

I. THE HIT LIST

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| <ul style="list-style-type: none"> • contin on compact \equiv unif contin • contin on compact \implies attains max • equivalent definitions of compactness: <ul style="list-style-type: none"> – limit-point compact – sequentially compact – open cover def – finite intersection def • Intermediate Value Thm, Mean Value Thm • FTOC • Stone-Weierstrass • Convergence theorems <ul style="list-style-type: none"> – Fatou’s Lemma – Monotone Convergence – Dominated Convergence – how each implies the others • Hahn & Lebesgue Decomposition • Radon-Nikodym Theorem • Riesz Representation • Hahn-Banach Theorem • Baire Category <ul style="list-style-type: none"> – Open Mapping – Closed Graph – Uniform Boundedness • Arzela-Ascoli (s&a only¹) • Fubini-Tonelli (s&a only) • Inverse & Implicit Function Theorems • Hölder and Minkowski Inequalities • Littlewood’s Three Principles | <ul style="list-style-type: none"> – measurable/open sets – Lusin’s Theorem – Egoroff’s Theorem • Modes of Convergence • Banach \iff
(abs. conv. \implies conv.) • Y Banach $\implies L(X, Y)$ Banach • Completeness of L^p • $(L^p)^* = L^q$ • $(L^1)^* = L^\infty$ but $(L^\infty)^* \neq L^1$ • T bounded $\iff T$ contin $\iff T$ contin at 0 • λ is the <i>unique</i> trans-inv meas on \mathbb{R} with $\lambda I = 1$ • step fns, simple fns dense in L^p • $X \cong X^{**}$ • l^∞ is Banach under sup norm • X complete in two norms \implies they are equiv • basic integral properties <ul style="list-style-type: none"> – $\int \chi_E d\mu = \mu E$ – $\int f d\mu = 0 \implies f = 0 \mu - ae$ • $C(X), \ \cdot\ _\infty$ is Banach • Cantor set: uncountability, measurability • $\ T\ < 1 \implies 1 - T$ is invertible • C_∞ is Banach under sup norm • Nasty Integrals • equiv definitions of absolute continuity |
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¹statement and application only, no proof required

II. CONVEXITY

A function $f : (a, b) \rightarrow \mathbb{R}$ is called *convex* iff

$$f(\lambda s + (1 - \lambda)t) \leq \lambda f(s) + (1 - \lambda)f(t)$$

for all $s, t \in (a, b)$ and $\lambda \in (0, 1)$. Geometrically, this states that the graph of f over the interval from s to t lies underneath the line segment joining $(s, f(s))$ to $(t, f(t))$.

1. A convex function on (a, b) is continuous.

2. f is convex iff

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2}$$

for all $x, y \in (a, b)$.

3. f is convex iff for all $s, t, s', t' \in (a, b)$ such that $s \leq s' < t'$ and $s < t \leq t'$,

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(t') - f(s')}{t' - s'}.$$

4. f is convex iff f is absolutely continuous on every every compact subinterval of (a, b) and f' is increasing (on the set where it is defined).

5. If f is convex and $t_0 \in (a, b)$, then there exists $\beta \in \mathbb{R}$ such that $f(t) - f(t_0) \geq \beta(t - t_0)$ for all $t \in (a, b)$.

6. (Jensen's Inequality) Let (X, \mathcal{M}, μ) be a probability space (i.e., a measure space with $\mu X = 1$). If $g : X \rightarrow (a, b)$ is in $L^1(\mu)$ and f is convex on (a, b) , then

$$f\left(\int g d\mu\right) \leq \int (f \circ g) d\mu.$$

7. Use Jensen's Inequality to prove that the (weighted) geometric mean is never more than the (weighted) arithmetic mean, i.e., for $\{\alpha_i\}$ such that $\sum \alpha_i = 1$,

$$\prod y_i^{\alpha_i} \leq \sum \alpha_i y_i,$$

for any collection of positive numbers $\{y_i\}$.

8. When does equality hold in the Minkowski inequality? (The answer is different for $p = 1$ and for $1 < p < \infty$. What about $p = \infty$?)
9. If $f \in L^p \cap L^\infty$ for some $p < \infty$, then $\|f\|_\infty = \lim_{q \rightarrow \infty} \|f\|_q$ for $q > p$.
10. A variation on the previous question: let (X, \mathcal{M}, gm) be a finite measure space and let f be a \mathcal{M} -measurable real- or complex-valued function on X .
- (1) Show that f belongs to L^∞ iff
 - (a) $f \in L^p, \forall p \in [0, \infty)$, and
 - (b) $\sup\{\|f\|_p : 1 \leq p \leq \infty\} < \infty$.
 - (2) Show that if these conditions hold, then $\|f\|_\infty = \lim_{p \rightarrow \infty} \|f\|_p$.
11. Let (X, \mathcal{M}, μ) be a probability space and suppose that $1 \leq p_1 < p_2 < \infty$.
- (1) Use Hölder's Inequality to show that if $f \in L^{p_2}$, then $f \in L^{p_1}$ and $\|f\|_{p_1} \leq \|f\|_{p_2}$.
 - (2) Use Jensen's Inequality to show that if $f \in L^{p_2}$, then $f \in L^{p_1}$ and $\|f\|_{p_1} \leq \|f\|_{p_2}$.
 - (3) Show that if $f, f_1, f_2, \dots \in L^{p_2}$, then

$$\|f_n - f\|_{p_2} \xrightarrow{n \rightarrow \infty} 0 \quad \implies \quad \|f_n - f\|_{p_1} \xrightarrow{n \rightarrow \infty} 0.$$

III. APPROXIMATION IN L^p

We have the following:

- for $1 \leq p \leq \infty$, the simple functions \mathcal{S} on X form a dense subspace of $L^p(X, \mathcal{A}, \mu)$,
 - for $1 \leq p < \infty$, the step functions \mathcal{S} on $[a, b]$ form a dense subspace of $L^p([a, b], \mathcal{A}, \mu)$,
 - for $1 \leq p < \infty$, the continuous functions $C[a, b]$ form a dense subspace of $L^p([a, b], \mathcal{A}, \mu)$.
- (1) Show that the subspace of step functions $\mathcal{S}[a, b]$ is not dense in $L^\infty([a, b], \mathcal{A}, \mu)$.
(Hint: Construct a Borel subset A of $[a, b]$ such that $\|\chi_A - f\|_\infty \geq \frac{1}{2}$ whenever $f \in \mathcal{S}[a, b]$.)
 - (2) Show that the subspace of continuous functions $C[a, b]$ is not dense in $L^\infty([a, b], \mathcal{A}, \mu)$.
(Hint: Let $A = [a, c]$, where $a < c < b$. How small can $\|\chi_A - f\|_\infty$ be for $f \in C[a, b]$?)

Suppose we expand the definition of *step function*. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be called a *step function on \mathbb{R}* iff its restriction to any finite interval $[a, b]$ is a step function on $[a, b]$ in the previous sense.

- (3) Show that the set of step functions on \mathbb{R} that vanish outside some bounded interval are dense in $L^p([a, b], \mathcal{A}, \mu)$.
- (4) Show that the set of continuous functions on \mathbb{R} that vanish outside some bounded interval are dense in $L^p([a, b], \mathcal{A}, \mu)$.

IV. MODES OF CONVERGENCE

Pointwise:

$$f_n \xrightarrow{\text{pw}} f \quad \text{iff} \quad \forall \varepsilon, \forall x \in X, \exists N \text{ s.t. } n \geq N \implies |f_n(x) - f(x)| < \varepsilon.$$

Almost everywhere:

$$f_n \xrightarrow{\text{ae}} f \quad \text{iff} \quad \forall \varepsilon, \exists A \subseteq X \text{ with } \mu(X \sim A) = 0, \text{ and} \\ \forall x \in A, \exists N \text{ s.t. } n \geq N \implies |f_n(x) - f(x)| < \varepsilon.$$

Uniform:

$$f_n \xrightarrow{\text{unif}} f \quad \text{iff} \quad \forall \varepsilon, \exists N \text{ s.t. } \forall x \in X, \quad n \geq N \implies |f_n(x) - f(x)| < \varepsilon.$$

Almost uniform:

$$f_n \xrightarrow{\text{au}} f \quad \text{iff} \quad \forall \varepsilon, \exists A \subseteq X \text{ with } \mu(X \sim A) = 0, \text{ and} \\ \exists N \text{ s.t. } \forall x \in A, \quad n \geq N \implies |f_n(x) - f(x)| < \varepsilon.$$

L^p (or “convergence in p^{th} mean”):

$$f_n \xrightarrow{L^p} f \quad \text{iff} \quad \forall \varepsilon, \exists N \text{ s.t. } n \geq N \implies \|f_n - f\|_p < \varepsilon.$$

Measure:

$$f_n \xrightarrow{\mu} f \quad \text{iff} \quad \forall \varepsilon, \exists N \text{ s.t. } n \geq N \implies \mu\{x : |f_n(x) - f(x)| \geq \varepsilon\} < \varepsilon.$$

We also say $\{f_n\}$ is **Cauchy in measure** iff

$$\forall \varepsilon, \exists N \text{ s.t. } m, n \geq N \implies \mu\{x : |f_n(x) - f_m(x)| \geq \varepsilon\} < \varepsilon.$$

The following functions are the archetypal (counter)examples for the modes of convergence:

- (i) $f_n = \frac{1}{n} \chi_{(0,n)}$.
- (ii) $f_n = \chi_{[n,n+1]}$.
- (iii) $f_n = n \chi_{[0,1/n]}$.
- (iv) $f_n = \chi_{[j/2^k, (j+1)/2^k]}$ where $n = 2^k + j$ with $0 \leq j < 2^k$. Thus,

$$\begin{array}{llll} f_1 = \chi_{[0,1]} & & & \\ f_2 = \chi_{[0, \frac{1}{2}]} & f_3 = \chi_{[\frac{1}{2}, 1]} & & \\ f_4 = \chi_{[0, \frac{1}{4}]} & f_5 = \chi_{[\frac{1}{4}, \frac{1}{2}]} & f_6 = \chi_{[\frac{1}{2}, \frac{3}{4}]} & f_7 = \chi_{[\frac{3}{4}, 1]} \\ \vdots & & & \\ \text{etc.} & & & \end{array}$$

Exercises:

1. Check if $f_n \rightarrow f$ in each of the modes listed above, for the $\{f_n\}$ given in (i)-(iv).

2. For the following, $1 \leq p < \infty$.
- If μ is finite, then $f_n \xrightarrow{\text{ae}} f \implies f_n \xrightarrow{\mu} f$.
 - If $f_n \xrightarrow{\mu} f$, then there is a subsequence $\{f_{n_k}\} \subseteq \{f_n\}$ with $f_{n_k} \xrightarrow{\text{ae}} f$.
 - If $\{f_n\}$ is Cauchy in measure, then there is a measurable f such that $f_n \xrightarrow{\mu} f$.
 - If μ is finite, then $f_n \xrightarrow{\text{ae}} f \implies f_n \xrightarrow{\text{au}} f$. (Egoroff's Thm)
 - $f_n \xrightarrow{\text{au}} f \implies f_n \xrightarrow{\mu} f$.
 - Let μ be counting measure on $X = \mathbb{Z}$. Then $f_n \xrightarrow{\mu} f \iff f_n \xrightarrow{\text{unif}} f$.
 - For $f, f_n \in L^p$, $f_n \xrightarrow{L^p} f \implies f_n \xrightarrow{\mu} f$.
 - For $f, f_n \in L^p$, if $f_n \xrightarrow{L^p} f$, then $\exists \{f_{n_k}\} \subseteq \{f_n\}$ with $f_{n_k} \xrightarrow{\text{ae}} f$.
 - For $f, f_n \in L^p$, let $g : X \rightarrow [0, \infty]$ such that $|f_n| \leq_{\text{ae}} g$ and $|f| \leq_{\text{ae}} g$. If either
 - $f_n \xrightarrow{\text{ae}} f$, or
 - $f_n \xrightarrow{\mu} f$, then $f_n \xrightarrow{L^p} f$.
 - For $f, f_n \in L^p$, show that if $f_n \xrightarrow{L^p} f$ so fast that $\sum_n \int |f_n - f|^p d\mu < \infty$, then $f_n \xrightarrow{\text{ae}} f$.
3.
 - Interpret these results for the functions (i)-(iv), in light of problem 1.
 - Which of (d)-(f) remain true for $p = \infty$?
4. If f_n, f are measurable and φ is continuous and $f_n \xrightarrow{\text{ae}} f$, then $\varphi \circ f_n \xrightarrow{\text{ae}} \varphi \circ f$.
5. Suppose $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$.
- $f_n + g_n \xrightarrow{\mu} f + g$
 - $f_n g_n \xrightarrow{\mu} fg$ if $\mu X < \infty$, but not necessarily if $\mu X = \infty$.

Function toolbox: how to make $\{f_n\}$ such that $f_n \rightarrow f$.

- $f_n := f \chi_{X_n}$, where $X_n \nearrow X$. For example, $f_n := f \chi_{[-n, n]}$.
- $f_n(x) := \begin{cases} f(x) & f(x) \leq n \\ 0 & \text{else} \end{cases}$, or $f_n(x) := \begin{cases} f(x) & f(x) \leq n \\ n & \text{else} \end{cases}$.
- Find $\{f_n\}$ that converge to a continuous function g and use 4, above.

Note: if you use (1), you can bound the domain to avoid tails. If you use (2), your f_n are bounded and you avoid spikes.

V. THE “BAEZ BUMPS”

Assigned when Dr. Baez taught 209C in the spring of 2000, this collection of problems is really helpful for combining the concepts of Modes of Convergence with relations between different L^p spaces.

1. Spikes: for which values of k, p does the function lie in $L^p(\mathbb{R}, dx)$? (Here, $k > 0$, and $1 \leq p \leq \infty$.)

$$f(x) = \begin{cases} |x|^{-k} & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$$

2. Tails: for which values of k, p does the function f lie in $L^p(\mathbb{R}, dx)$? (Again, $k > 0$, and $1 \leq p \leq \infty$.)

$$f(x) = \begin{cases} |x|^{-k} & |x| > 1 \\ 0 & |x| \leq 1 \end{cases}$$

3. Moving spikes: let $f_n : \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$f(x) = \begin{cases} n^k & |x - n| \leq \frac{1}{2n} \\ 0 & \text{otherwise.} \end{cases}$$

So f_n is a spike of height n^k and width $\frac{1}{n}$ centered at n . For what values of $k > 0$ does this sequence $\{f_n\}$ converge to zero

- a) pointwise?
- b) pointwise a.e.?
- c) uniformly?
- d) in measure?
- e) in L^p ? ($1 \leq p \leq \infty$)

4. Flattening bumps: let $f_n : \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{1}{n^k} & |x| \leq \frac{n}{2} \\ 0 & \text{otherwise.} \end{cases}$$

So f_n is a bump of height $\frac{1}{n^k}$ and width n centered at n . For what values of $k > 0$ does this sequence $\{f_n\}$ converge to zero in senses (a)-(e) as above?

5. If $p \neq q$ ($1 \leq p, q \leq \infty$), find a function $f : \mathbb{R} \rightarrow \mathbb{R}$ with $f \in L^p$ but $f \notin L^q$. (Hint: if $p > q$, use a function with a tail; if $p < q$, use a function with a spike.)
 Let (X, \mathcal{M}, μ) be a measure space and $f : X \rightarrow \mathbb{R}$ be a measurable function.
 Here, $L^p = L^p(X, \mu)$ and $L^q = L^q(X, \mu)$.

6. Show that if X is a *finite* measure space (i.e., $\mu(X) < \infty$) and $p > q$, then $f \in L^p \implies f \in L^q$. (Note: since X is a *finite* measure space, there cannot be tails.)

7. Show that if f is *bounded* and $p < q$, then $f \in L^p \implies f \in L^q$. (Note: since f is *bounded*, it cannot have spikes.)

8. Suppose $f, f_n : \mathbb{R} \rightarrow \mathbb{R}$ are measurable functions. Prove or give a counterexample:

a) $f_n \xrightarrow{\text{a.e.}} f \implies f_n \xrightarrow{\mu} f$ (in measure).

b) $f_n \xrightarrow{\mu} f \implies f_n \xrightarrow{\text{a.e.}} f$.

c) $f_n \xrightarrow{\mu} f \implies f_n \xrightarrow{L^2} f$.

d) $f_n \xrightarrow{L^2} f \implies f_n \xrightarrow{\mu} f$.

For those which are false, which become true when we consider $f, f_n : [0, 1] \rightarrow \mathbb{R}$ instead? No proof is necessary. (Note: those that are true will remain true, since we can think of functions on $[0, 1]$ as a special case of functions on \mathbb{R} .)

9. For $1 \leq p \leq q < \infty$, show that

$$|f| \geq 1 \implies \|f\|_p \leq \|f\|_q.$$

10. For $1 \leq p \leq q < \infty$, show that

$$|f| \leq 1 \implies \|f\|_p \geq \|f\|_q.$$

11. For $1 \leq p \leq q \leq r < \infty$, show that

$$f \in L^p, f \in L^r \implies f \in L^q.$$

(Note: In 9, the function f cannot have tails. In 10, it cannot have spikes. In 11, it can have both, but $f \in L^p$ controls how bad the tails are, and $f \in L^r$ controls how bad the spikes are.)

VI. BAIRE CATEGORY PROBLEMS

1. State precisely and sketch the proof of Baire's Thm. Give at least one significant application. [2003, 2000, 1999]
2. a) State precisely the Closed Graph Theorem and the Open Mapping Theorem.
b) Assuming the Open Mapping Theorem, deduce the Closed Graph Theorem.
3. Let X be a linear space which is a Banach space under each of the norms $\|\cdot\|_1, \|\cdot\|_2$. If there exists a $a > 0$ such that $\|x\|_1 \leq a\|x\|_2, \forall x \in X$, then the two norms are equivalent. [1998, 1997, 1995]

4. Use the Open Mapping Theorem to show that if the vector space V is complete in $\|\cdot\|_1, \|\cdot\|_2$, then there are constants $c_1, c_2 > 0$ such that

$$c_1\|x\|_1 \leq \|x\|_2 \leq c_2\|x\|_1, \forall x \in X.$$

State the Closed Graph Theorem and use this result to prove it. [1996]

5. Let $L(X)$ be the space of bounded linear operators on the Banach space X , and let $T \in L(X)$. [2003, 1999, 1996, 1995]
 - a) Show $\|T\| < 1 \implies I - T$ is invertible with bounded inverse.
 - b) Deduce (using proof of (a)) that the set G of invertible elements of $L(X)$ is an open set of $L(X)$. Also, show that $\varphi : G \rightarrow G$ by $\varphi(T) = T^{-1}$ is continuous.
6. Let X, Y be Banach, $S : X \rightarrow Y$ an unbounded linear map, and $\Gamma(S) = \{(x, y) \in X \times Y : y = Tx\}$. Then
 - a) $\Gamma(S)$ is not complete.
 - b) $T : X \rightarrow \Gamma(S)$ by $Tx = (x, Sx)$ is closed but not bounded.
 - c) $T^{-1} : \Gamma(S) \rightarrow X$ is bounded and surjective, but not open.

7. Let $Y = L^1(\mu)$ where μ is counting measure on \mathbb{N} , and let

$$X = \left\{ f \in Y : \sum_{n=1}^{\infty} n|f(n)| < \infty \right\},$$

equipped with the L^1 norm.

- a) X is a proper dense subspace of Y ; hence X is not complete.
- b) Define $T : X \rightarrow Y$ by $Tf(n) = nf(n)$. Then T is closed but not bounded.
- c) Let $S = T^{-1}$. Then $S : Y \rightarrow X$ is bounded and surjective but not open.

8. Show that the completeness of the range is necessary in the corollary to the Open Mapping Theorem, i.e., find a Banach space X , a normed linear space Y , and a continuous linear bijection $f : X \rightarrow Y$ such that f^{-1} is not continuous.
9. Show that the completeness of the domain is necessary in the Closed Graph Theorem, i.e., find a normed linear space X a Banach Space Y , and a closed linear map $T : X \rightarrow Y$ such that T is not bounded.
10. There exist meager subsets of \mathbb{R} whose complements have Lebesgue measure zero.
11. The Baire Category Theorem remains true if X is assumed to be a locally compact Hausdorff space instead of a complete metric space. (Hint: the proof is similar; the substitute for completeness is the finite-intersection-property definition of compactness.)
12. Let $C^k([0, 1])$ be the space of functions on $[0, 1]$ possessing continuous derivatives up to order k on $[0, 1]$, including one-sided derivatives at the endpoints.
 - a) If $f \in C([0, 1])$, then $f \in C^k([0, 1])$ iff f is k times continuously differentiable on $(0, 1)$ and $\lim_{x \rightarrow 0^+} f^{(j)}(x)$ and $\lim_{x \rightarrow 1^-} f^{(j)}(x)$ exist for $j \leq k$. (Hint: try the Mean Value Theorem.)
 - b) $\|f\| = \sum_{n=0}^k \|f^{(n)}\|_u$ is a norm on $C^k([0, 1])$ that makes $C^k([0, 1])$ into a Banach space. (Hint: use induction on k . The essential point is that if $\{f_n\} \subseteq C^1([0, 1])$, $f_n \xrightarrow{\text{unif}} f$, and $f'_n \xrightarrow{\text{unif}} g$, then $f \in C^1([0, 1])$ and $f' = g$. One way to prove this is to show that $f(x) - f(0) = \int_0^x g(t) dt$.)
13. Let $Y = C([0, 1])$ and $X = C^1([0, 1])$, both equipped with the uniform norm.
 - a) X is not complete.
 - b) The map $\frac{d}{dx} : X \rightarrow Y$ is closed but not bounded. (Hint: use the previous exercise to show closed.)
14. Why doesn't a 1-point space violate the Baire Theorem?

VII. THE HAHN-BANACH THEOREM(S)

There are no problems here (unless you want to prove the equivalences - good luck!), this is just a collection of different versions of the HBT. This is to give some idea of how rich this theorem is, and give some different perspectives.

The following are equivalent:

- (HB 1) Existence of Banach Limits. Let I be a directed set, and let $\{f_\alpha\}$ be bounded functions from X into \mathbb{R} . Then there exists a real-valued Banach limit L for $\{f_\alpha\}$ – that is, a linear map that satisfies $L(f_\alpha) = \lim_{\alpha} L(f_\alpha)$ for each α .
- (HB 2) Convex Extension Theorem and
- (HB 3) Sublinear Extension Theorem. Suppose X is a real vector space, X_0 is a linear subspace, f_0 is a linear map, p is a convex (or sublinear) function, and $f_0 \leq p$ on X_0 . Then f_0 can be extended to a linear map that satisfies $f \leq p$ on X .
- (HB 4) Convex Support Theorem and
- (HB 5) Sublinear Support Theorem. Any convex (or sublinear) function p from a real vector space X into \mathbb{R} is the pointwise maximum of the affine functions that lie below it. That is, if f is convex (respectively, sublinear), then for each x_0 there exists some affine function g that satisfies $g(x_0) = p(x_0)$ and $g \leq p$ on X .
- (HB 6) Sandwich Theorem. Let C be a convex subset of a real vector space. Suppose that f is a concave function, g is a convex function, and $f \leq g$ everywhere on C . Then there exists an affine function h satisfying $f \leq h \leq g$ on C .
- (HB 7) Norm-Preserving Extensions. Let X be a normed space, and let Y be a linear subspace of X . Let f – that is, let f be a bounded linear map from Y into the scalar field, where Y is normed by the restriction of $\|\cdot\|$ to Y . Then f can be extended to some F satisfying $\|F\| = \|f\|$.
- (HB 8) Functionals for Given Vectors. Let X be a normed vector space other than the degenerate space $\{0\}$, and let $\{x_\alpha\}$. Then there exists some f such that $f(x_\alpha) = \|x_\alpha\|$ and $\|f\| = 1$. Hence the norm of a vector in X can be characterized in terms of the values of members of $\{x_\alpha\}$:
- (We emphasize that this is a maximum, not just a supremum; contrast that with 28.41.7.) Therefore each x_α acts as a bounded linear functional by the rule $Tx(f) = f(x)$, with norm $\|x_\alpha\|$.

- (HB 9) Separation of Points. If X is a normed space, then separates the points of X . That is, if x and y are distinct points of X , then there exists some f such that $f(x) \neq f(y)$. Equivalently, if $x \neq y$, then there exists some f such that $f(x) < f(y)$.
- (HB 10) Variational Principle. Let X be a normed space. Let V be a closed linear subspace, let f vanish on V , and let $f(x) = 1$. Then $\|x\|$ is nonempty, and $\|x\| = 1$.
- (HB 11) Separation of Subspaces. Let B be a closed linear subspace of a Banach space X , and let $x \notin B$. Then there exists a member f of X^* that vanishes on B but not on x .
- (HB 12) Luxemburg's Measure. For every nonempty set Ω and every proper filter of subsets of Ω , there exists a probability charge on Ω that takes the value 1 on elements of \mathcal{F} .
- (HB 13) On every Boolean algebra there exists a probability charge.
- (HB 14) Let X be a Boolean algebra. Then for every proper ideal I in X there exists a probability on X that vanishes on I .
- (HB 15) Riesz Seminorms and
- (HB 16) Positive Functionals. Let X be a Riesz space, let S be a Riesz subspace, and suppose either
 (a) S is a Riesz subspace and f is a positive linear functional on S ,
 (b) f is a positive linear functional on S .
 Then f extends to a positive linear functional on X .
- (HB 17) Continuous Support Theorem. Let X be a real TVS. Then any continuous convex function from X into \mathbb{R} is the pointwise maximum of the continuous affine functions that lie below it. That is, if f is continuous and convex, then for each x_0 there exists some continuous affine function that satisfies $f(x) \leq p(x)$ for all x and $f(x_0) = p(x_0)$.
- (HB 18) Separation of Convex Sets in TVS's. Let A and B be disjoint nonempty convex subsets of a real topological vector space X , and suppose A is open. Then there exists f such that for every $x \in A$, $f(x) < 1$ and for every $y \in B$, $f(y) \geq 1$.
- (HB 19) Separation of Convex Sets in LCS's. Let A and B be disjoint nonempty convex subsets of a real, locally convex topological vector space X . Suppose A is compact and B is closed. Then there exists f such that for every $x \in A$, $f(x) < 1$ and for every $y \in B$, $f(y) \geq 1$.

- (HB 20) Separation of Points from Convex Sets. Let B be a nonempty closed convex subset of a real, locally convex topological vector space X . Let $x \notin B$. Then there exists such that $x \in \text{int}(B)$.
- (HB 21) Intersection of Half-Spaces. Let X be a real, locally convex topological vector space. Then any closed convex subset of X is the intersection of the closed half-spaces that contain it. (By a closed half-space we mean a set of the form $\{x \in X : f(x) \leq r\}$, for some continuous linear functional and some real number r .)
- (HB 22) Separation of Points. If X is a Hausdorff LCS, then separates points of X . That is, if x and y are distinct points of X , then there exists some such that $x \in \text{int}(B)$ and $y \notin B$. Equivalently, if $x \neq y$, then there exists some such that $x \in \text{int}(B)$ and $y \notin B$.
- (HB 23) Separation of Subspaces. Let B be a closed linear subspace of a locally convex space X , and let $x \notin B$. Then there exists a member of that vanishes on B but not on x .
- (HB 24) Weak closures. In a locally convex space, every σ -closed, convex set is σ -closed. In brief, every closed convex set is weakly closed.
- (HB 25) Banach's Generalized Integral. Let μ , etc., be as in 29.30, with μ . Let ν be a bounded charge on \mathcal{A} . Then the Bartle integral $\int \nu d\mu$, already defined on in 29.30, can be extended (not necessarily uniquely) to a continuous linear map $\nu \mapsto \int \nu d\mu$, satisfying $\int \nu d\mu = \int \nu d\mu$. If ν is a positive charge, then $\int \nu d\mu$ can be chosen so that it is also a positive linear functional.
- (HB 26) Banach's Charge. Let \mathcal{A} be an algebra of subsets of a set X , and let ν be a bounded real-valued charge on \mathcal{A} . Then ν can be extended to a real-valued charge on all of \mathcal{A} . If ν is a positive charge, then we can also choose to be a positive charge.

Dang! There's 26 of them!

VIII. UNDERGRADUATE MATERIAL

All problems are from recent quals.

1. [1996] Find

$$\overline{\lim}_{n \rightarrow \infty} \sin(nx)$$

where $\overline{\lim}$ denotes the limit superior (or ‘lim sup’). Prove your answer is true.

2. [1997] Prove or give a counterexample: the set \mathbb{R} of real numbers with its usual topology is a union of nowhere dense subsets.
3. [2003s] Let $\{x_n\}$ be a Cauchy sequence in a metric space. Show that if $\{x_n\}$ admits a convergent subsequence, then the entire sequence is convergent.
4. [1996] Prove that there is no value of k such that the equation $x^3 - 3x + k = 0$ has two distinct roots in $[0, 1]$.
5. [1998] Let X be a compact topological space. Suppose that $f : X \rightarrow \mathbb{R}$ has the property that $\{x : f(x) \geq a\}$ is closed for each $a \in \mathbb{R}$. Prove that f is bounded above and that it attains its least upper bound.
6. [1997] If $a_n \xrightarrow{n \rightarrow \infty} a$ (for $a_n, a \in \mathbb{R}$) show that $\lim_{n \rightarrow \infty} \left(1 + \frac{a_n}{n}\right)^n = e^a$ by first establishing that $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$ and then using it.
7. [2002] Show that if $\{x_n\}$ is a convergent sequence in \mathbb{R}^k with limit b , then

$$A := \{x_n : n \in \mathbb{N}\} \cup \{b\}$$

is a compact subset of \mathbb{R}^k . Would this result be true in an arbitrary metric space, rather than in \mathbb{R}^k ?

8. [1996] Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.
9. [2003s] Show that the series $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}$ is convergent, and calculate its sum.
10. [1996] Determine the radius of convergence of the series $f(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$. Prove that within the radius of convergence, $f''(z) = f(z)$.

11. [1997] Prove that if X and Y are topological spaces, $f : X \rightarrow Y$ is continuous, and $K \subseteq X$ is compact, then $f(K) \subseteq Y$ is compact.
12. [1997, 1995] Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous, but not necessarily differentiable function. Show that it attains a maximum at some point of the closed unit interval.
13. [2000]
- Give the definition of a connected set in Euclidean space \mathbb{R}^n .
 - Show that if $C \subseteq \mathbb{R}^n$ is connected, then any continuous $f : C \rightarrow \mathbb{Z}$ is constant. ($\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ is the set of integers.) Is the converse true?
14. [1999] Show that a (scalar-valued) continuous function on the interval $[0, 1]$ is necessarily uniformly continuous.

15. [1998, 1995] Let

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{\cos nx}{n}, \quad x \in \mathbb{R}.$$

At which points is f well-defined? At which points is it continuous?

16. [1996] Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $\lim_{x \rightarrow \pm\infty} f(x) = \infty$. Using the fact that every Cauchy sequence in \mathbb{R} converges, show that f attains a minimum value.
17. [1996] Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is one-to-one and onto, and let f^{-1} denote the inverse function (not the reciprocal).
- If f is continuous, is f^{-1} necessarily continuous? Give a proof or a counterexample.
 - If f is differentiable, is f^{-1} necessarily differentiable? Give a proof or a counterexample.
18. [2001] A function f is said to satisfy a Lipschitz condition on an interval $[a, b]$ if there is a constant M such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in [a, b]$.
- Show that a function satisfying a Lipschitz condition is absolutely continuous.
 - Show that an absolutely continuous function f satisfies a Lipschitz condition if and only if $|f'|$ is bounded.

- c) Prove or give a counterexample: f satisfies a Lipschitz condition if one of its derivatives (say D^+) is bounded.
19. [2000]
- a) Let $f(x) = \|x\|$ for $x \in \mathbb{R}^n$, where $\|x\| = (\sum_{k=1}^n x_k^2)^{1/2}$ denotes the Euclidean length of $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Compute $f'(x)$, the differential of f at x , whenever it exists. If $f'(x)$ does not exist for some x , explain why.
- b) Answer the same question as in (a) for the function $f = \|x\|^2$, for $x \in \mathbb{R}$.
20. [1999] Let $f(x)$ be a continuous function from $[a, b]$ into itself.
- a) Prove from basic principles that $f(x_0) = x_0$ for some $x_0 \in [a, b]$.
- b) Assume in addition that the derivative f' exists on (a, b) and that $|f'(x)| \leq \alpha$, for some $0 < \alpha < 1$. Prove that the fixed point x_0 is unique and state and prove an algorithm for finding x_0 .
21. [1999] State the Fundamental Theorem of Calculus (FTOC) (relating a function F to the integral of its derivative) in its most general form, stating the necessary and sufficient condition (C) that F must satisfy in order that the theorem hold. Finally, consider $F(x) = |x|$ on \mathbb{R} . Illustrate the truth or falsity of the FTOC in this case; i.e., show that $F(x)$ does or does not satisfy (C) and also that $F(x)$ does or does not satisfy the statement of the FTOC.
22. [2003s] Calculate the volume of the unit ball in \mathbb{R}^3 . (Show your work). Explain briefly how you would proceed to extend this result to \mathbb{R}^n , for any $n > 1$.
23. [1998, 1995] Let f be a continuous function on $[0, 1]$ such that
- $$\int_0^1 x^n f(x) dx = 0, \text{ for all } n = 0, 1, 2, \dots$$
- Show that $f(x) = 0$ for all $x \in [0, 1]$.
24. [2003s, 1998, 1995] Show that the series $S_1 = \sum_{n=1}^{\infty} nq^n$ and $S_2 = \sum_{n=1}^{\infty} n^2q^n$ are uniformly convergent for $0 < q < 1$ (compact intervals). Show that $S_1 = q(1 - q)^{-2}$. Using this, find a similar expression for S_2 .
25. [2002] Let $f_n(x) = \cos(x + \frac{1}{n})$, for $x \in \mathbb{R}$ and $n \geq 1$. Is the sequence $\{f_n\}_{n=1}^{\infty}$ uniformly convergent on \mathbb{R} ? If so, what is its limit?

26. [2002] Let $\{f_n\}$ be a uniformly convergent sequence of continuous function on $[a, b]$ and let $c \in [a, b]$. Prove directly that

$$\lim_{n \rightarrow \infty} \lim_{x \rightarrow c} f_n(x) = \lim_{x \rightarrow c} \lim_{n \rightarrow \infty} f_n(x).$$

Prove the result directly; do not quote a known theorem.

27. [2000] Let $\zeta(x) = \sum_{n=1}^{\infty} n^{-x}$, for $x > 1$.

- a) Show that ζ is uniformly convergent on (a, ∞) for every $a > 1$.
- b) Show that ζ is continuous on $(1, \infty)$.

28. [2000]

- a) Give an example of a sequence of real-valued functions $\{f_n\}$ which converges pointwise but not uniformly on $[0, 1]$.
- b) Give an example of a series of differentiable functions $\sum_{n=1}^{\infty} f_n$ which converge uniformly on $[0, 1]$ but such that there exists x_0 in $[0, 1]$ at which the series of derivatives $\sum_{n=1}^{\infty} f'_n(x_0)$ does not converge.

29. [1999] Give an example where the integral and sum of an infinite sequence of continuous functions cannot be interchanged. State (without proof) a theorem guaranteeing that this interchange can be carried out.

30. [2002] Suppose that f is a real-valued continuous function on $[a, b]$. Show that

$$\lim_{n \rightarrow \infty} \int_a^b f(x) \sin(nx) dx = 0.$$

31. [2000]

- a) State (without proof) the Stone-Weierstrass Theorem (for real-valued functions).
- b) Show that the algebra of real-valued functions generated by the set $\{1, x^2\}$ is dense in $C[0, 1]$ but is not dense in $C[-1, 1]$. (Here, $C(X)$ denotes the space of real-valued continuous functions on I equipped with the uniform topology.)

32. [2003s] Let \mathcal{A} be the real vector space spanned by the functions

$$1, \sin x, \sin^2 x, \dots, \sin^n x, \dots$$

defined on $[0, 1]$. Show that \mathcal{A} is dense in $C[0, 1]$, the space of real-valued continuous function on $[0, 1]$ equipped with the sup norm.

33. [2001] Let $f : [0, \pi] \rightarrow \mathbb{R}$ be a continuous function. Show that for every $\varepsilon > 0$, there is a trigonometric function T_n defined as

$$T_n(x) = \sum_{k=0}^n a_k \cos kx$$

such that $\sup_{0 \leq x \leq \pi} |f(x) - T_n(x)| < \varepsilon$, and explain why this conclusion no longer holds if the cosine function is replaced by the sine function.

34. [2002] Let

$$\varphi(t) = \int_0^\infty e^{-tx^2} dx, \text{ for } t > 0.$$

Find $t_0 > 0$ such that $\varphi(t_0) = 1$. Is such a point unique? Justify your computation.

35. [1998, 1995] Let $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuously differentiable function. Define for $t \in [0, 1]$ and $n = 1, 2, 3, \dots$

$$y_0(t) = e^{-t}$$

$$y_{n+1}(t) = y_n(t) + \int_0^t f(t, y_n(t)) dt$$

Show that the sequence y_n converges uniformly to a continuous function on $[0, 1]$. Hint: use the Cauchy criterion.

36. [1999] If $\{f_n\}$ is a sequence of pointwise bounded functions on $[a, b]$, show that there exists a subsequence of $\{f_n\}$ which converges on a dense subset of $[a, b]$. (Assume that the functions are \mathbb{R}^k -valued.)

37. [1999] Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a differentiable function that is homogeneous of order k (i.e., $f(\lambda x_1, \dots, \lambda x_n) = \lambda^k f(x_1, \dots, x_n)$, $\forall \lambda \in \mathbb{R}$). Show that $x_1 \frac{\partial f}{\partial x_1} + \dots + x_n \frac{\partial f}{\partial x_n} = kf$ (i.e., $\vec{x} \cdot f'(x) = kf(\vec{x})$).

38. [1999] Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a differentiable function. Define the directional derivative of f (at \vec{x}) in the direction of the unit vector $\vec{v} \in \mathbb{R}^n$. Show that this derivative has maximum modulus when \vec{v} is in the direction of the gradient (f') of f .

39. [2002]

- Give the definition of differentiability of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, at the point $a \in \mathbb{R}^n$.
- Show that if f is differentiable at a , then it is continuous at a . Is the converse true? Explain your answer.

40. [2003s] Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by

$$f(x, y) = \left(\cos \left(\sqrt{x^2 + y^2} \right), \sin \left(\sqrt{x^2 + y^2} \right) \right).$$

Show that f is continuously differentiable on $\{(x, y) \in \mathbb{R}^2 : x, y > 0\}$ and determine both directly and by using a well-known theorem whether f is locally invertible near the point $(x_0, y_0) = (1, 1)$.

41. [1998, 1997, 1995] State the Inverse Function Theorem for functions from \mathbb{R}^n into \mathbb{R}^n . What conclusion can be drawn from this theorem about the function $f(x, y) = (x + y, x^2 - y)$ near the point $(0, 0)$ in \mathbb{R}^2 ?

42. [1999, 1998] Let X be any compact subset of \mathbb{R} containing an interval (of positive length). Is it possible that $\{f \in C(X) : |f(x)| \leq 1, \forall x \in X\}$ is a compact subset of $C(X)$? Prove your assertion.

43. [1995] State the Arzela-Ascoli Theorem. Give reasons why it does or does not apply to the following collections of functions:

a) $A_1 = \{f_n : f_n(x) = x - n, n \geq 1, x \in [0, 1]\}$

b) $A_2 = \{f_n : f_n(x) = x^n, n \geq 1, x \in [0, 1]\}$

c) $A_3 = \{f_n : f_n(x) = (1 + (x + n)^2)^{-1}, n \geq 1, x \in [0, \infty)\}$

44. [1994] For $a > 0$, let f_n be a sequence of functions defined on the interval $[0, a]$ by $f_n(x) = (x/a)^n$, $n = 1, 2, \dots$. Verify that the Arzela-Ascoli theorem fails for this sequence, and explain why.

IX. NASTY INTEGRALS

1. $f : X \rightarrow [0, \infty]$ is measurable and $\int_X f d\mu = c$ where $0 < c < \infty$. Let $\alpha \in \mathbb{R}$ be a constant. Show that

$$\lim_{n \rightarrow \infty} \int_X n \log \left[1 + \left(\frac{f(x)}{n} \right)^\alpha \right] d\mu = \begin{cases} \infty & 0 < \alpha < 1 \\ c & \alpha = 1 \\ 0 & \alpha > 1 \end{cases}$$

2. Define $F(t) = \int_0^\infty \frac{e^{-xt}}{1+x^2} dx$, for $t > 0$.
- Show that F is well-defined as an improper Riemann integral and as a Lebesgue integral.
 - Show $F''(t)$ exists on $(0, \infty)$ and satisfies $F''(t) + F(t) = \frac{1}{t}$.
 - (Extra credit) Compute $F(t)$.

3. Let I be an open interval of \mathbb{R} and suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $x \mapsto e^{xt}f(x)$ is integrable for each fixed $t \in I$. Define $F : I \rightarrow \mathbb{R}$ by

$$F(t) = \int_{\mathbb{R}} e^{xt} f(x) dx.$$

Show that F is differentiable with derivative $F'(t) = \int_{\mathbb{R}} x e^{xt} f(x) dx$ at each $t \in I$.

4. [2000] Show $F(t) = \int_{-\infty}^\infty \frac{\sin(x^2 t)}{1+x^2} dx$ is continuous on \mathbb{R} .
5. [1998] $f \in C[0, 1]$ is such that $\int_0^1 x^n f(x) dx = 0$ for $n = 0, 1, 2, \dots$. Show that $f \equiv 0$.

6. Compute the limits

a) $\lim_{n \rightarrow \infty} \int_0^\infty \left(1 + \frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right) dx$

b) $\lim_{n \rightarrow \infty} \int_a^\infty n(1 + n^2 x^2)^{-1} dx$

7. a) Find the smallest constant c such that $\log(1 + e^t) < c + t$ for $0 < t < \infty$.
- b) Does $\lim_{n \rightarrow \infty} \frac{1}{n} \int_0^1 \log(1 + e^{nf(x)}) dx$ exist for every real $f \in L^1[0, 1]$, if $f > 0$?

X. THE RADON-NIKODYM THEOREM

Theorem. (Radon-Nikodym) Let (X, \mathcal{A}) be a measurable space, and let μ and ν be σ -finite positive measures on (X, \mathcal{A}) . If $\nu \ll \mu$, then there is an \mathcal{A} -measurable function $f : X \rightarrow [0, \infty)$ such that $\nu(A) = \int_A f d\mu$ holds for each $A \in \mathcal{A}$. The function f is unique up to μ -ae equality. We denote $f = \frac{d\nu}{d\mu}$.

1. (The Sum Rule) $d(\nu_1 + \nu_2)/d\mu = \frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu}$.
2. (The Chain Rule) Suppose ν_1 is a σ -finite signed measure and ν_2, ν_3 are σ -finite positive measures on (X, \mathcal{A}) such that $\nu_1 \ll \nu_2$ and $\nu_2 \ll \nu_3$.
 - a) If $g \in L^1(\nu_1)$, then $g(\frac{d\nu_1}{d\nu_2}) \in L^1(\nu_2)$ and $\int g d\nu_1 = \int g \frac{d\nu_1}{d\nu_2} d\nu_2$.
 - b) We have $\nu_1 \ll \nu_3$, and $\frac{d\nu_1}{d\nu_3} = \frac{d\nu_1}{d\nu_2} \frac{d\nu_2}{d\nu_3}$.
 - c) If $\nu_2 \ll \nu_1$ and $\nu_1 \ll \nu_2$, then $\frac{d\nu_1}{d\nu_2} \frac{d\nu_2}{d\nu_1} \stackrel{\text{ae}}{=} 1$ (with respect to μ or ν).
3. Let μ and ν be measures on (X, \mathcal{A}) , and suppose μ is σ -finite.
 - a) If ν is σ -finite, show that the following are equivalent:
 - (i) $\nu \ll \mu$ and $\mu \ll \nu$,
 - (ii) μ and ν have exactly the same sets of measure zero, and
 - (iii) there is a \mathcal{A} -measurable function g that satisfies $0 < g(x) < \infty$ at each $x \in X$ and is such that $\nu(A) = \int_A g d\mu$ holds $\forall A \in \mathcal{A}$.
 - b) Show that if μ is σ -finite measure on (X, \mathcal{A}) , then there is a finite measure ν on (X, \mathcal{A}) such that $\nu \ll \mu$ and $\mu \ll \nu$.
4. Let μ be counting measure on \mathbb{Q} . Show μ is σ -finite but $\mu(a, b) = \infty, \forall a < b$.
5. Let $\{A_k\}$ be a sequence of measurable sets such that $\sum_n \mu(A_n) < \infty$. Then the set of points that belong to A_k for infinitely many values of k has measure 0 under μ . (Hint: consider $\cap_{n=1}^\infty \cup_{k=n}^\infty A_k$ and note that $\mu(\cap_{n=1}^\infty \cup_{k=n}^\infty A_k) \leq \mu(\cup_{k=p}^\infty A_k) \forall p$.)
6. Let $\{q_n\}$ be an enumeration of the rational numbers, and for each $n \in \mathbb{N}$, let $f_n : \mathbb{R} \rightarrow \mathbb{R}$ be a nonnegative Borel function that satisfies $\int f_n d\lambda = 1$ and vanishes outside the closed interval of length $1/2^n$ centered at q_n . Define μ on $\mathcal{B}(\mathbb{R})$ by $\mu A = \int_A \sum_n f_n d\lambda$.
 - a) $\sum_n f_n(x) < \infty$ holds at λ -ae x in \mathbb{R} . (Hint: see prev. exercise.)
 - b) μ is σ -finite, that $\mu \ll \lambda$, and each non-empty open subset of \mathbb{R} has infinite measure under μ .