Orthogonal Polynomials on the Sierpinski Gasket

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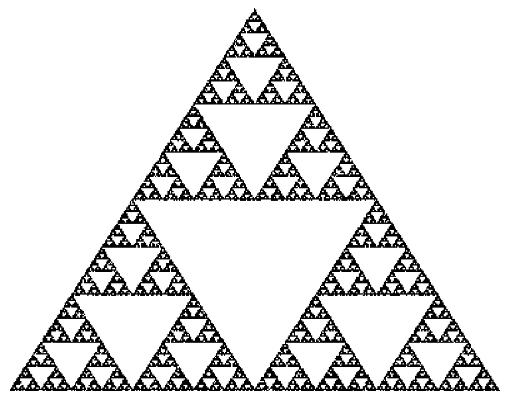
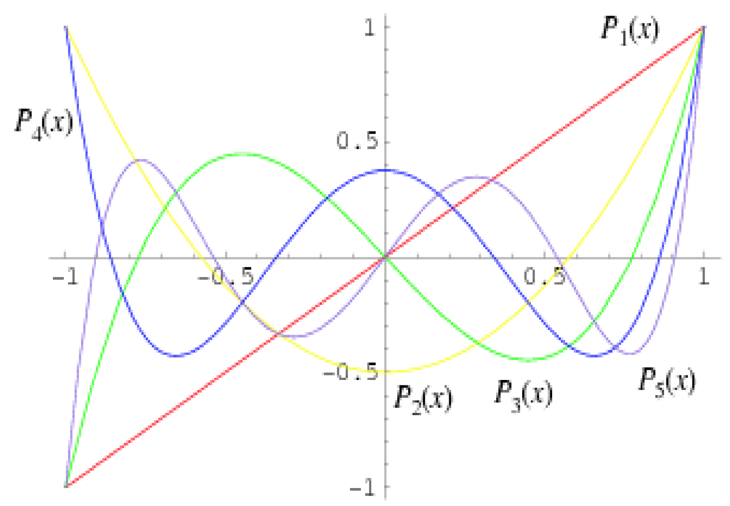
1. Analysis on Fractals

This is part of an ongoing effort to extend classical analysis to fractals, specifically the Sierpinski Gasket (SG).





Examples of orthogonal polynomials on the unit interval are the Legendre Polynomials, we seek their analogs on the SG.

2. Polynomials on SG

Let $u(x)$ be a function defined on the set of all vertices in the SG. The graph Laplacian on the m[th] level graph approximation of SG is:

$$\Delta_m u(x) = \sum_{y \tilde{m} x} (u(y) - u(x)) \qquad \text{for } x \epsilon V_m \backslash V_0$$

The Laplacian is defined as the renormalized limit

$$\Delta u(x) = \lim_{m\to\infty} \frac{3}{2} 5^m \Delta_m u(x)$$

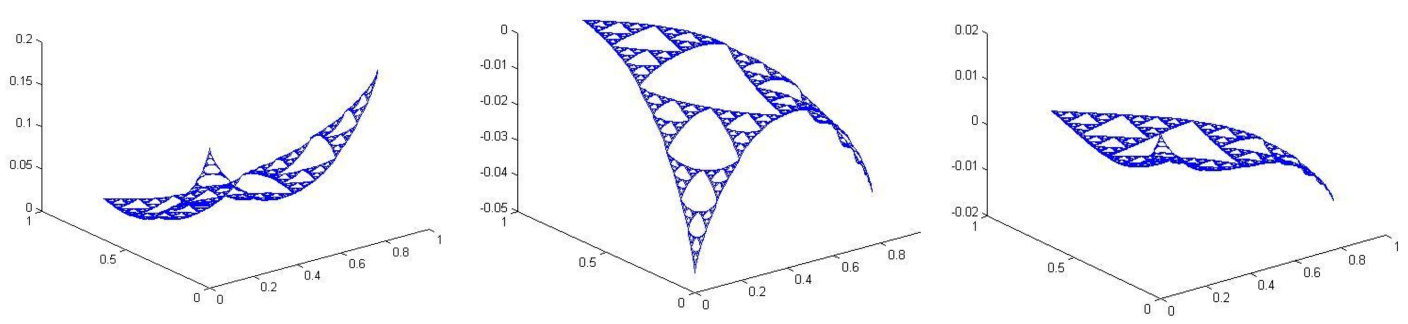
Fix a boundary point q_l. The polynomials $P_{jk}^{(l)}$ for k=1,2,3, and j=0,1,2,.. Are the functions satisfying:

$$\Delta^m P_{jk}{}^{(l)}(q_l) = \delta_{mj}\delta_{k1}$$

$$\partial_n \Delta^m P_{jk}{}^{(l)}(q_l) = \delta_{mj}\delta_{k2}$$

$$\partial_T \Delta^m P_{jk}{}^{(l)}(q_l) = \delta_{mj}\delta_{k3}$$

Fix the boundary point q_0, the polynomials $P_{1k}^{(0)}$ are:



$P_{11}^{(0)}$ $P_{12}^{(0)}$ $P_{13}^{(0)}$

These can be separated into two types: symmetric and anti-symmetric.

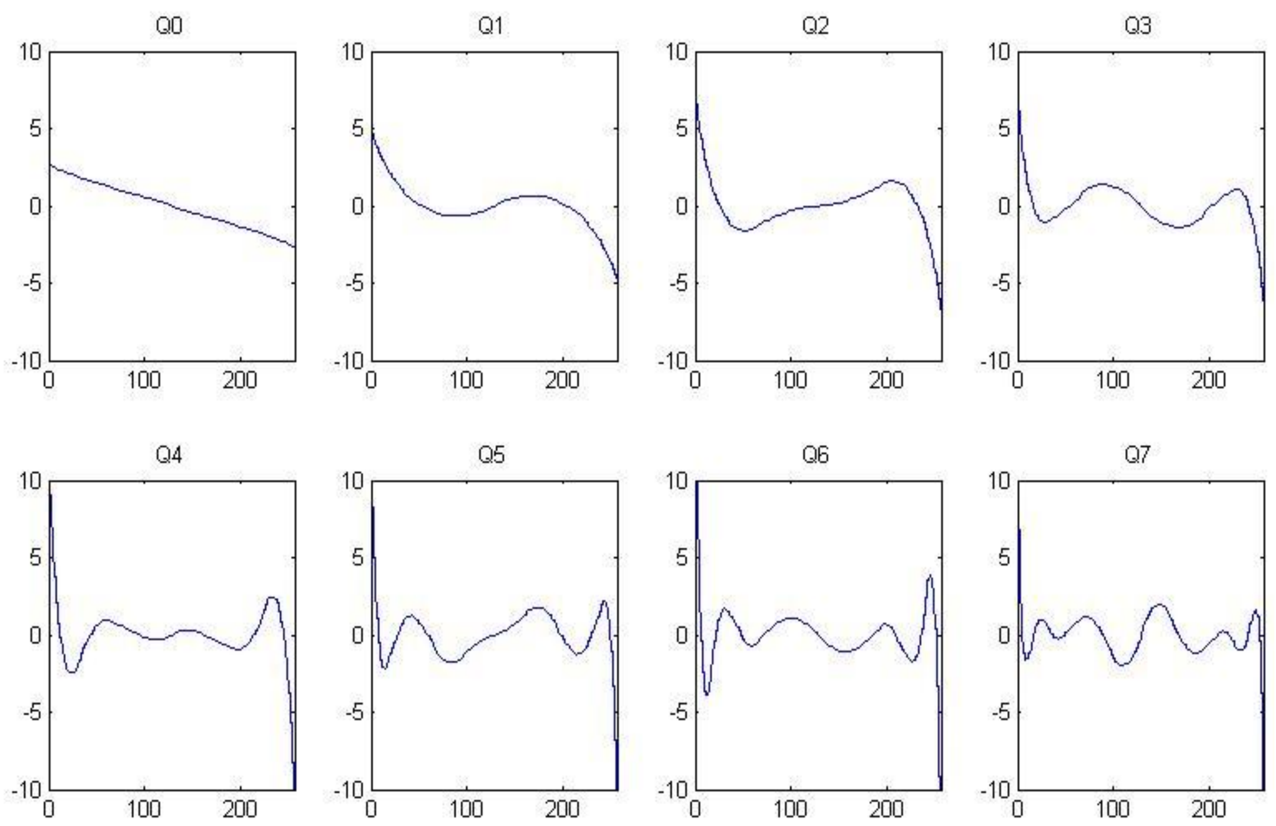
3. Gram- Schmidt

Using the Gauss-Green formula, we can define the inner product as a sum of boundary terms:

$$\int P_{ja} P_{kb} = \sum_{l=0}^{j} \sum_{i=0}^{2} P_{ja}(q_i)\partial_n P_{kb}(q_i) - P_{ja}(q_i)\partial_n P_{kb}(q_i)$$

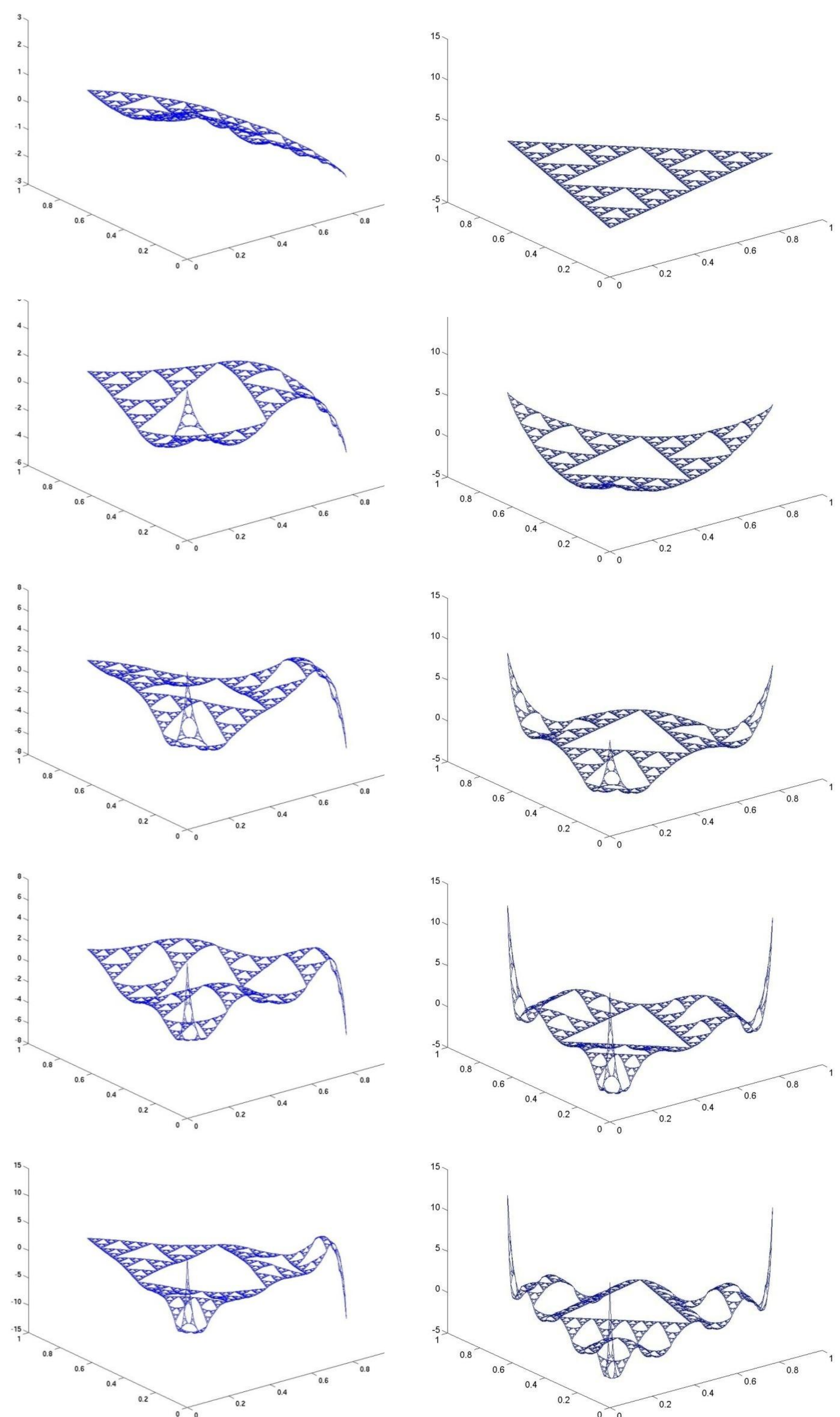
Now that we have an inner product, we can perform Gram-Schmidt on the polynomials.

Restriction to the edge



Result 1: Rotations of the Anti-Symmetric Orthogonal Polynomials form a tight frame for its span.

Anti-symmetric Orthogonal Polynomials | **Fully -Symmetric Orthogonal Polynomials**

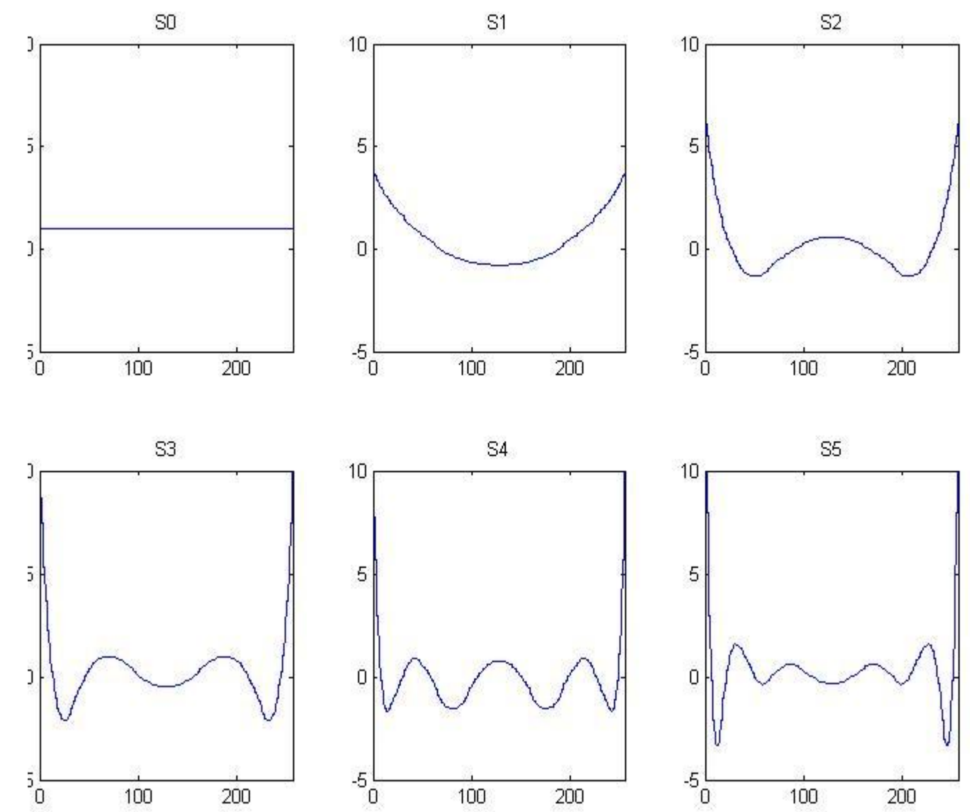


Now we want to define a fully symmetric polynomial, and perform Gram-Schmidt on that.

$$\rho_j = P_{j,1}^{(0)} + P_{j,1}^{(1)} + P_{j,1}^{(2)}$$

Gram-Schmidt is an unstable algorithm, and the inner products include small numbers, so there is a lot of noise accumulation.

Restriction to the edge



Result 2: Rotations of the Anti-Symmetric orthogonal polynomials added to the Symmetric orthogonal polynomials are an orthogonal basis.

4. Recursion Relation

There is a three-term recursion relation for finding the orthogonal polynomials.

$$Q_{k+1}(x) = \frac{1}{a_k}\left(\int G(x,y)Q_k(y)\,d\mu(y) - b_k Q_k - a_{k-1}Q_{k-1}\right)$$

The recursion relation has an advantage over Gram-Schmidt in that it should not accumulate errors as quickly.

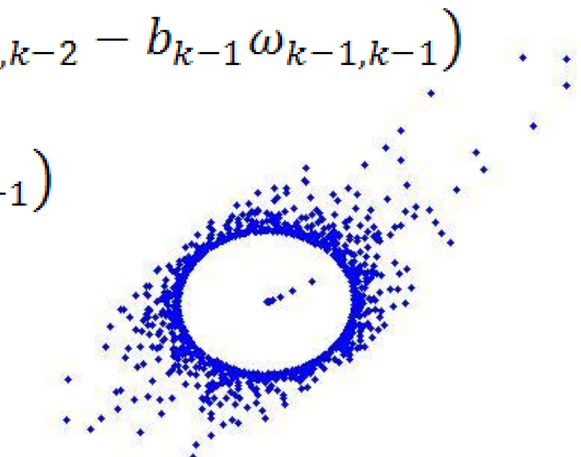
Define the anti-symmetric orthogonal polynomial as :

$$Q_k = \sum_{\ell=0}^{k} \omega_{k,\ell} P_{\ell,3}$$

Recursion Algorithm:

1. $\omega_{k,0} = \frac{1}{a_{k-1}}(\zeta_{k-1} - b_{k-1}\omega_{k-1,0} - a_{k-2}\omega_{k-2,0})$
2. $\omega_{k,\ell} = \frac{1}{a_{k-1}}(\omega_{k-1,\ell-1} - b_{k-1}\omega_{k-1,\ell} - a_{k-2}\omega_{k-2,\ell})$ $for \;\; 0 \le \ell \le k-2$
3. $\omega_{k,k-1} = \frac{1}{a_k - 1}(\omega_{k-1,k-2} - b_{k-1}\omega_{k-1,k-1})$
4. $\omega_{k,k} = \frac{1}{a_k - 1}(\omega_{k-1,k-1})$

For given recursive sequences a_k and b_k



Dynamics of the Legendre polynomials

One of the goals is to do dynamics on the orthogonal polynomials, by plotting the points:

$$(Q_{k+1}(x), Q_k(x))$$

For a large number of k, and fixed point x in SG

It seems that the recursion relation will be the best way to find these points.