I. Qualifier: September 1999

1. Show that a (scalar-valued) continuous function on the interval $[0, 1]$ is necessarily uniformly continuous.

   See Baby Rudin, Theorem 4.19, p.90.

2. Give an example where the integral and sum of an infinite sequence of continuous functions cannot be interchanged. State (without proof) a theorem guaranteeing that this interchange can be carried out.

   Define $f_n(x) = ae^{-nax} - be^{-bnx}$. Then $\{f_n\}$ are obviously continuous. For simplicity, we'll use $a = 1, b = 2$. Then

$$\begin{aligned}
\sum_{n=1}^{\infty} \int_0^{\infty} f_n(x) &= \sum_{n=1}^{\infty} \int_0^{\infty} (e^{-nx} - 2e^{-2nx}) dx \\
&= \sum_{n=1}^{\infty} \left(\int_0^{\infty} e^{-nx} dx - 2 \int_0^{\infty} e^{-2nx} dx \right) \\
&= \sum_{n=1}^{\infty} \left(-\tfrac{1}{n} \left[e^{-nx} \right]_0^{\infty} + \tfrac{1}{n} \left[e^{-2nx} \right]_0^{\infty} \right) \\
&= \sum_{n=1}^{\infty} \left(-\tfrac{1}{n}(0 - 1) + \tfrac{1}{n}(0 - 1) \right) \\
&= \sum_{n=1}^{\infty} 0 = 0,
\end{aligned}$$

but

$$\begin{aligned}
\sum_{n=1}^{\infty} f_n(x) &= \sum_{n=1}^{\infty} (e^{-nx} - 2e^{-2nx}) \\
&= \sum_{n=1}^{\infty} (e^{-x})^n - 2 \sum_{n=1}^{\infty} (e^{-2x})^n \qquad \text{(I.1)} \\
&= \sum_{n=0}^{\infty} (e^{-x})^n - 1 - 2 \sum_{n=0}^{\infty} (e^{-2x})^n + 2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{1-e^{-x}} - \frac{2}{1-e^{-2x}} + 1 \\
&= \frac{(1+e^{-x})-2}{(1-e^{-x})(1+e^{-x})} + 1 \\
&= \frac{e^{-x}-1+(1-e^{-2x})}{(1-e^{-x})(1+e^{-x})} \\
&= \frac{e^{-x}(1-e^{-x})}{(1-e^{-x})(1+e^{-x})} \\
&= \frac{e^{-x}}{1+e^{-x}} \\
&= \frac{1}{e^{x}+1}
\end{aligned}
$$

where the rearrangement in line (I.1) follows by absolute convergence of the sum. This shows that

$$
\begin{aligned}
\int_0^\infty \sum_{n=1}^\infty f_n(x) &= \int_0^\infty \frac{e^{-x}}{1+e^{-x}} dx \\
&= \left[-\log|1+e^{-x}|\right]_0^\infty \\
&= -\left(\log 0 - \log 2\right) \\
&= \log 2.
\end{aligned}
$$

In general, $\int_0^\infty \sum_{n=1}^\infty \left(ae^{-nax} - be^{-bnx}\right) = \log\left(\frac{b}{a}\right)\, dx$.

As for a theorem, the Beppo-Levi theorem states that this interchange can be carried out if

$$
\sum_{n=1}^\infty \int_0^\infty |f_n(x)|\, dx < \infty.
$$

Note that this doesn't hold for this example, since

$$
\begin{aligned}
\sum_{n=1}^\infty \int_0^\infty |f_n(x)|\, dx &= \sum_{n=1}^\infty \int_0^\infty |2e^{-2nx} - e^{-nx}|\, dx \\
&= \sum_{n=1}^\infty \tfrac{1}{2n} = \tfrac{1}{2} \sum_{n=1}^\infty \tfrac{1}{n} = \infty
\end{aligned}
$$

3. Describe the Cantor ternary set $\mathcal{C}$ via "removing middle thirds" and show that $\mathcal{C}$ is uncountable and has measure 0.

To see $\mathcal{C}$ is uncountable,

a) Note that every element of $\mathcal{C}$ has a ternary expansion consisting entirely of 0's and 2's, and that every number in $[0, 1]$ with ternary expansion consisting entirely of 0's and 2's is an element of $\mathcal{C}$. This "address" is not unique — any rational point of $\mathcal{C}$ has two equivalent expansions. However, if we just take the set of expansions which do not terminate, there is still an uncountable number of them by Cantor's diagonalization argument.

b) Show that $\mathcal{C}$ is perfect. As an intersection of closed sets, it is clearly closed, so it suffices to show that it has no isolated points. Pick any point of $\mathcal{C}$ and fix $\varepsilon > 0$. Use the construction of $\mathcal{C}$ to find another point within ε. This shows $\mathcal{C}$ is perfect. Then by Baby Rudin Thm 2.43 (p.41), every nonempty perfect set is uncountable.

To see $\mathcal{C}$ has measure zero, take the measure of the unit interval and subtract the measures of all the intervals removed during its construction:

$$\begin{aligned}
\mu(\mathcal{C}) &= 1 - \left(\tfrac{1}{3} + 2\tfrac{1}{9} + 4\tfrac{1}{27} + \dots\right) \\
&= 1 - \sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} \\
&= 1 - \tfrac{1}{3}\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \\
&= 1 - \tfrac{1}{3}\left(\frac{1}{1-2/3}\right) = 1 - \tfrac{1}{3}\cdot 3 = 1 - 1 = 0
\end{aligned}$$

4. Prove or disprove the following statement:
A real-valued continuous function on the interval $[0, 1]$ that is differential with derivative $= 0$ everywhere but on a set of (one-dimensional) measure 0 is necessarily constant.

This is false: the standard counterexample is the Cantor-Lebesgue function or "Devil's Staircase". Define $f(x)$ in the successive intervals removed during the construction of $\mathcal{C}$ by

$$\begin{array}{llll}
f(x) = \frac{1}{2} & \text{for } \frac{1}{3} < x < \frac{2}{3} & & \\
f(x) = \frac{1}{4} & \text{for } \frac{1}{9} < x < \frac{2}{9} & f(x) = \frac{3}{4} & \text{for } \frac{7}{9} < x < \frac{8}{9} \\
f(x) = \frac{1}{8} & \text{for } \frac{1}{27} < x < \frac{2}{27} & f(x) = \frac{3}{8} & \text{for } \frac{7}{27} < x < \frac{8}{27} \\
f(x) = \frac{5}{8} & \text{for } \frac{19}{27} < x < \frac{20}{27} & f(x) = \frac{7}{8} & \text{for } \frac{25}{27} < x < \frac{26}{27} \\
\vdots & & &
\end{array}$$

This function is differential with derivative 0 everywhere except on $\mathcal{C}$. It is clearly not constant. Use the construction of $\mathcal{C}$ to show that it is continuous.
This function motivates the definition of absolute continuity (it is not absolutely continuous).

5. Given a Banach space X, denote by $\mathcal{L}(X)$ the space of bounded linear operators from X to itself, equipped with its usual norm, and let G denote the set of invertible elements in $\mathcal{L}(X)$.

   a) Show that if $A \in \mathcal{L}(X)$ and $\|A\| < 1$, then $I - A \in G$; further, compute $(I - A)^{-1}$.

      The request to compute $(I - A)^{-1}$ is actually a hint. Do this first. Like many results in analysis, first figure out what it must be algebraically, then use analysis to prove it makes sense.
      We have
      $$(I - A)(x) = x - Ax,$$
      So do the obvious thing and consider $(I + A)$:
      $$(I + A)(x - Ax) = x - Ax + Ax - A^2x = x - A^2.$$
      That didn't work; we need to get rid of the A^2 terms. Try $(I + A + A^2)$:
      $$\begin{aligned}(I + A)(x - Ax) &= x - Ax + Ax - A^2x + A^2x - A^3x \\ &= x - A^3x.\end{aligned}$$

Hmm ... guess where this is going. Try $\sum_{n=0}^{\infty} A^n$:

$$(\sum_{n=0}^{\infty} A^n)(x - Ax) = x - Ax + Ax - A^2x + A^2x - \cdots = x.$$

Bingo! And the other way:

$$(\sum_{n=0}^{\infty} A^n)(x) = x + Ax + A^2x + A^3x + \ldots,$$

so

$$\begin{aligned}(I - A)(x + Ax + A^2x + A^3x + \ldots)\\ &= (x + Ax + A^2x + A^3x + \ldots)\\ &\qquad - (Ax + Ax^2 + A^3x + A^4x + \ldots)\\ &= x.\end{aligned}$$

So we have the candidate for the inverse. Then

$$\begin{aligned}\|(I - A)^{-1}\| &= \|\sum_{n=0}^{\infty} A^n\|\\ &\le \sum_{n=0}^{\infty} \|A\|^n\\ &< \infty \qquad \text{because } \|A\| < 1, \text{ (geom series)}\end{aligned}$$

Thus $\sum_{n=0}^{\infty} A^n$ is a well-defined operator and

$$(I - A)^{-1} = \sum_{n=0}^{\infty} A^n \in \mathcal{L}(X)$$

shows $I - A \in G$.

b) Show that G is an open subset of $\mathcal{L}(X)$ and that the map $\varphi : G \to G$ defined by $\varphi(Q) = Q^{-1}$ is continuous.

6. Sketch the proof of the Hahn-Banach Theorem.

The idea of the proof is to first show that if $x \in X$ but $x \notin S$, then we can extend f to a functional having all the right properties on the space spanned by x and S. We then use a Zorn's Lemma / Hausdorff

Maximality argument to show that this process can be continued to extend f to the whole space X.

1. Consider the family

$$\mathcal{G} := \{g : D \to \mathbb{R} \,\vdots\, g \text{ is linear; } g(x) \leq p(x), \ \forall x \in D; \ g(s) = f(s), \ \forall s \in S\},$$

   where D is any subspace of X which contains S. So $\mathcal{G}$ is roughly the collection of "all linear extensions of f which are bounded by p".

   Now $\mathcal{G}$ is a poset under

$$g_1 \prec g_2 \qquad \iff \qquad \text{Dom}(g_1) \subseteq \text{Dom}(g_2) \text{ and } g_2 \Big|_{\text{Dom}(g_1)} = g_1.$$

2. Use Hausdorff maximality Principle (or Zorn) to get a maximal linearly ordered subset $\{g_\alpha\} \subseteq \mathcal{G}$ which contains f. Define F on the union of the domains of the $\{g_\alpha\}$ by $F(x) = g_\alpha(x)$ for $x \in \text{Dom}(g_\alpha)$.

3. Show that this makes F into a well-defined linear functional which extends f, and that F is maximal in that $F \prec G \implies F = G$.

4. Show F is defined on all of X using the fact that F is maximal. Do this by showing that a linear functional defined on a *proper* subspace has a *proper* extension. (Hence F must be defined on all of X or it wouldn't be maximal.)

5. State and prove the Baire Category Theorem. Moreover, give at least one significant application of this theorem in real or functional analysis.

   Proof of Baire's Theorem:

6. If $\{f_n\}$ is a sequence of pointwise bounded functions on $[a, b]$, show that there exists a subsequence of $\{f_n\}$ which converges on a dense subset of $[a, b]$. Assume that the functions are $\mathbb{R}^k$-valued.

7. Let X be any compact subset of $\mathbb{R}$ containing an interval (of positive length). Is it possible that

$$D := \{f \in C(X) : |f(x)| \leq 1, \forall x \in X\}$$

is a compact subset of $C(X)$? Prove your assertion.

X contains an interval of positive length, so wlog let X contain the interval $[0, 1]$. Compactness implies limit point compactness, so if D is compact, any Cauchy sequence in D should converge to a limit point in D. We will construct a Cauchy sequence of D which does not converge.

Note: the topology of $C(X)$ is that of *uniform convergence*, i.e., limits are considered with respect to the *sup norm.*

Define $f_n(x) = \max\{0, 1 - n|x - \frac{1}{2}|\}$ so that f_n looks like a peak at $\frac{1}{2}$ with sides of slope $|n|$, and f_n vanishes elsewhere. I.e.,

$$f_n(\tfrac{1}{2}) = 1,\ \forall n,\ \text{ and for } x \neq \tfrac{1}{2}, f_n(x) \xrightarrow{n\to\infty} 0.$$

Thus $sup|f_n(x) - f_m(x)| \xrightarrow{n,m\to\infty} 0$, $\forall x$, and $\{f_n\}$ is Cauchy with respect to t he sup norm.

But

$$f_n(x) \xrightarrow{n\to\infty} f(x) = \begin{cases} 0 & x \neq \frac{1}{2} \\ 1 & x = \frac{1}{2}, \end{cases}$$

and $f \notin C(X) \implies f \notin D$.

8. State Fatou's Lemma. Prove the Lebesgue Dominated Convergence Theorem using Fatou's Lemma.

9. State the Fundamental Theorem of Calculus (FTOC) (relating a function F to the integral of its derivative) in its most general form, stating the necessary and sufficient condition (C) that F must satisfy in order that the theorem hold. Finally, consider $F(x) = |x|$ on $\mathbb{R}$. Illustrate the truth or falsity of the FTOC in this case; i.e., show that $F(x)$ either does or does not satisfy the statement of the FTOC.

10. Let L^1 be the Banach space of Lebesgue integrable functions on $[0, 1]$. Let F be a bounded linear functional on L^1. Prove that there is a bounded measurable function g so that $F(f) - \int_0^1 f(x)g(x)dx, f \in L^1$. (You may use properties of absolutely continuous functions, density of step functions, etc. You may NOT use the fact that the dual space of L^1 is L^∞.)

11. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differential function that is homogeneous of order k (i.e., $f(\lambda x_1, \ldots, \lambda x_n) = \lambda^k f(x_1, \ldots, x_n), \forall \lambda \in \mathbb{R}$). Show that $x_1 \frac{\partial f}{\partial x_1} + \cdots + x_n \frac{\partial f}{\partial x_n} = kf$ (i.e., $\vec{x} \cdot f'(\vec{x}) = kf(\vec{x})$).

12. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differential function. Define the directional derivative of f (at $\vec{x}$) in the direction of the unit vector $\vec{v} \in \mathbb{R}^n$. Show that this derivative has maximum modulus when v is in the direction of the gradient (f') of f.

13. Let $f(x)$ be a continuous function from $[a, b]$ to itself.

    a) Prove from basic principles that $f(x_0) = x_0$ for some $x_0 \in [a, b]$.

    b) Assume in addition that the derivative f' exists on (a, b) and that $|f'(x)| \leq \alpha$, for some $0 < \alpha < 1$. Prove that the fixed point x_0 is unique and state and prove an algorithm for finding x_0.