depend on any choices.

Let $\mathbf{x} \in \mathbb{R}^d$. Define an element of $(\mathbb{R}^d)^{\star\star}$ by

$$\mathbf{x}^{\star\star}(\mathbf{a}) = \mathbf{a}(\mathbf{x})$$

for $\mathbf{a} \in (\mathbb{R}^d)^\star$. While it is immediate that $\mathbf{x}^{\star\star}$ is a function from $(\mathbb{R}^d)^\star \to \mathbb{R}$, the real value of this definition is the following.

Problem 36 *Prove that $\mathbf{x} \to \mathbf{x}^{\star\star}$ is a linear isomorphism between $\mathbb{R}^d$ and $(\mathbb{R}^d)^{\star\star}$.*

Now we can make precise one of the most useful properties of $K^\star$.

Proposition 5.2.2 *If K is a closed convex subset which contains the origin, then under the above identification $K = K^{\star\star}$.*

Before we begin the proof we observe that the hypothesis that K contains the origin is unavoidable since $K^\star$, and hence $K^{\star\star}$, contains the origin.

Proof: Using the isomorphism from Problem 36 and Definition 5.1.1

$$\begin{aligned} K^{\star\star} &= \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a}(\mathbf{x}) \le 1 \text{ for all } \mathbf{a} \in K^\star\} \\ &= \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a}(\mathbf{x}) \le 1 \text{ for all } \mathbf{a} \in (\mathbb{R}^d)^\star \text{ such that } \mathbf{a}(\mathbf{y}) \le 1 \text{ for all } \mathbf{y} \in K.\} \end{aligned}$$

Hence any $\mathbf{x} \in K$ is in $K^{\star\star}$. To see that $K^{\star\star} \subseteq K$ we show that if $\mathbf{x} \notin K$, then $\mathbf{x} \notin K^{\star\star}$. So suppose that $\mathbf{x}$ is not in K. Then, since the origin is in K, there exists a hyperplane $H^{=}_{\mathbf{a},1}$ that separates K and $\mathbf{x}$ with $K \subseteq H^{-}_{\mathbf{a},1}$. In particular, $\mathbf{a} \in K^\star$. However, $\mathbf{a} \cdot \mathbf{x} = \mathbf{a}(\mathbf{x}) > 1$. Thus $\mathbf{x} \notin K^{\star\star}$. □

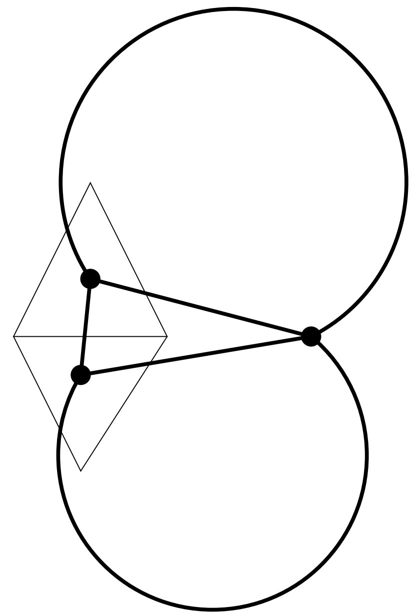

Figure 5.1: A planar graph and its **dual**

In addition to linear duality, the reader has probably seen graph duality. For any (embedded) planar graph G it dual is defined by assigning a vertex to each region of the plane that G divides $\mathbb{R}^2$ into and an edge between any two regions separated by a single edge of G. See Figure 5.1 for a typical example.

One way to view this process is as a way of changing the dissection of $\mathbb{R}^2$ induced by the graph G. Zero-dimensional faces (the vertices of G) are replaced by two-dimensional faces (regions) of its dual. Similarly, edges are replaced by other edges and two-dimensional faces are replaced by vertices. Under mild conditions this is another kind of duality encoded by $K^\star$.

Let K be a compact convex subset of $\mathbb{R}^d$ with the origin in its interior. For each subset F of K define $\Psi(F) \subseteq K^\star$ by

$$\Psi(F) = \{\mathbf{a} \in K^\star : \mathbf{a}(\mathbf{x}) = 1 \text{ for all } \mathbf{x} \in F.\}$$

Theorem 5.2.3 *Let K be as above.*

1. *If $G \subseteq F$, then $\Psi(G) \supseteq \Psi(F)$.*
2. *If F is a face of K, then $\Psi(F)$ is a face of $K^\star$.*
3. *If F is a face of K, then $\Psi(\Psi(F)) = F$.*

Proof: The first property follows easily from the definition. For the second, we first consider two extreme cases. If $F = K$, then, since K contains the origin, $\Psi(F) = \Psi(K) = \emptyset$. On the other hand, if $F = \emptyset$, then it is immediate that $\Psi(F) = \Psi(\emptyset) = K^\star$. So from here on we assume that F is neither of these extremes. Let $\mathbf{x}$ be in the relative interior of F (equal to F when F is just a point).

Problem 37 *$H_{-\mathbf{x}^{\star\star},-1}$ is a valid inequality for $K^\star$ and $\Psi(F) = K^\star \cap H_{-\mathbf{x}^{\star\star},-1}$. Hence $\Psi(F)$ is a face of $K^\star$.*

The last property we leave as a problem. □

Problem 38 *Prove that if F is a face of K, then $\Psi(\Psi(F)) = F$.*

One way to sum up the above result would be to observe that if K is a compact subset of $\mathbb{R}^d$ with the origin in its interior, then Ψ induces an inclusion reversing bijection between $\mathcal{F}(K)$ and $\mathcal{F}(K^\star)$. The **order dual** of a poset $(\Pi, \leq)$, which we denote by $(\Pi^\star, \leq^\star)$, is the poset whose ground set is also Π and whose order relation is given by $x \leq^\star y$ if and only if $y \leq x$. Hence, $\mathcal{F}(K)$ is isomorphic to $\mathcal{F}(K^\star)^\star$.

What consequences can we derive for face posets of polytopes? To begin with, we need the following.

Problem 39 *If P is a polytope, F a face of P and G a face of F, then G is a face of P.*

Problem 40 *If P is a d-polytope in $\mathbb{R}^d$, then $P^\star$ is also a d-polytope.*

A **chain** in a poset is a sequence

$$x_0 < x_1 < \cdots < x_m$$

of distinct elements of the poset. The **length** of the chain is one less than the number of elements in the chain. The above chain has length m. A chain is **maximal** if it is not contained in any strictly longer chain. For instance, in D_{30}, $1 < 3 < 6 < 30$ is a maximal chain as is $1 < 5 < 10 < 30$. (Recall the definition of D_n in Chapter 2 However, in B_3, $\emptyset < \{2\} < \{1, 2, 3\}$ is not a maximal chain. A poset is **graded** if the length of every maximal chain is the same. If Π is a graded poset, then the **rank** of an element $x \in \Pi$ is the maximum length among all chains which end with x. The rank of a graded poset is the maximum rank obtained by its elements. Equivalently, the common length of all of its maximal chains. For examples of graded posets see Exercises 2 and 3

Problem 41 *If P is a d-polytope, then $\mathcal{F}(P)$ is a graded poset of rank $d+1$. More generally, the rank of any face F of P in $\mathcal{F}(P)$ is $1 + \dim F$.*

The dual of a simplicial polytope is called a **simple** polytope. A simple combinatorial characterization of simple polytopes is contained in the exercises below. By Problem 41, Ψ interchanges i-dimensional faces of a rank d-polytope with $(d-i)$-dimensional faces of $P^\star$. Thus the complete description of possible f-vectors of simplicial polytopes with n vertices given in Chapter 4 also gives a complete description of the possible f-vectors of simple polytopes with n facets. In addition, Problem 34 gives an exact formula for the maximum number of vertices for such a polytope.

Exercise 5.3

1. *Let K be a closed convex subset of $\mathbb{R}^d$ which contains the origin. Then $K^\star$ contains an affine subspace of dimension i if and only if $\dim K \leq d - i$.*
2. *B_n is a graded poset and the rank of $A \subseteq [n]$ is equal to $|A|$.*
3. *D_n is a graded poset. If $a \in n$ and the prime factorization of a is*
$$a = p_1^{e_1} \cdot p_2^{e_2} \cdot \cdots \cdot p_m^{e_m},$$
then the rank of a is $e_1 + \cdots + e_m$.
4. *Let P be a d-polytope. If F is a $(d-2)$-dimensional face, then F is contained in exactly two facets.*
5. *P is a simple d-polytope if and only if every vertex is incident to d edges.*
6. *Any simplex is both simple and simplicial. Are there any other polytopes with this property?*
7. *Are there any polytopes such that $P = P^\star$? Polytopes with the property that $\mathcal{F}(P) \simeq \mathcal{F}(P^\star)$ are called self-dual. Simplices are self dual. Are there other self dual polytopes?*
8. *What can you say about self dual convex subsets?*

Chapter 6

The Möbius function

In order to further advance our study of face posets of polytopes, their connections to hyperplane arrangements and graph coloring, we now turn to Möbius inversion. This is now one of the standard techniques of modern combinatorics. Introduced by Weisner [18], Möbius inversion is a poset based generalization of inclusion-exclusion. Its real value as a powerful approach to many enumerative problems was pioneered by Rota [13]. Throughout this chapter Π is a finite poset. While the theory applies to locally finite posets, those such that every interval $[x, y]$ is finite, we will have no need of this generality.

6.1 The Möbius function

The Möbius function is a simultaneous generalization of inclusion-exclusion and its more famous number theory namesake. It is the function $\mu : \Pi \times \Pi \to \mathbb{Z}$ defined inductively using the following properties:

1. If $x \not\leq y$, then $\mu(x, y) = 0$.
2. $\mu(x, x) = 1$.
3. If $x < y$, then
$$\sum_{x \leq z \leq y} \mu(x, z) = 0.$$

As long as $[x, y]$ is finite, $\mu(x, y)$ can be computed using the last two properties inductively. See Figure 6.1 for a simple example where $\mu(x, y)$ is computed for a specific poset and a fixed x in the poset.

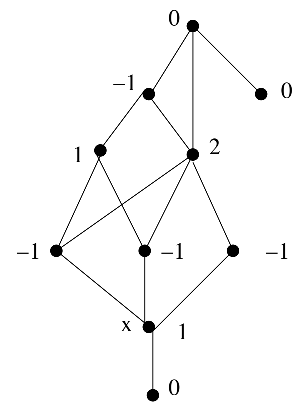

Figure 6.1: Computing $\mu(x, y)$.

While the above inductive procedure gives a finite algorithm to compute the Möbius function for any two elements in Π, it is known that for general posets and elements it is not computationally fast. However, in many specific cases it can be computed easily. Here are two examples. (Recall the definitions of B_n and D_n in Chapter 2,)

Problem 42 *Compute $\mu(A, B)$ for any $A, B \in B_n$.*

Problem 43 *Compute $\mu(s, t)$ for any $s, t \in D_n$.*

Another class of posets for which the Möbius function is (very) easy to compute are Eulerian posets. A poset is called **Eulerian** if it is graded, contains a unique minimal element and a unique maximal element, and its Möbius function satisfies

$$\mu(x, y) = (-1)^{\rho(y)-\rho(x)}$$

for all $x \leq y$. Here, $\rho(x)$ is the rank of x. The easiest example of an Eulerian poset is B_n. Perhaps the most interesting class of Eulerian posets are face posets of polytopes.

Problem 44 *If P is a polytope, then $\mathcal{F}(P)$ is an Eulerian poset.*

Another large class of Eulerian posets are the face posets of simplicial complexes homeomorphic to spheres or odd-dimensional manifolds.

The Möbius inversion formula is probably best understood by introducing the incidence algebra of a poset. The **incidence algebra** of Π, denoted by $I(\Pi)$, is the space of all functions $\eta : \Pi \times \Pi \to \mathbb{R}$ such that $\eta(x,y) = 0$ whenever $x \not\leq y$. It comes with the usual real vector space structure associated to spaces of functions to the reals. It also has a multiplicative structure which looks suspiciously like matrix multiplication (if you are an algebraist) or convolution (if you are an analyst). For $\eta, \psi \in I(\Pi)$ define $\eta \circ \psi$ in $I(\Pi)$ by

$$(\eta \circ \psi)(x,y) = \begin{cases} 0 & \text{if } x \not\leq y \\ \sum_{x \leq z \leq y} \eta(x,z) \cdot \psi(z,y) & \text{if } x \leq y. \end{cases}$$

This multiplicative structure is easily deciphered from the point of view of matrix multiplication. Let $n = |\Pi|$ and let $\beta : [n] \to \Pi$ be a bijection so that the induced order relation $\leq_{[n]}$, given by $i \leq_{[n]} j$ if and only if $\beta(i) \leq \beta(j)$, is natural. (See Section 2.3.)

Problem 45 *The incidence algebra $I(\Pi)$ is isomorphic to the matrix algebra of $n \times n$ real matrices M such that $M_{i,j} = 0$ if $i \not\leq_{[n]} j$.*

All of the properties of $I(\Pi)$ below follow easily from the above problem.

Corollary 6.1.1

1. $(\eta \circ \psi) \circ \nu = \eta \circ (\psi \circ \nu)$ *(associativity).*
2. *Define* $\delta(x,y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}.$
   Then $\eta \circ \delta = \delta \circ \eta$ *(existence of identity)*
3. *Define* $\zeta(x,y) = \begin{cases} 1 & x \leq y \\ 0 & x \not\leq y \end{cases}.$
   Then $\mu \circ \zeta = \zeta \circ \mu = \delta$ *(inverse of the Möbius function).*

Let f and g be functions from Π to $\mathbb{Z}$. Möbius inversion is a method for analyzing the situation when

$$g(y) = \sum_{x \leq y} f(x)$$

or its order dual relation

$$g(x) = \sum_{x \leq y} f(y).$$

It might seem that such a relationship would be quite rare. As we will see, it occurs quite frequently. Here is a simple example. Let $\phi(n)$ be the Euler totient - $\phi(n)$ is the number of positive integers less than or equal to n which are relatively prime to n. (Have no common factors other than one.) For instance, $\phi(12) = 4$ since $1, 5, 7, 11$ are the only such integers.

Proposition 6.1.2

$$\sum_{d|n} \phi(d) = n.$$

Proof: Write down the fractions $\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}$. Then put them in lowest terms. For instance, for $n = 12$ we end with

$$\frac{1}{12}, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{5}{12}, \frac{1}{2}, \frac{7}{12}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \frac{11}{12}, \frac{1}{1}.$$

Notice that only denominators d which divide n occur, and each such denominator occurs exactly $\phi(d)$ times. □

Theorem 6.1.3 *(Möbius inversion) Let f and g be functions from Π to $\mathbb{Z}$.*

$$g(y) = \sum_{x \le y} f(x)$$

if and only if

$$f(y) = \sum_{x \le y} g(x)\mu(x, y).$$

Dually,

$$g(x) = \sum_{x \le y} f(y)$$

if and only if

$$f(x) = \sum_{x \le y} \mu(x, y)\ g(y).$$

The above formulas look like multiplication of vectors and matrices and that is the main idea behind the proof of the theorem.

Problem 46 *Prove the Möbius inversion formulas.*

The number theoretic Möbius function, for which the poset version gets its name is defined for $n \in \mathbb{Z}^+$ by

$$\mu(n) = \begin{cases} 1 & n = 1 \\ (-1)^k & n = p_1 \cdots p_k, \text{ the } p_i \text{ distinct primes} \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 6.1.4

$$\phi(n) = \sum_{d|n} \mu(n/d) \cdot d.$$

Proof: Problem 43 and Möbius inversion. □

Our first combinatorial application of Möbius inversion involves graph coloring. Let G be a graph without loops. Denote the vertices of G by $V(G)$. A **λ-coloring** of G is a function $f : V(G) \to [\lambda]$. The coloring f is **proper** if no vertices with the same color (i.e. image) share an edge. Define $\chi_G(\lambda)$ to be the number of proper λ-colorings of G. Evidently χ_G is only defined for $\lambda \geq 1$.

What kind of function is χ_G? It is called the **chromatic polynomial**, so it is a good guess that χ_G is a polynomial! Any λ-coloring of G induces a partition of the vertices which has at most λ many blocks. The **partition-induced** subgraph of G corresponding to the coloring f is the graph whose vertices are the same as G, but whose edges are those whose end points have the *same* color. Thus a proper coloring corresponds to the graph consisting of the vertices of G and no edges. For another example, consider the graph in Figure 6.2. If f mapped vertices A and D to color 1, and vertices B and C to color 2, then the partition-induced subgraph would consist of the vertices and the single edge between B and C. Notice that the same thing would happen if f mapped vertex A to 1, vertex D to 2 and vertices B and C to 3.

The poset of vertex-induced subgraphs of G, which we denote by L_G, consists of all possible partition-induced subgraphs of G (over all possible colorings, not just those of a fixed λ) ordered by inclusion. Its least element is the graph with no edges and its maximal element is G itself. The Hasse diagram for the graph in Figure 6.2 can be seen in Figure 6.3. What else can we say about L_G?

Problem 47 *L_G is graded. The rank of a partition-induced subgraph H of G is $|V| - \#$ components of H.*

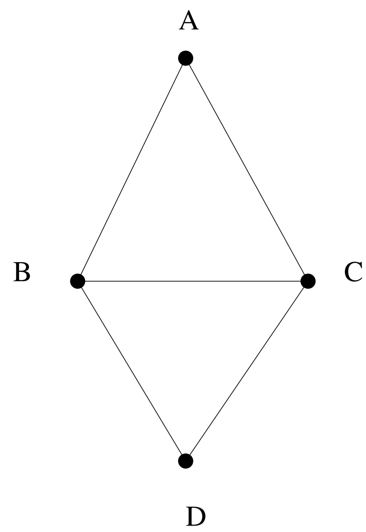

Figure 6.2: A simple graph

In order to connect L_G with the chromatic polynomial, for $H \in L_G$ define $f_\lambda(H)$ to be the number of λ-colorings of G such that H is the associated partition-induced subgraph. By definition, $\chi_G(\lambda)$ is $f_\lambda(\hat{0})$, where $\hat{0}$ is the least element of L_G. Can we apply Möbius inversion? For any graph H let $c(H)$ be the number of components of H.

Problem 48

$$\chi_G(\lambda) = \sum_{H \in L_G} \mu(\hat{0}, H)\lambda^{c(H)} = \lambda^{c(G)} \sum_{H \in L(G)} \mu(\hat{0}, H)\lambda^{\rho(G)-\rho(H)}.$$

From this we easily see that $\chi_G(\lambda)$ is a polynomial of degree $|V|$, contains $\lambda^{c(G)}$ as a factor and the coefficient of $\lambda^{c(G)}$ is $\mu(\hat{0}, G)$. The last term of the right-hand side of the above equality makes sense for any (finite) graded poset with a unique minimal element $\hat{0}$.

Definition 6.1.5 *Let* Π *be a (finite) graded poset with unique minimal element* $\hat{0}$ *and let* r *be the rank of* Π*. The* **chracteristic polynomial** *of* Π *is*

$$\chi_\Pi(\lambda) = \sum_{x \in \Pi} \mu(\hat{0}, x)\lambda^{r-\rho(x)}.$$

We will meet the characteristic polynomial again in the next chapter on hyperplane arrangements.

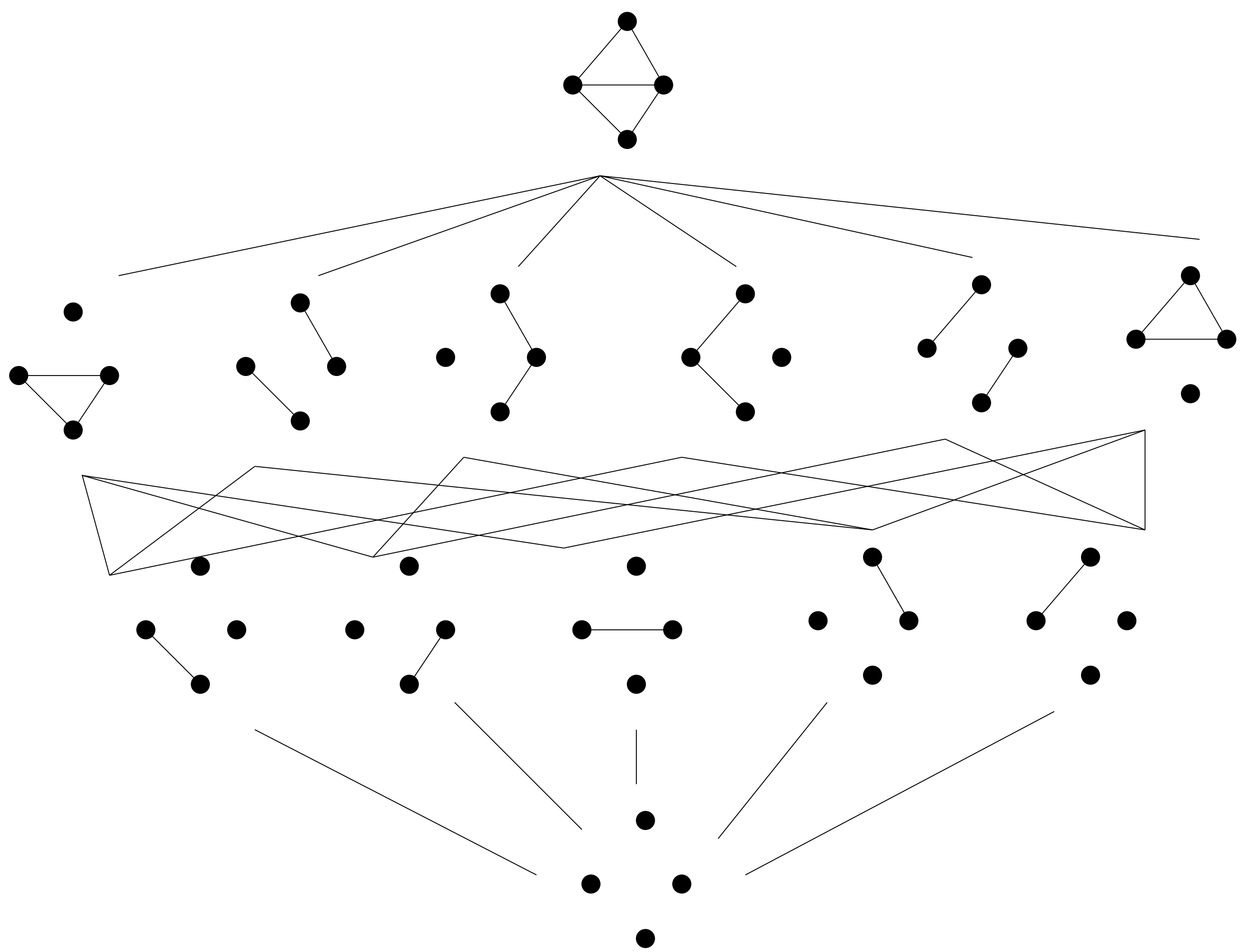

Figure 6.3: L_G

Exercise 6.2

1. *An element $\nu \in I(\Pi)$ is multiplicatively invertible if and only if $\nu(x,x) \neq 0$ for all $x \in \Pi$.*

2. *Inclusion-exclusion is the following combinatorial principle. Let $A_1, \ldots, A_m$ be finite sets. For each nonempty subset I of $[m]$ define $A_I = \cap_{i \in I} A_i$. Then*

$$|A_1 \cup \cdots \cup A_m| = \sum_{i=1}^{m} \sum_{|I|=i} (-1)^{i+1} |A_I|.$$

Prove this by showing it is a consequence of Möbius inversion on B_m.

3. *Let G be a connected graph. Prove that if λ does not divide $\mu_{L_G}(\hat{0}, G)$, then G has a proper λ-coloring. Prove the converse holds if $\lambda = 2$.*

4. *Fix $n \geq 1$. The* **partition lattice**, *π_n, is the poset whose elements are partitions of $[n]$ and whose order relation is given by refinement. For instance, in π_3 the least element is the partition with three blocks, $\{1\}, \{2\}, \{3\}$, the greatest element is the partition with one block, $\{1, 2, 3\}$, and there are a total of five elements. In general π_n has a unique minimum, $\hat{0} = \{1\}, \ldots, \{n\}$ and a unique maximum $\hat{1} = \{1, \ldots, n\}$. Compute $\mu(\hat{0}, \hat{1})$ in π_n.*

5. *A different approach to the chromatic polynomial is through deletion-contraction. Let G be a graph. It may have parallel edges (more than one edge between a pair of vertices) and/or loops. Define $\chi_G(\lambda)$ as before. If G has a loop, then $\chi(G) \equiv 0$. Let e be an edge of G. The deletion of e, denoted $G - e$, is the graph obtained by removing the edge e. The contraction of G along e, denoted G/e is the graph obtained by contracting the edge down to a vertex and identifying the two vertices of the edge down to one vertex. This may introduce loops and/or parallel edges. For instance, if G is a triangle, then the contraction of G along any of its edges is a graph with two vertices and two parallel edges. Prove that for any edge e of a loopless graph,*

$$\chi_G(\lambda) = \chi_{G-e}(\lambda) - \chi_{G/e}(\lambda).$$

6. *Prove that the coefficients of $\chi_G(\lambda)$ alternate in sign.*

7. *Compute χ_G when G is a tree, a connected graph with no circuits.*

Chapter 7

Hyperplane arrangements and Zonotopes

On the surface, hyperplane arrangements are easy to describe. They are simply a finite collection of hyperplanes in $\mathbb{R}^d$. As noted earlier, any hyperplane $H^{=}_{\mathbf{a},b}$ divides space into three disjoint convex sets, the hyperplane itself and two open half spaces, $H^{+}_{\mathbf{a},b}$ and $H^{-}_{\mathbf{a},b}$. The two full-dimensional sets are called regions. As we add hyperplanes there will be more regions. The main question we will address is, "How many regions are there?" Along the way we will discover a connection between this question, "Does $\chi(-1)$ mean anything", and how many pieces a d-dimensional cake can be cut into using n slices.

7.1 Hyperplane arrangements

A **hyperplane arrangement** in $\mathbb{R}^d$ is a finite collection $\mathcal{A} = \{H_1, \ldots, H_n\}$ of hyperplanes in $\mathbb{R}^d$. The arrangement is called **central** if the hyperplanes are all linear, i.e. all contain the origin. Otherwise $\mathcal{A}$ is called **affine**. If the intersection of all the hyperplanes in a central arrangement is exactly the origin, then it is **essential**.

A central arrangement divides space into various subsets which are called cones.

Definition 7.1.1 *C is a* **cone** *in $\mathbb{R}^d$ if for all $\mathbf{x} \in C$ and $t \geq 0$, $t\mathbf{x} \in C$.*

By definition, the empty set and $\mathbb{R}^d$ are cones in $\mathbb{R}^d$. Examples of cones include closed half-spaces of the form $H_{\mathbf{a},0}, \mathbf{a} \neq \vec{0}$ and their intersections. A cone is **polyhedral** if it is the intersection of a finite number of such half-spaces. In general, the intersection of cones is a cone. As usual, this means that for any subset Y of $\mathbb{R}^d$ there is a smallest cone containing Y which we denote by $co(Y)$. It consists of all of the rays starting at the origin and containing a nonzero point of Y. Hence nonempty cones in $\mathbb{R}^d$ are in one-to-one correspondence with subsets of the unit sphere in $\mathbb{R}^d$.

Let $\mathcal{A} = \{H^=_{\mathbf{a}_1}, \ldots, H^=_{\mathbf{a}_n}\}$ be a central arrangement in $\mathbb{R}^d$. For $\mathbf{x} \in \mathbb{R}^d$ we can record using $\{+, 0, -\}^n$ whether $\mathbf{a}_i \cdot \mathbf{x}$ is positive, zero or negative. The resulting sign vector $s_{\mathcal{A}}(\mathbf{x})$ is in $\{+, 0, -\}^n$. As long as there is not confusion we will suppress the subscript $\mathcal{A}$. Note that $s_{\mathcal{A}}(\mathbf{x})$ depends on the choice of direction of $\mathbf{a}_i$. For a potential sign vector $s \in \{+, 0, -\}^n$ we want to describe the points of $\mathbb{R}^d$ with the given sign vector. For this purpose, define

$$c(s) = \left\{ \mathbf{x} \in \mathbb{R}^d : \begin{cases} \mathbf{a}_i \cdot \mathbf{x} = 0, & s(i) = 0 \\ \mathbf{a}_i \cdot \mathbf{x} \leq 0, & s(i) = - \\ \mathbf{a}_i \cdot \mathbf{x} \geq 0, & s(i) = + \end{cases} \right\}.$$

Figure 7.1 shows a typical example in $\mathbb{R}^2$. The **covectors** of $\mathcal{A}$ are those sign vectors $s \in \{+, 0, -\}^n$ such that

$$\left\{ \mathbf{x} \in \mathbb{R}^d : \begin{cases} \mathbf{a}_i \cdot \mathbf{x} = 0, & s(i) = 0 \\ \mathbf{a}_i \cdot \mathbf{x} < 0, & s(i) = - \\ \mathbf{a}_i \cdot \mathbf{x} > 0, & s(i) = + \end{cases} \right\}$$

is not empty. For instance, for the arrangement in Figure 7.1, $c(+, -, +) = c(+, -, 0) = c(-, 0, 0) = c(0, 0, 0) = \{\vec{0}\}$, but only $(0, 0, 0)$ is a covector of the arrangement. With this notation, part 3 of the next proposition shows us that our original question can be rephrased, "How many covectors are there with no zeros in their sign vector?"

Proposition 7.1.2

1. *$c(s)$ is a closed convex cone.*

2. *Suppose that s and t are covectors of $\mathcal{A}$. Then $c(s) \subseteq c(t)$ if and only if s can be obtained from t by changing some $+$'s and/or $-$'s to zeros and this holds if and only if $c(s)$ is a face of $c(t)$.*

3. *If s is a covector of $\mathcal{A}$, then $\dim c(s) = \dim \cap_{i:s(i)=0} H_i$.*

Problem 49 *Prove the above proposition.*

Looking at Figure 7.1 and Proposition 7.1.2 suggests that the various cones $c(s)$ are arranged in a way somewhat similar to the way polytopal complexes are configured. To make this precise requires the notion of a fan.

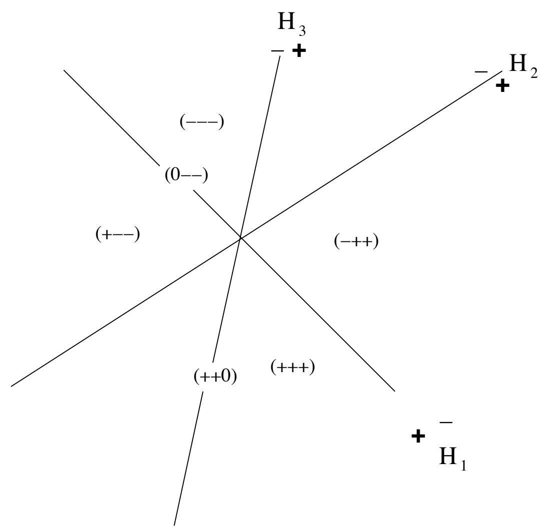

Figure 7.1: Several sign vectors

Definition 7.1.3 *A finite set of cones $\mathcal{C} = \{C_1, \ldots, C_n\}$ in $\mathbb{R}^d$ is a* **fan** *if*

1. *Any nonempty face of a cone in the fan is in the fan.*
2. *For all C_i and C_j in the fan, $C_i \cap C_j$ is a face of both C_i and C_j.*

The fan is called **complete** *if its union is all of $\mathbb{R}^d$.*

Problem 50 *If $\mathcal{A}$ is a central hyperplane arrangement, then the collection of the cones $c(s), s$ a covector of $\mathcal{A}$, is a complete fan.*

A second example of a complete fan is the face fan of a polytope. Let P be a d-polytope in $\mathbb{R}^d$ with the origin in its interior. We form a fan by collecting all subsets of the form $co(F)$, F a proper face of P and then including the origin. See Figure 7.2 for an example of the face fan of a polygon. Since the dimension of $co(F)$ is the dimension of F plus one, the number of cones of dimension $i, 1 \leq i \leq d$ in the face fan of P is just f_{i-1} of P.

Problem 51 *Let $\mathcal{C}$ be the face fan of a d-polytope P with the origin in its interior and let $f_i(\mathcal{C})$ be the number of i-dimensional cones in $\mathcal{C}$. Prove that*

$$\sum_{i=0}^{d} (-1)^i f_i(\mathcal{C}) = (-1)^d.$$

The key to relating face fans to the fans coming from hyperplane arrangements are zonotopes.

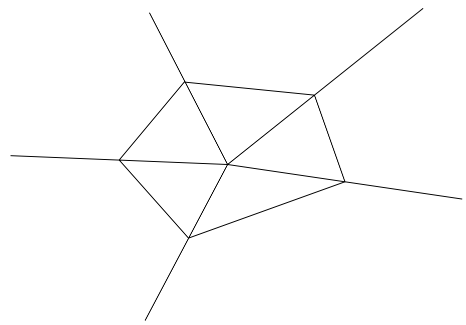

Figure 7.2: Face fan of a polygon

7.2 Zonotopes

An polytope Z in $\mathbb{R}^e$ is a **zonotope** if it is the affine projection of a cube. Specifically, there exists an affine map $f : \mathbb{R}^d \to \mathbb{R}^e$ such that $f(\Box^d) = Z$. Since faces of cubes are translates of cubes, it is not unreasonable to hope that faces of zonotopes are zonotopes.

Problem 52 *Let f be a projection of a polytope P onto a polytope Q. If F is a face of Q, then $f^{-1}(F) \cap P$ is a face of P.*

Corollary 7.2.1 *Faces of zonotopes are zonotopes.*

At first sight it is not apparent how zonotopes might be related to hyperplane arrangements. A hint that there might be a connection comes from an alternative characterization of zonotopes via Minkowski sums.

Definition 7.2.2 *Let X and Y be nonempty subsets of $\mathbb{R}^d$. The* **Minkowski sum** *of X and Y is*

$$X + Y = \{\mathbf{x} + \mathbf{y} : \mathbf{x} \in X, \mathbf{y} \in Y.\}$$

Since addition is commutative and associative Minkowski sums are commutative and associative. Furthermore, it is easy to see that $\{\vec{0}\}$ acts as the identity with respect to Minkowski sums. Several operations we have seen can be described in terms of Minkowski sums. Figure 7.3 suggests how to construct cubes, translate subsets, and form prisms using Minkowski sums.

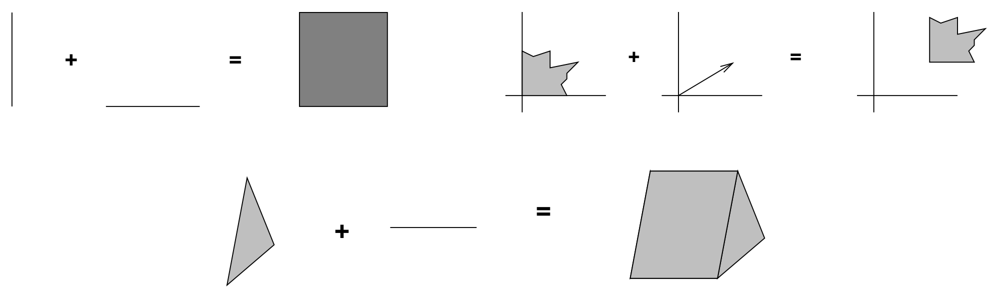

Figure 7.3: Cube, translation and prisms as Minkowski sums

Problem 53 *Z is a zonotope in $\mathbb{R}^d$ if and only if there exists $\mathbf{x}_1, \ldots, \mathbf{x}_n$ in $\mathbb{R}^d$ such that Z can be written as a translation of a Minkowsi sum*

$$Z = [-\mathbf{x}_1, \mathbf{x}_1] + \cdots + [-\mathbf{x}_n, \mathbf{x}_n].$$

(Recall that $[-\mathbf{x}_i, \mathbf{x}_i]$ is the line segment from $-\mathbf{x}_i$ to $\mathbf{x}_i$.)

Now suppose $\mathcal{A} = \{H^{=}_{\mathbf{a}_1,0}, \ldots, H^{=}_{\mathbf{a}_n,0}\}$ is an essential central hyperplane arrangement. Consider the zonotope in $Z_{\mathcal{A}}$ in $(\mathbb{R}^d)^\star$ given by

$$[-\mathbf{a}_1, \mathbf{a}_1] + \cdots + [-\mathbf{a}_n, \mathbf{a}_n].$$

Since $\mathcal{A}$ is essential, $Z_{\mathcal{A}}$ is a d-polytope with the origin in its interior. What does the face fan of $Z^\star_{\mathcal{A}}$ in $(\mathbb{R}^d)^{\star\star} = \mathbb{R}^d$ look like?

Theorem 7.2.3 *The face fan of $Z^\star_{\mathcal{A}}$ equals the fan of the hyperplane arrangement $\mathcal{A}$. Specifically, F is a proper face of $Z_{\mathcal{A}}$ if and only if the smallest cone containing $\Psi(F)$ equals $c(s)$ for some covector $s \neq (0, \ldots, 0)$ of $\mathcal{A}$.*

Proof: We begin by considering the faces of $Z_{\mathcal{A}}$. By Problems 52 and 53 we know that a proper face F of Z is the image of a face of the cube $\square^n$. So there is a partition of $[n]$ into three blocks $[n] = Pl \cup Mi \cup Ze$ such that

$$F = \left\{ \mathbf{a} : \mathbf{a} = \sum_{i=0}^{n} \lambda_i \mathbf{a}_i, \begin{cases} \text{if } i \in Pl, & \text{then } \lambda_i = 1 \\ \text{if } i \in Mi, & \text{then } \lambda_i = -1 \\ \text{if } i \in Ze, & \text{then } -1 \leq \lambda_i \leq 1 \end{cases} \right\}. \tag{7.1}$$

Note that while every face is of this form, it is not true that every partition corresponds to a face.

When is $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ in $\Psi(F)$ (the face in $Z^\star_{\mathcal{A}}$ dual to F)? When $\mathbf{a} \cdot \mathbf{x} = 1$ for all $\mathbf{a} \in F$ and $\mathbf{a} \cdot \mathbf{x} \leq 1$ for all $\mathbf{a}$ in Z. From this we can see that $t\mathbf{x}, t > 0$ is in $co(\Psi(F))$ if and only if $\mathbf{x} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{x}$ is maximized over all $\mathbf{a} \in Z_{\mathcal{A}}$ on F. Looking at F we first notice that this means that $\mathbf{a}_i \cdot \mathbf{x}$ must be zero for all $i \in Ze$. Otherwise, for $\mathbf{a} \in F$ of the form specified in (7.1), $\mathbf{a} \cdot \mathbf{x}$ would not be constant. Similarly, $\mathbf{a}_i \cdot \mathbf{x} \geq 0$ for $i \in Pl$, and $\mathbf{a}_i \cdot \mathbf{x} \leq 0$ for $i \in Mi$. Otherwise $\mathbf{a} \cdot \mathbf{x}$ is not maximized over $Z_{\mathcal{A}}$ on the face F. Thus, each $co(F)$ is equal to $c(s)$ for the covector s which is $+$ for all $i \in Pl, -$ for all $i \in Mi$ and 0 for all $i \in Ze$. Conversely, any nonzero covector s determines F by reversing the above reasoning. □

Corollary 7.2.4 *Let $\mathcal{A}$ be a central arrangement. Let $\mathcal{C}$ be the fan of the arrangement.*

$$\sum_{i=0}^{d} (-1)^i f_i(\mathcal{C}) = (-1)^d.$$

Proof: When $\mathcal{A}$ is essential this follows immediately from Theorem 7.2.3 and Corollary 51.

Problem 54 *Prove the above formula when $\mathcal{A}$ is not essential.*

7.3 The intersection poset

In order to use the above result on the number of cones in the fan of a central hyperplane arrangement we introduce the intersection poset of $\mathcal{A}$. It consists of all intersections of hyperplanes ordered by **reverse** inclusion. This includes the intersection of no hyerplanes, $\mathbb{R}^d$. The intersection poset is denoted $L_{\mathcal{A}}$ and always has a minimal element $\hat{0} = \mathbb{R}^d$, and a maximal element $\hat{1}$ equal to the intersection of all of the hyperplanes. One way to get a feeling for $L_{\mathcal{A}}$ is to look at graphic arrangements.

Let G be a graph without loops or parallel edges, whose vertex set is $[d]$ and edge set is E. For each edge $e = \{i, j\} \in E$ define the hyperplane H_e in $\mathbb{R}^d$ by $x_i - x_j = 0$. The set $\mathcal{A}_G = \{H_e\}_{e \in E}$ of all the H_e is the **graphic arrangement** associated to G. Evidently $\mathcal{A}_G$ is a central arrangement. However, it is never essential (see Exerc. 5).

Problem 55 $L_{\mathcal{A}_G} \simeq L_G$.

The most well known graphic arrangements are the braid arrangements - the graph arrangements associated to complete graphs, K_n. There is a close connection between the fan of the arrangement $\mathcal{A}_{K_n}$ and the permutahedron (see Exerc. 6).

This suggests several properties of intersection posets of all central hyperplane arrangements. A poset is a **lattice** if every pair of elements of the poset have a least upper bound and a greatest lower bound. Suppose L is a lattice. The least upper bound of x and y in L is called their **join** and is denoted by $x \vee y$. The greatest lower bound of x and y is called their **meet** and is denoted $x \wedge y$. The operations of meet and join are commutative and associative. When L is finite we can take the join of all of the elements to see that L has a unique maximal element $\hat{1}$. Similarly, the meet of all the elements of L shows that there is a unique minimal element $\hat{0}$. In any poset which contains a unique minimal element $\hat{0}$ the elements which cover $\hat{0}$ are called the **atoms** of the poset.

Problem 56 *Let $\mathcal{A}$ be a central arrangement in $\mathbb{R}^d$.*

1. *$L_\mathcal{A}$ is graded. The rank of $A \in L_\mathcal{A}$ is $d - \dim A$.*
2. *$L_\mathcal{A}$ is a lattice.*
3. *Let $x, y \in L$. Then $\rho(x) + \rho(y) \geq \rho(x \vee y) + \rho(x \wedge y)$. ($L_\mathcal{A}$ is* **semi-modular.***)*
4. *Every $x > \hat{0}$ in $L_\mathcal{A}$ is the join of atoms of $L_\mathcal{A}$. ($L_\mathcal{A}$ is* **atomic***.)*

Any finite lattice which satisfies the last two properties of the above problem is called a **geometric lattice**.

Problem 57 *Are there any geometric lattices which are not isomorphic to $L_\mathcal{A}$ for some central hyperplane arrangement?*

We are finally ready to prove the promised formula for the number of regions of a central hyperplane arrangement. For $A \in L_\mathcal{A}$ define $f(A)$ to be the number of cones in the fan of $\mathcal{A}$ whose affine span equals A. Thus the number of regions of the hyperplane arrangement equals $f(\hat{0})$.

Theorem 7.3.1 *[19], [11] Let $\mathcal{A}$ be an essential central hyperplane arrangement in $\mathbb{R}^d$. Then*

$$f(\hat{0}) = (-1)^d \chi_{L_\mathcal{A}}(-1).$$

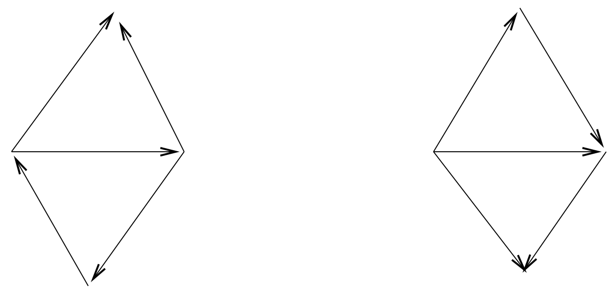

Figure 7.4: Two orientations of a graph, one acyclic

Problem 58 *Prove the above formula.*

What if $\mathcal{A}$ is not essential?

Problem 59 *Let $\mathcal{A}$ be a nonessential central arrangement,* $\dim \cap H_i = n > 0$. *Then there exists an essential central arrangement $\overline{\mathcal{A}}$ in $\mathbb{R}^{d-n}$ such that $L_{\mathcal{A}} \simeq L_{\overline{\mathcal{A}}}$ and the covectors of the two arrangements are identical.*

7.4 Acyclic orientations

At the beginning of the chapter we asked whether or not $\chi_G(-1)$ meant anything. Since $\chi_G(\lambda) = \lambda^{c(G)}\chi_{L_G}(\lambda)$, $\chi_{L_G} = \chi_{L_{\mathcal{A}_G}}$ and Theorem 7.3.1 tells us that the number of regions of the hyperplane arrangement $\mathcal{A}_G$ is $(-1)^{|V(G)|}\chi_{L_{\mathcal{A}_G}}(-1)$, we could say that $\chi_G(-1)$ is $(-1)^{|V(G)|-c(G)}$ times the number of regions of the hyperplane arrangement $\mathcal{A}_G$. This does not seem like a very satisfactory answer. At the very least an interpretation of $\chi_G(-1)$ should involve objects which are 'obviously' graph invariants. In 1973 Richard Stanley showed how to do this via acyclic orientations.

An **orientation** of a graph is a choice of direction for every edge of the graph. If G has n edges, then there are 2^n possible orientations for G. An orientation of G is **acyclic** if there are no *directed* circuits. See Figure 7.4 for examples.

Let $\mathcal{O}$ be an orientation of G. Define a binary relation $\leq_{\mathcal{O}}$ on the vertices of G by $v_1 \leq v_2$ if there is a directed path from v_1 to v_2. The empty path is permitted (as a directed path from v_1 to v_1).

Problem 60 *The orientation $\mathcal{O}$ is acyclic if and only if $\leq_{\mathcal{O}}$ is a partial order on the vertices of G. Conversely, if $(\Pi, \leq)$ is a poset, then there exists a graph G and an acyclic orientation $\mathcal{O}$ on the graph so that $(\Pi, \leq)$ is isomorphic to $(V(G), \leq_{\mathcal{O}})$.*

Stanley's original observation was the following.

Problem 61 *[16] If G is a graph, then the number of acyclic orientations of G is $(-1)^{|V|-c(G)}\chi_G(-1)$.*

Acyclic orientations occur in a variety of contexts. We will see a simple application of them in the next chapter as we examine the graphs of polytopes.

Exercise 7.5

1. *Classify all open cones in $\mathbb{R}^d$.*
2. *Classify all convex cones in $\mathbb{R}^2$.*
3. *An n-gon is a zonotope if and only if n is even.*
4. *The Minkowski sum of two polytopes is a polytope.*
5. *If G is a simple graph, then $\dim \cap_{e\in E} H_e = c(G)$.*
6. *Let $\overline{\mathcal{A}}$ be the arrangement guaranteed by Problem 59 for $\mathcal{A}_{K_n}$. Show that $Z_{\overline{\mathcal{A}}}$ is combinatorially equivalent to a permutahedron.*
7. *Give an example of a lattice L with no maximal element.*
8. *Let L be a finite poset such that $x \wedge y$ exists for all $x, y \in L$. Show that $\hat{L}$, the poset obtained by adding a unique maximal element to L, is a lattice. Does this work if L is infinite?*
9. *Let E be a finite set of vectors in a vector space V. Define L_E to be poset of all subsets of the form $E \cap W$, where W is a linear subspace of V, ordered by inclusion. Prove that L_E is a geometric lattice.*

Chapter 8

Graphs of polytopes

The simplex algorithm optimizes a linear function over a polytope by examining a vertex and then considering the vertices adjacent to that vertex to see if a better choice is available. This requires an understanding of the graph of the polytope - the graph given by the vertices and edges of the polytope. For a polytope P we denote this graph by $G(P)$.

How much information does $G(P)$ tell us about P? A natural question to ask is whether or not $G(P)$ determines the face poset of P. Can we determine which subsets of vertices of the graph correspond to faces of the polytope? Looking at Exercise 1 we can see that $G(P)$ does not even determine the dimension of P! There are at least two situations where $G(P)$ does determine $\mathcal{F}(P)$.

One is when P is three-dimensional. In this case we already know the vertices and edges, so all that remains is to determine which circuits of the graph are the boundaries of facets. If we remove the boundary of a facet from the boundary of the polytope, then the graph will still be connected. However, if we remove a circuit that is not the boundary of a 2-face then the graph will be disconnected. This gives us an algorithm for figuring out which circuits correspond to facets.

The other situation when $G(P)$ determines $\mathcal{F}(P)$ is when P is a simple polytope. A remarkably simple argument due to Kalai shows that $G(P)$ does indeed determine the face poset of P. The acyclic orientations we met at the end of the last chapter play a crucial role in Kalai's proof.

8.1 Acyclic orientations of $G(P)$

The most common method for obtaining an acyclic orientation of the graph of a polytope is by using a generic element of $(\mathbb{R}^d)^\star$.

Definition 8.1.1 *Let P be a polytope in $\mathbb{R}^d$. An element $\mathbf{a} \in (\mathbb{R}^d)^\star$ is* **generic** *for P if for all distinct vertices v_1 and v_2 of P, $\mathbf{a} \cdot v_1 \neq \mathbf{a} \cdot v_2$.*

Let $\mathbf{a}$ be generic for the polytope P. The orientation of the graph of the polytope induced by $\mathbf{a}$ is given by orienting an edge $\{v_1, v_2\}$ so that it points toward v_1 if $\mathbf{a} \cdot v_1 > \mathbf{a} \cdot v_2$ and points toward v_2 otherwise. Since all of the arrows point toward vertices whose dot product with $\mathbf{a}$ is greater, there cannot be any directed circuits and the orientation is acyclic. For this to be useful we need to know that there are generic $\mathbf{a}$.

Problem 62 *Let P be a d-polytope and let $\mathcal{G}_P$ be the linear maps in $(\mathbb{R}^d)^\star$ which are generic for P. Then $\mathcal{G}_P$ is an open subset and has nonvoid intersection with every nonempty open ball.*

Corollary 8.1.2 *Let F be a face of a polytope P and let $V(F)$ be the vertices of F. Then there exists an acyclic orientation $\mathcal{O}$ of $G(P)$ such that the vertices of F form an initial segment of $\leq_{\mathcal{O}}$.*

An **initial segment** of a poset with minimal element $\hat{0}$ is a subset of the poset of the form $[\hat{0}, x]$ for some x in the poset.

Proof: Let $H_{\mathbf{a},b}$ be a closed half-space which shows that F is a face of P. By Problem 62 there exist generic $\mathbf{a}'$ arbitrarily close to $\mathbf{a}$. In the partial order associated to $-\mathbf{a}'$ the vertices of F must come first. □

8.2 Simple polytopes

Most properties of simple polytopes are either self-evident or follow easily by looking at their simplicial duals.

Problem 63 *Let P be a simple d-polytope.*

- *Any k-dimensional face of P is the intersection of exactly $d - k$ facets.*
- *Let v be a vertex of P and $\{e_1, \ldots, e_m\}$ edges incident to v. There exists a unique m-dimensional face of P which contains $\{e_1, \ldots, e_m\}$ and v.*

How can we identify the faces of a simple polytope knowing only the graph of the polytope? The key to Kalai's algorithm are special class of acyclic orientations of $G(P)$. An acyclic orientation of $G(P)$ is **good** if the graph of every nonempty every face of P has a unique sink. The prototypical example of a good orientation is one associated to a generic linear map. Is it possible to know which acyclic orientations $\mathcal{O}$ are good without knowing the faces of P? For an acyclic orientation $\mathcal{O}$ define $h_i^{\mathcal{O}}$ to be the number of vertices with indegree i. Now set

$$f^{\mathcal{O}} = h_0^{\mathcal{O}} + 2h_1^{\mathcal{O}} + \cdots + 2^d h_d^{\mathcal{O}}. \tag{8.1}$$

Problem 64 *Prove that $f^{\mathcal{O}}$ is greater than or equal to the number of faces P. Furthermore, $f^{\mathcal{O}}$ equals the number of nonempty faces of P if and only if $\mathcal{O}$ is a good acyclic orientation.*

Notice that this allows us to determine which acyclic orientations of $G(P)$ are good without knowing the faces of P. We compute $f^{\mathcal{O}}$ for all of them and $\mathcal{O}$ is good if it is a minimizer.

Are there any other properties of the vertices of a face of P that could help us identify them? Every face of a simple polytope is a simple polytope and the graph of any simple polytope is regular. A graph is **regular** if the degree of every vertex is the same.

Theorem 8.2.1 *[10] Let P be a simple polytope. Then a vertex induced subgraph H of $G(P)$ equals $G(F)$ for some face of P if and only if*

- *H is regular.*
- *The vertices of H form an initial segment of $\leq_{\mathcal{O}}$ for some good acyclic orientation of $G(P)$.*

Problem 65 *Prove the above theorem.*

Theorem 8.2.1 gives us an algorithm for determining the face poset of a simple polytope from its graph. However, this is not the end of the story. The first complete proof that $\mathcal{F}(P)$ was determined by $G(P)$ for simple polytopes was due to Blind and Mani [3]. However, it was not constructive. Kalai's algorithm, while elegant and transparent, is exponential in time and it was an open question whether or not a polynomial time algorithm for determining the faces of a simple polytope from its graph exists. In [8] Friedman provided such an algorithm.

8.3 Balinkski's theorem

Another property of polytope graphs is that they are connected. In fact, they are highly connected. A simple graph G is **k-connected** if it has at least k vertices and is connected whenever $(k-1)$ or fewer vertices (and their incident edges) are removed. The following is usually called Balinski's theorem.

Theorem 8.3.1 *[?] Let P be a d-polytope. Then $G(P)$ is d-connected.*

Before we begin the proof we state a lemma that we will need.

Lemma 8.3.2 *Let P be a polytope in $\mathbb{R}^d, \mathbf{a} \in (\mathbb{R}^d)^\star$ and v a vertex of P. Then there exists a path in $G(P)$*

$$v = v_0 \to v_1 \to \cdots \to v_m$$

such that for all $0 \leq i \leq m-1$, $\mathbf{a} \cdot v_i > \mathbf{a} \cdot v_{i+1}$ and $\mathbf{a} \cdot v_m$ minimizes $\mathbf{a} \cdot \mathbf{x}$ for all $\mathbf{x} \in P$.

Problem 66 *Prove this lemma.*

By applying the lemma to $-\mathbf{a}$, the same result holds for paths with increasing $\mathbf{a} \cdot v_i$ instead of decreasing.

Proof (Balinski's theorem): Let S be a subset of the vertices of P with $|S| \leq d-1$. We consider two cases.

Case one: The vertices in S all lie on a proper face F of P. Then $F = P \cap H_{\mathbf{a},b}$ for some hyperplane $H^=_{\mathbf{a},b}$. Let c be the minimum of $\mathbf{a} \cdot \mathbf{x}, \mathbf{x} \in P$. By Exercise 3.5 (6) $F_{min} = P \cap H^=_{\mathbf{a},c}$ is a face of P containing at least one vertex. By Lemma 8.3.2 any vertex not in S is part of a path which ends in F_{min}. Since F_{min} is connected, all vertices not in S are connected by paths which do not go through S.

Case two: The vertices in S do not lie in a proper face of P. Let v be any vertex of P not in S. Consider $S \cup \{v\}$. Since $|S \cup \{v\}| \leq d$, this set lies in at least one hyperplane $H^=_{\mathbf{a},b}$. (More than one if $S \cup \{v\}$ is affinely dependent.) Furthermore, there are points in P on both sides of $H^=_{\mathbf{a},b}$. Otherwise, all of the vertices of S would lie in the proper face $P \cap H^=_{\mathbf{a},b}$. Define

$$H^+ = \{\mathbf{x} \in P : \mathbf{a} \cdot \mathbf{x} \geq b\}, \quad H^- = \{\mathbf{x} \in P : \mathbf{a} \cdot \mathbf{x} \leq b.\}$$

Now apply Lemma 8.3.2 to both H^+ and H^- in the appropriate direction to see that all the vertices of P not in S are connected to v by paths which do not contain any vertices of S. □

An immediate consequence of Balinski's theorem is that the graphs of 3-polytopes are simple, planar and 3-connected. The remarkable fact is that this is a characterization of such graphs.

Theorem 8.3.3 *(Steinitz' theorem) [**?**] A graph is the graph of a* 3*-polytope if and only if it is simple, planar and* 3*-connected.*

Exercise 8.4

1. *Give examples of two polytopes P and Q such that $G(P)$ is isomorphic to $G(Q)$, but* $\dim P \neq \dim Q$.
2. *Prove that $h_i^{\mathcal{O}}$ from Equation (8.1) equals $h_i(P^\star)$.*
3. *Does every good acyclic orientation of a $G(P)$ for a simple polytope come from a generic* $\mathbf{a}$?
4. *Let P be a* 3*-polytope with n facets. Then for any two vertices of P there exists a path of length at most $n - 3$ between them in $G(P)$. (The length of a path is the number of edges in the path.)*

Chapter 9

Hints

9.1 Affine and convex geometry

1. If W is a linear subspace, then $W + \mathbf{v}$ is a *linear* subspace if and only if $\mathbf{v} \in W$ and (hence) $W + \mathbf{v} = W$.

2. Translate.

3. Show by induction on the number of terms in the affine combination that every affine combination is in the affine span.

4. Equate to linear independence.

5. Write in terms of linear maps.

6. Just like Problem 2.

7. The set of all convex combinations of elements of A is convex.

8. Write $\mathbf{y}$ as a convex combination of elements of A with as few terms as possible. If there are more than $d + 1$ terms then there must be an affine dependence among the elements of A in the expression.

9.2 What is a polytope?

9. Compare to Problem 5.

10. This will be easier in the future.

11. $(x_1, \ldots, x_d) \in \diamond^d$ if and only if $|x_1| + \cdots + |x_d| \leq 1$.

12. See 10.

13. If $\mathbf{y}$ is not one of the $\mathbf{x}_i$, then it must lie on a line segment contained in P.

14. You have seen an example.

15. K lies in an affine space of dimension d.

16. Use the previous problem.

17. Use Exerc. 1.2.6 to show that $\gamma(t_{i_1}), \ldots, \gamma(t_{i_{d+1}})$ are affinely independent.

18. For a potential set $\{\gamma(t_{i_1}), \ldots, \gamma(t_{i_d})\}$ of vertices for a facet of $C(n, d)$ use determinants to find a formula for the hyperplane containing those vertices.

9.3 Convexity

19. Compactness for existence. If K is convex and there are two points $\mathbf{x}_1, \mathbf{x}_2$ which minimize the distance to $\mathbf{y}$, consider the triangle $\Delta(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y})$.

20. Consider the line segment $[\mathbf{y}, \mathbf{x}]$ where $\mathbf{y}$ is the projection of $\mathbf{x}$ onto K.

21. For a fixed affine hyperplane H_ϵ there are many possible ways to write $H_\epsilon = H^{=}_{\mathbf{a}_\epsilon, b_\epsilon}$ Try to normalize them so that $\mathbf{a}_\epsilon$ and b_ϵ converge.

22. How do hyperplanes intersect line segments?

23. Induction on m.

24. Induction on dimension.

9.4 Shelling

25. When is a face new?

26. For connected graphs, start with a spanning tree. For the last, order the vertices any way you want, and list the facets lexicographically.

27. Induction on the number of facets.

28. Use the previous problem.

29. What is the effect on the coefficients of a polynomial in t when you multiply by $t - 1$?

30. Put the vertices in general position.

9.5 Duality

35. Mostly straightforward. For instance, if $B_r(\vec{0}) \subseteq K$, then $K^\star$ is contained in $B_{1/r}(\vec{0})$.

36. $\dim \mathbb{R}^d = \dim(\mathbb{R}^d)^\star = \dim(\mathbb{R}^d)^{\star\star} = d$.

37. Suppose that $\mathbf{a}$ is in $K^\star \cap H_{-\mathbf{x}^{\star\star},-1}$ but not in $\Psi(F)$. Then $\mathbf{a}\cdot\mathbf{x} = 1$, and $\mathbf{a}\cdot\mathbf{y} < 1$ for some $\mathbf{y} \in F$.

38. There is an $\mathbf{a}$ in $K^\star$ such that $F = H_{\mathbf{a},1} \cap K$.

39. Rotate the hyperplane which defines F 'correctly'.

40. What are the hyperplanes which define $P^\star$?

41. Problem 39 and Exercise 3.5.3

9.6 Möbius inversion

42. First compute $\mu(\emptyset, [n])$ in B_n.

43. First compute $\mu(1, n)$ in D_n. For this, consider two cases. When n is the product of distinct primes and when n has a factor which is a square of a prime.

44. Induction on dimension. Euler characteristic.

45. For $\nu \in I(\Pi)$ define $\beta(\nu)$ to be the matrix whose i, j entry is $\nu(\beta(i), \beta(j))$.

46. For $f : \Pi \to \mathbb{R}$ define $\beta(f)$ to be the vector in $\mathbb{R}^n$ whose i^{th} coordinate is $f(\beta(i))$. What is $\beta(\zeta) \cdot \beta(f)$? (Notation as in the previous hint.)

47. If $H \subseteq G$ can they have the same number of components?

48. Consider $f : L_G \to \mathbb{R}$, where $f(H)$ is the number of λ-colorings of G such that H is the associated partition-induced subgraph.

9.7 Hyperplane arrangements and zonotopes

49. If s is a covector, then the affine span of $c(s) = \cap_{s(i)=0} H_i$.

50. Proposition 7.1.2.

51. Polytope Euler characteristic.

52. Use the projection $f : P \to Q$ and the $\mathbf{a} \in (\mathbb{R}^d)^\star$ which defines F.

53. A linear map is determined by the image of the unit coordinate vectors.

54. Let $H = \cap H_i$. Consider $\mathcal{A} \cap H$.

55. $\mathbf{x}$ is in H_e, where $e = \{i, j\}$ if and only if the i^{th} and the j^{th} coordinates are equal.

56. Translate to linear algebra.

57. The Fano plane.

58. Corollary 7.2.4.

59. See the hint for Problem 54.

9.8 Graphs of polytopes

62. The union of a finite number of hyperplanes does not contain a nonempty ball.

63. Translate to the simplicial setting.

64. The second item of Problem 63.

65. Suppose H satisfies the conditions of the theorem and $\mathcal{O}$ is the good orientation guaranteed by these conditions. Look at the minimum vertex in H with respect to $\mathcal{O}_{\leq}$. Its incident edges correspond to a face F (Problem 63). Show that $H = G(F)$.

66. What does it mean if for all vertices w which share an edge with $v, \mathbf{a} \cdot w \geq \mathbf{a} \cdot v$?

Chapter 10

Solutions

10.1 Affine and convex geometry

1. Suppose that W is a linear subspace of $\mathbb{R}^d$ and $\mathbf{v}$ is a vector such that $W+\mathbf{v}$ is a linear subspace. Then $\mathbf{v} \in W$ and $W+\mathbf{v} = W$. To see this, note that since $\vec{0} \in W+\mathbf{v}$, we know that $-\mathbf{v}$, and hence $\mathbf{v}$, is in W. As linear subspaces are closed under vector addition, $W+\mathbf{v} \subseteq W$. Similarly, for any $y \in W, y = (y + -\mathbf{v}) + \mathbf{v} \in W + \mathbf{v}$, so $W \subseteq W + \mathbf{v}$.

Now assume that $W + \mathbf{v} = W' + \mathbf{v}'$. Then, $W = W' + (\mathbf{v}' - \mathbf{v})$ and the above discussion shows that $W = W'$.

2. Let $A = \cap_\alpha A_\alpha$ be an intersection of affine subspaces. If A is empty there the result is immediate. So, let $\mathbf{v} \in A$. Each $A_\alpha - \mathbf{v}$ is a linear subspace, hence their intersection W is a linear subspace. But $A = W + \mathbf{v}$.

3. Let Y be the set of all affine combinations of elements of A and let $\mathbf{y} \in Y$. We show by induction on n, the number of terms in the affine combination, that $\mathbf{y} \in aspan(A)$. For $n = 0$ or 1 this is obvious. From Problem 1 we know that $aspan(A) = W + \mathbf{v}$ for any $\mathbf{v} \in aspan(A)$ and W a fixed linear subspace. For the induction step we consider two cases. If $a_n = 1$, then $\mathbf{v} = \mathbf{y} - \mathbf{x}_n + \mathbf{x}_{n-1}$ is in $aspan(A)$ which implies that $\mathbf{y} = \mathbf{v} - \mathbf{x}_{n-1} + \mathbf{x}_n$ is also in $aspan(A)$. When $a_n \neq 1$, the induction hypothesis implies that

$$\mathbf{v} = \frac{1}{1-a_n}(a_1\mathbf{x}_1 + \cdots + a_{n-1}\mathbf{x}_{n-1})$$

is in A which implies that $\mathbf{v} - \mathbf{x}_n$ is in W. So $(1-a_n)(\mathbf{v} - \mathbf{x}_n)$ is also in W. Hence $\mathbf{y} = (1-a_n)(\mathbf{v} - \mathbf{x}_n) + \mathbf{x}_n$ is in $aspan(A)$.

Now that we know that $Y \subseteq aspan(A)$ it remains to show that Y is an affine subspace. Equivalently, $Y - \mathbf{v}$ is a linear subspace for any element of A. However, $Y - \mathbf{v}$ consists of all linear combinations of elements of A whose coefficients sum to zero. As this set of linear combinations is a linear subspace we are done.

4. Let $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$. Suppose that $\mathbf{x}_1 = a_2\mathbf{x}_2 + \cdots + a_n\mathbf{x}_n$ with $a_2 + \cdots + a_n = 1$. Then $a_2(\mathbf{x}_2 - \mathbf{x}_1) + \cdots + a_n(\mathbf{x}_n - \mathbf{x}_1)$ is a nontrivial linear dependence in $aspan(X) - \mathbf{x}_1$. So, the dimension of X is less than $n - 1$.

Conversely, if $\dim X < n - 1$, then there exists $\mathbf{x}_2, \ldots, \mathbf{x}_n$ in X such that $\mathbf{x}_2 - \mathbf{x}_1, \ldots, \mathbf{x}_n - \mathbf{x}_1$ have a nontrivial linear independence relation. If the sum of the coefficients of this relation is zero, then one of $\mathbf{x}_2, \ldots, \mathbf{x}_n$ can be written as an affine combination of the others. If the coefficients do not sum to zero, then by first multiplying the linear relation by the reciprocal of the sum of the coefficients we can reverse the process of the first part to write $\mathbf{x}_1$ as an affine combination of $\mathbf{x}_2, \ldots, \mathbf{x}_n$.

5. Write $f = T + \mathbf{v}$ with $T : \mathbb{R}^d \to \mathbb{R}^e$ a linear map. Then $f(A) = f(W + \mathbf{v}) = f(W) + f(\mathbf{v})$ which is an affine subspace.

Now suppose that B is an affine subspace of $\mathbb{R}^3$. If $f^{-1}(B)$ is empty then by definition it is affine. Suppose that $B = \mathbf{x}$ is a single point and $f^{-1}(B)$ is not empty. Then $f^{-1}(B) = f^{-1}(\mathbf{x}) = T^{-1}(\mathbf{x} - \mathbf{v}) = W + \mathbf{y}$, where $\mathbf{y}$ is any element of $T^{-1}(\mathbf{x} - \mathbf{v})$ and W is the kernel of T. So, in this case $f^{-1}(B)$ is affine. For the general case, write $B = U + \mathbf{z}$, so $f^{-1}(B) = f^{-1}(U + \mathbf{z}) = T^{-1}(U + \mathbf{z} - \mathbf{v}) = T^{-1}(U) + T^{-1}(\mathbf{z} - \mathbf{v})$ which is an affine subspace.

6. Let $C = \bigcap_{A \in \mathcal{A}} A$ and let $x, y \in C$. Since each $A \in \mathcal{A}$ is convex, $[x, y] \subseteq A$ for every $A \in \mathcal{A}$. Hence $[x, y] \subseteq C$.

7. If A is empty there is nothing to prove. Set C to be the set of all convex combinations of elements of A. First we show that $C \subseteq ch(A)$. Let $\mathbf{x} \in C$ and write it as a convex combination

$$\mathbf{x} = a_1\mathbf{x}_1 + \cdots + a_n\mathbf{x}_n$$

of elements of A. If $n = 1$ or 2, then by definition $\mathbf{x}$ is in every convex set containing A and hence is in $ch(A)$. For larger n we proceed by induction by observing that

$$\mathbf{x} = (1 - a_n)\left[\frac{1}{1 - a_n}(a_1\mathbf{x}_1 + \cdots + a_{n-1}\mathbf{x}_{n-1})\right] + a_n\mathbf{x}_n.$$

To see that $ch(A) \subseteq C$ it is sufficient to show that C is convex. Let $\mathbf{y}$ and $\mathbf{z}$ be elements of C. By setting several coefficients to zero if necessary, we can write both $\mathbf{y}$ and $\mathbf{z}$ as convex combinations of the same finite number of elements of A.

$$\mathbf{y} = \sum_{i=1}^{n} a_i\mathbf{x}_i, \qquad \mathbf{z} = \sum_{i=1}^{n} b_i\mathbf{x}_i.$$

Hence for any $0 \leq t \leq 1$,

$$(1-t)\mathbf{y} + t\mathbf{z} = \sum_{i=1}^{n} [(1-t)a_i + tb_i]\mathbf{x}_i,$$

is a convex combination of elements of A.

8. Write $\mathbf{y}$ as a convex combination of the elements of A with as few terms as possible. We want to show a contradiction if there are $d+2$ or more terms. So, suppose

$$\mathbf{y} = a_1\mathbf{x}_1 + \cdots + a_j\mathbf{x}_j$$

is a minimal expression with $j \geq d+2$. Since $\mathbb{R}^d$ is d-dimensional $\mathbf{x}_1, \ldots, \mathbf{x}_j$ must be affinely dependent. Let

$$b_1\mathbf{x}_1 + \cdots + b_j\mathbf{x}_j = 0$$

be an affine dependence. Without loss of generality we can assume that $b_j > 0$ and among all b_i with $b_i > 0$, $\frac{a_j}{b_j} \leq \frac{a_i}{b_i}$. For $1 \leq i \leq j-1$ set $c_i = a_i - \frac{a_j}{b_j} b_i$. Now check that $\sum_{i=1}^{j-1} c_i = 1, c_i \geq 0$ and

$$\mathbf{y} = c_1\mathbf{x}_1 + \cdots + c_{j-1}\mathbf{x}_{j-1}.$$

This gives our desired contradiction.

10.2 What is a polytope?

9. Yes. The affine image of convex combinations is a convex combination. So, $f(P)$ is the convex hull of $f(\mathbf{x}_1), \ldots, f(\mathbf{x}_n)$, where P is the convex hull of $\mathbf{x}_1, \ldots, \mathbf{x}_n$. However, if f is a projection of $\mathbb{R}^2$ onto $\mathbb{R}^1$, then the inverse image of any set is unbounded. In this case $f^{-1}(P)$ can not be a $\mathcal{V}$-polytope.

10. As we will see in Chapter 3 the answer is yes.

11. The d-cube is the intersection of all closed half-spaces of the form $x_i \leq 1, x_i \geq -1$. The d-crosspolytope is the intersection of all closed half-spaces of the form

$$\varepsilon_1 x_1 + \cdots + \varepsilon_d x_d \leq 1,$$

where $\varepsilon_i = \pm 1$. The d-simplex is the intersection of $x_1 + \cdots + x_d \leq 1, x_1 + \cdots + x_d \geq 1$ and all inequalities of the form $x_i \geq 0$ and $x_i \leq 1$. This is definitely not the most efficient way of presenting Δ^d as an $\mathcal{H}$-polytope.

12. See 10.

13. For our previous work we know that any element $\mathbf{y}$ of P is a convex combination of $\mathbf{x}_1, \ldots, \mathbf{x}_n$. If $\mathbf{v}$ is not one of the $\mathbf{x}_i$, then it must lie on a line segment ℓ which is completely contained in P. We know from Exerc. 1.2.4 that the intersection of any hyperplane $H_{\mathbf{a},b}$ with ℓ must be empty, all of ℓ or just one point. In the last case the line segment must be on both sides of the hyperplane. But this makes it impossible for $\mathbf{y}$ to be a vertex of P.

14. The upper left hand corner of Figure 1.1

15. Let A be the affine span of K. By definition $\dim A = \dim K$. Since A is an affine subspace it is equal to $W + \mathbf{v}$ where $\dim W = \dim K$. Let $f : \mathbb{R}^{\dim K} \to W$ be any linear isomorphism. Then $\phi(\mathbf{x}) = f(\mathbf{x}) + \mathbf{v}$ is an affine map which is a bijection on $K' = \phi^{-1}(K)$.

16. By the previous problem we may assume that P and Q both lie in $\mathbb{R}^d$ and the affine map given by the definition of affine equivalence is an affine isomorphism. Hence it maps hyperplanes to hyperplanes. Now it is easy to see that it maps faces of P to faces of Q and vice versa.

17. It is sufficient to show that any collection of $d+1$ distinct $\gamma_(t_{i_1}), \ldots, \gamma(t_{i_{d+1}})$ are affinely independent. (Why? Is it necessary?) To do this we apply Exerc. 1.2.6. For notational convenience we use $t_1, \ldots, t_{d+1}$. Consider the $(d+1) \times (d+1)$ matrix

$$\begin{bmatrix} 1 & 1 & \ldots & 1 & 1 \\ t_1 & t_2 & \ldots & t_d & t_{d+1} \\ t_1^2 & t_2^2 & \ldots & t_d^2 & t_{d+1}^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t_1^{d-1} & t_2^{d-1} & \ldots & t_d^{d-1} & t_{d+1}^{d-1} \\ t_1^d & t_2^d & \ldots & t_d^d & t_{d+1}^d \end{bmatrix}$$

This matrix is usually called the Vandermonde matrix and its determinant is called the Vandermonde determinant and it is

$$\prod_{1 \le i < j \le d+1}^{d+1} t_j - t_i.$$

To see this, proceed by induction and perform elementary row operations to show that the above determinant is equal to the determinant of

$$\begin{bmatrix} 1 & 1 & \ldots & 1 & 1 \\ 0 & t_2 - t_1 & \ldots & t_d - t_1 & t_{d+1} - t_1 \\ 0 & t_2(t_2 - t_1) & \ldots & t_d(t_d - t_1) & t_{d+1}(t_{d+1} - t_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & t_2^{d-2}(t_2 - t_1) & \ldots & t_d^{d-2}(t_d - t_1) & t_{d+1}^{d-2}(t_{d+1} - t_1) \\ 0 & t_2^{d-1}(t_2 - t_1) & \ldots & t_d^{d-1}(t_d - t_1) & t_{d+1}^{d-1}(t_{d+1} - t_1) \end{bmatrix}.$$

18. Let $A = \{\gamma(t_{i_1}), \ldots, \gamma(t_{i_d})\}$ be the vertices of a potential facet $F = ch(A)$ of $C(n,d)$. For $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ define

$$\delta(\mathbf{x}) = \det \begin{bmatrix} 1 & 1 & \ldots & 1 & 1 \\ x_1 & t_{i_1} & \ldots & t_{i_{d-1}} & t_{i_d} \\ x_2 & t_{i_1}^2 & \ldots & t_{i_{d-1}}^2 & t_{i_d}^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{d-1} & t_{i_1}^{d-1} & \ldots & t_{i_{d-1}}^{d-1} & t_{i_d}^{d-1} \\ x_d & t_{i_1}^d & \ldots & t_{i_{d-1}}^d & t_{i_d}^d \end{bmatrix}.$$

The theory of determinants (for instance, Laplace expansion) tells us that $\delta(\mathbf{x}) = 0$ is a hyperplane H in $\mathbb{R}^d$. It also tells us that $\gamma(t_{i_1}), \ldots, \gamma(t_{i_d})$ are all contained in H. Hence, H is the hyperplane determined by the γ_{i_j}. In addition, we now know that F is a facet of $C(n,d)$ if and only if for all $t_j \notin A$ the signs of $\delta(\gamma(t_j))$ are all the same.

Now consider the function $p(t) = \delta(\gamma(t))$. This is a polynomial of degree at most d. On the other hand, $t_{i_1}, \ldots, t_{i_d}$ are d-distinct real roots of the polynomial. Therefore they are the only roots and the sign of $\delta(t)$ changes between each root. This implies that the number of vertices in A between any two vertices not in A must be even.

10.3 Convexity

19. Let $\mathbf{z}$ any element of K and set $r = ||\mathbf{z} - \mathbf{y}||$. Since $f(\mathbf{v}) = ||\mathbf{v} - \mathbf{y}||$ is a continuous function and the closed ball $\overline{B}_r(\mathbf{y})$ intersected with K is compact, there exists at least one point $\mathbf{x}$ in K which minimizes the distance to $\mathbf{y}$. To show that $\mathbf{x}$ is unique when K is convex, suppose there are two such points, $\mathbf{x}_1$ and $\mathbf{x}_2$. Consider the triangle $\Delta(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y})$. Since K is convex the line segment $[\mathbf{x}_1, \mathbf{x}_2] \subseteq K$. However, elementary Euclidean geometry tells us that the distance from the midpoint of the line segment to $\mathbf{y}$ is strictly less than the distance to either $\mathbf{x}_1$ or $\mathbf{x}_2$. Contradiction.

20. Let $\mathbf{y}$ the projection of $\mathbf{x}$ onto K. The hyperplane H which is the perpendicular bisector of the line segment $[\mathbf{y}, \mathbf{x}]$ separates $\mathbf{x}$ and K. Indeed, if $\mathbf{z} \in H \cap K$, then so is $[\mathbf{z}, \mathbf{x}]$ and by elementary Euclidean geometry there would exist $\mathbf{v} \in K$ closer to $\mathbf{x}$ than $\mathbf{y}$.

21 Since $\mathbf{0}$ is in the interior of K there exists a ball $B_r(\mathbf{0})$ contained in K. To see that $\mathbf{x} \notin (1 - \epsilon K)$ suppose it is. Then $\mathbf{y} = 1/(1-\epsilon)\mathbf{x}$ is in K and hence $\mathbf{x} \in [\mathbf{0}, \mathbf{y}]$. But this implies that $\mathbf{x}$ is in the interior of the convex hull of $\mathbf{y} \cup B_r(\mathbf{0})$ which is contained in K, contrary to the assumption that $\mathbf{x}$ is on the boundary of K.

Now, for each $\epsilon > 0$ let $H_\epsilon^=$ be a hyperplane which separates $(1-\epsilon)K$ and $\mathbf{x}$. Write each $H_\epsilon^= = H_{\mathbf{a}_\epsilon,b_\epsilon}^=$, where we normalize so that $\mathbf{a}_\epsilon$ is a unit vector pointing away from K (this determines b_ϵ). The unit sphere in $\mathbb{R}^d$ is compact, so there exists a subsequence of the $\mathbf{a}_\epsilon$ which converge to a limit $\mathbf{a}$. Set $b = \mathbf{a}\cdot\mathbf{x}$. By definition $\mathbf{x} \in H_{\mathbf{a},b}^= \cap K$, and an easy limit argument shows that $H_{\mathbf{a},b}$ is a valid inequality for K.

22 First observe that the intersection of any hyperplane and a line segment is either empty, the whole line segment, or a single point.

Now, let $\mathbf{x}$ be a vertex of K. Since $\mathbf{x}$ is a vertex, $\mathbf{x} = K \cap H_{\mathbf{a},,b}^=$, where $H_{\mathbf{a},b}$ is a valid inequality for K. If $\mathbf{x}$ is not an extreme point, then it is in the interior of a line segment $[\mathbf{y},\mathbf{x}] \subseteq K$. By the above observation $H_{\mathbf{a},b}^= \cap [\mathbf{y},\mathbf{z}] = \mathbf{x}$. But this is impossible since $H_{\mathbf{a},b}$ is a valid inequality for K.

To see that $\operatorname{ext} K \subseteq A$ whenever $K = ch(A)$, it is enough to show that if $\mathbf{x} = a_1\mathbf{x}_1 + \cdots + a_n\mathbf{x}_n,\ n > 1$ is a convex combination with distinct $\mathbf{x}_i$ and all a_i nonzero, then $\mathbf{x}$ is in the interior of a line segment contained in $ch(\mathbf{x}_1,\ldots,\mathbf{x}_n)$. This is easily established by induction on n.

For the last item, it is immediate that $F \cap \operatorname{ext} K \subseteq \operatorname{ext} F$. For the reverse containment, suppose $\mathbf{x} \in \operatorname{ext} F$ but is not an extreme point of K. Then $\mathbf{x}$ is in the relative interior of a line segment $[\mathbf{y},\mathbf{z}]$ in K. Since F is a face of K, there exists a valid inequality $H_{\mathbf{a},b}$ for K such that $F = K \cap H_{\mathbf{a},b}^=$. Once again, consideration of $[\mathbf{y},\mathbf{z}] \cap H_{\mathbf{a},b}$ shows that either $H_{\mathbf{a},b}$ is not a valid inequality, or $[\mathbf{y},\mathbf{z}] \subseteq F$, contrary to the assumption that $\mathbf{x}$ is an extreme point of F.

23

24 We show that if A is the union $A_1 \cup \cdots \cup A_n$ of a finite number of affine subspaces of $\mathbb{R}^d$ each of which is of dimension $(d-1)$ or less, then A does not contain any nonempty ball $B_r(\mathbf{x})$. The proof is by induction on d, where the initial case $d = 1$ is obvious.

For the induction step, suppose $B_r(\mathbf{x}) \subseteq A$. Let H be any affine hyperplane which contains $\mathbf{x}$. For each A_i either $H \cap A_i$ is an affine subspace of dimension at most $d-2$, the empty set, or all of H. As there are only finitely many A_i and infinitely many potential hyperplanes H, we can eliminate the last possibility by a judicious choice of H. But now the induction hypothesis implies that $B_r(\mathbf{x}) \cap H \not\subseteq (A \cap H)$.

10.4 Shelling

25. Let G be a face of F_j which is not contained in the union of the previous facets. Then each v_{j_k} must be a vertex of G, otherwise $G \subseteq F_{j_k}$. Hence every new face

contains M_j. Conversely, any face containing M_j was not in $\cup_{i=0}^{j-1} F_i$.

26.

27. We proceed by induction t, where $F_1, \ldots, F_t$ is a shelling order on the facets of Δ. If $t = 1$, then $h_\Delta(t) = t^d$ and $f_\Delta(t) = t^d + \binom{d}{1} t^{d-1} + \cdots + \binom{d}{d-1} t + 1 = (t+1)^d = h_\Delta(t+1)$. For the induction step, let M_j be the minimal new face, $|M_j| = i$. Set $\Delta_s = F_1 \cup \cdots \cup F_s$. The main point is that $f_k(\Delta_j) = f_k(\Delta_{j-1})$ plus the number of new k-faces. How many of these are there? In dimension $i-1$ there is one - M_j. In dimension i there are $\binom{d-i}{1}$ - those faces of F_j which contain the i vertices of M_j and one other. We now see that in dimension $i-1+k$ there are $\binom{d-i+k}{k}$ new faces. Thus, for $0 \le k \le d-i$,

$$f_{i-1+k}(\Delta_j) = f_{i-1+k}(\Delta_{j-1}) + \binom{d-i+k}{k}.$$

In terms of f_Δ this means that $f_{\Delta_j}(t) = f_{\Delta_{j-1}}(t) + (t+1)^{d-i}$. On the other hand, by definition, $h_{\Delta_j}(t) = h_{\Delta_j -1}(t) + t^{d-i}$. Hence, by the inductive hypothesis,

$$f_{\Delta_j}(t) = f_{\Delta_{j-1}}(t) + (t+1)^{d-i} = h_{\Delta_{j-1}}(t+1) + (t+1)^{d-i} = h_{\Delta_j}(t+1).$$

Appendix A

Analysis

For the sake of completeness we include the basic facts from analysis that we need. More detailed information can be found in any number of textbooks.

A.1 Compactness

The notions of open, closed, bounded and compact sets are some of the most fundamental ideas behind understanding the geometry of $\mathbb{R}^d$.

Definition A.1.1 *$A \subseteq \mathbb{R}^d$ is* **open** *if for all $\mathbf{x} \in A$, there exists an open ball centered at $\mathbf{x}$, $B_r(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^d : ||\mathbf{x} - \mathbf{y}|| < r\}, r > 0$, which is contained in A.*

Examples of open subsets of $\mathbb{R}^2$ are open balls, the empty set, all of A$\mathbb{R}^2$ and the interior of any polygon. A line segment and a polygon are not open. The union of open sets is always open.

Definition A.1.2 *A subset A of $\mathbb{R}^d$ is* **closed** *if it is the complement of an open set.*

Polygons, affine subspaces, any finite number of points and $\{(0, 1/n) : n \in \mathbb{Z}^+\} \cup \{(0, 0)\}$ are examples of closed subsets of $\mathbb{R}^2$. The intersection of closed sets is closed. As usual, this means there is a smallest closed set containing a given set A. It is called the **closure** of A. For instance, the closure of the rationals in $\mathbb{R}$ is all of $\mathbb{R}$.

A **bounded** subset of $\mathbb{R}^d$ is any subset which is contained in some ball $B_r(\mathbf{0})$. Compact sets form one of the most well-behaved classes of subsets of $\mathbb{R}^d$.

Definition A.1.3 *$A \subseteq \mathbb{R}^d$ is* **compact** *if it is closed and bounded.*

Two of the most useful properties of compact sets are described by the following theorem.

Theorem A.1.4 *Let C be a nonempty compact subset of $\mathbb{R}^d$.*

- *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous function. Then there exists $\mathbf{x} \in C$ which maximizes f on C. Similarly, there exists $\mathbf{y} \in C$ which minimizes f on C.*
- *Let $(\mathbf{x}_1, \mathbf{x}_2, \dots)$ be an infinite sequence in C. Then there exists a subsequence $(\mathbf{x}_{i_1}, \mathbf{x}_{i_2}, \dots)$ which converges to some $\mathbf{x} \in C$.*

The hypotheses that C be closed and bounded are both necessary.

Exercise A.2

1. *Polytopes are compact.*
2. *Construct a nonempty subset $A_c \subseteq \mathbb{R}^d$ which is closed and a continuous function $f : \mathbb{R}^d \to \mathbb{R}$ such that A_c does not contain a point which maximizes f on A_c.*
3. *Construct a nonempty subset $A_b \subseteq \mathbb{R}^d$ which is bounded and a continuous function $f : \mathbb{R}^d \to \mathbb{R}$ such that A_b does not contain a point which maximizes f on A_b.*
4. *Construct a nonempty subset $A_c \subseteq \mathbb{R}^d$ which is closed and an infinite sequence of elements of A_c which does not contain a convergent subsequence.*
5. *Construct a nonempty subset $A_b \subseteq \mathbb{R}^d$ which is bounded and an infinite sequence of elements of A_b which does not contain a convergent subsequence.*

Intuitively it seems obvious that the proper faces of a convex set lie on its 'boundary'. To make this precise requires we define what the boundary is!

Definition A.2.1 *The* **boundary** *of a subset A of $\mathbb{R}^d$ consist of all points $\mathbf{x} \in \mathbb{R}^d$ such that any nonempty ball $B_r(\mathbf{x})$ contains points of A and points not in A.*

The boundary of A is denoted by ∂A. Note that it is not required that the points in the boundary of a set are actually in the set. For instance, the boundary of the half-open interval $(0, 1]$ in $\mathbb{R}$ is $\{0, 1\}$. One of many properties of the boundary of a set is the following.

Complementary to the boundary of a set is its interior. A point $\mathbf{x}$ in A is an **interior** point of A if there exists a nonempty ball $B_r(\mathbf{x})$ contained in A. Any closed set is the (disjoint) union of its interior and boundary.

Proposition A.2.2 $\mathbf{x}$ *is in the boundary of* A *if and only if there exists a sequence of points in* A *which converge to* $\mathbf{x}$*, and there exists a sequence of points in the complement of* A *which converge to* $\mathbf{x}$.

Bibliography

[1] L. Billera and C. Lee, *A proof of the sufficiency of McMullen's conditions for f-vectors of simplicial polytopes*, J. Comb. Theory, Ser. A, **31** (1981), 227–255.

[2] A. Björner, *The unimodality conjecture for convex polytopes*, Bull. Amer. Math. Soc., **4** (1981), 187–188.

[3] R. Blind and P. Mani-Levitska, *On puzzles and polytope isomorphism*, Aeq. Math., **34** (1987), 287–297.

[4] H. Bruggesser and P. Mani, *Shellable decompositions of cells and spheres*, Math. Scan., **29** (1971), 197–205.

[5] G. Dantzig, *Linear Programming and Extensions*, Princeton University Press, Princeton 1963.

[6] M. Dehn, *Die Euersche Formel im Zusammenhang mit dem Inhalt in der nicht-Euklidischen Geometrie,* Math. Ann., **61** (1905), 561–568.

[7] J. Eckhoff, *Normat*, **54** (2006), 146–159.

[8] E. Friedman, *Finding a simple polytope from its graph in polynomial time*, Disc. Comp. Geom., **41** (2009), 249–256.

[9] B. Grünbaum, *Convex Polytopes* 2nd. ed., Springer-Verlag, New York, 2003.

[10] G. Kalai, *A simple way to tell a simple polytope from its graph*, J. Comb. Theory Ser. A, **49** (1988), 381–383.

[11] M. Las Vergnas, *Convexity in oriented matroids*, J. Comb. Theory, Ser. A, **29** (1980), 231–243.

[12] C. Lee, *Counting the faces of simplicial polytopes*, Ph.D. thesis, Cornell University, 1981.

[13] G.C. Rota, *On the foundations of combinatorial theory I. Theory of Möbius functions*, Z. Wahrscheinlichkeitstheorie 2 (1964), 340–368.

[14] M. Rudin, *An shellable triangulation of a tetrahedron*, Bull. Amer. Math. Soc., **64** (1958), 90–91.

[15] D. Sommerville, *The relations connecting the angle-sums and volume of a polytope in space of n dimensions.* Proc. Royal Soc. London, Ser. A, **115** (1927), 103–119.

[16] R. Stanley, *Acyclic orientations of graphs*, Disc. Math., **5** (1973), 171–178.

[17] R. Stanley, *The number of faces of a simplicial convex polytope*, Adv. Math., **35** (1980), 236–238.

[18] L. Weisner, *Abstract theory of inversion of finite series*, Trans. Amer. Math. Soc., **38** (1935), 474–484.

[19] T. Zaslavsky, *facing up to arrangements: face-count formulas for partitions of space by hyperplanes*, Mem. Amer. Math. Soc., **1 (1)** no. 154 (1975).