RESEARCH STATEMENT

ERIN PEARSE

http://www.math.cornell.edu/~erin/

INTRODUCTION

In [La-vF1], the authors lay the foundations for a theory of complex dimensions with a rather thorough investigation of the theory of fractal strings; that is, fractal subsets of the real line. Such an object may be represented by a sequence of bounded open intervals which are labeled according to their length. The authors are able to relate geometric and physical properties of such objects through the use of zeta functions which encode geometric and spectral information about the given string. This information includes dimension and measurability properties and yields precise asymptotic expansions for the geometric and spectral counting functions; capturing the intrinsic oscillations in the geometry, spectrum, and dynamics. Indeed, one also obtains a tube formula for the fractal object being studied.

The common thread that ties these ideas together is the notion of 'complex dimensions'; a generalization of real-valued dimension defined as the set of poles of the geometric zeta function associated with the fractal (i.e. a generating function for the geometry of the string). This is in accordance with the result that the Minkowski dimension of a fractal string is the abscissa of convergence of its geometric zeta function [La1]. The resulting theory sheds new light on the Riemann hypothesis, including a geometric interpretation, and provides connections to further reaches of mathematics, including spectral geometry, noncommutative geometry, and dynamical systems.

PUBLICATIONS

A tube formula for the Koch snowflake curve, with applications to complex dimensions (with M. L. Lapidus), J. London Math. Soc., accepted Sept. 2005. 18 pages. arXiv: math-ph/0412029.

Canonical self-similar tilings by IFS, submitted to Indiana Univ. Math. J., July 2006. 16 pages. arXiv: math.MG/0606111.

Tube formulas and complex dimensions of self-similar tilings (with M. L. Lapidus), submitted to Memoirs Amer. Math. Soc., July 2006. 61 pages. arXiv: math.DS/0605527.

Curvature measures and tube formulas for compact sets (with M. L. Lapidus), in prep.

Complex Dimensions of Self-Similar Systems. Ph.D. Thesis, June 2006.

DESCRIPTION OF DISSERTATION

My dissertation extends many of the concepts and significant findings of [La-vF1] to fractal subsets of higher-dimensional Euclidean spaces. I intend to use my results as a foundation for the study of curvature measures of fractal sets, and the analysis of measurability properties in the higher-dimensional case.

The dissertation begins with a direct computation of the tube formula for the Koch snowflake curve, via geometric considerations and extensive calculations; see [LaPe1]. The resulting formula is a sum taken over the numbers $\{\log_3 4 + 2\pi i n/\log 3\}$ and a sum taken over $\{2\pi i n/\log 3\}$ (here n is an integer), which we expect to be the complex dimensions of the Koch snowflake curve. This preliminary result serves as a way to check the theory of the ensuing chapters; the general tube formula should match this one when applied to the Koch curve. Additionally, the investigations leading to the results obtained in this chapter suggest a different approach to the study of fractal strings than that outlined in [La-vF1]. In particular, it indicates that one should consider the function system which generates the Koch curve as the primary object, rather than the curve itself. As a side benefit, the tube formula shows that the Koch curve is not Minkowski measurable, as has long been conjectured.

Next, the dissertation develops a framework suitable for the general analysis of self-similar objects, the self-similar tiling; see [Pe2]. The tiling is the geometric object which is the natural higher-dimensional counterpart of the fractal strings studied in [La-vF1]. Roughly speaking, the tiling is the obtained in 4 steps. (1) Begin with an iterated function system (IFS) where the functions are contraction similitudes. (2) Take the convex hull of the attractor of this system. (3) The images of the hull under the mappings forms a subset of the hull. One takes the connected components of the complement of this subset as the generators of the tiling. (4) Iterated application of the mappings to the generators produces a tiling of the complement of the original attractor. Typically, the tiling yields a decomposition of the complement of the fractal into relatively simple polyhedra to which the results of convex geometry may then be applied.

The tiling is used to define a measure as a method of encoding certain geometric data about the tiling (and hence also about the function system and associated attractor set). Then, the relationship between measures and zeta functions, as studied in [La-vF1], allows one to construct an associated zeta function (see also [LaPe3]) which acts like a generating function for the geometry of the tiling. The approach here is guided by the 1-dimensional case, with an eye towards the general tube formula (which is obtained as an expansion over the poles of the zeta function).

More precisely, for a given tiling we have a *scaling measure* which encodes all the scaling ratios which occur under repeated application of the self-similar system; and we have a *geometric measure* which comes from the scaling measure and encodes the "sizes" of all the tiles. Here, the size is measured by *inradius*, the radius of the largest Euclidean ball contained in the set. The idea is that the geometric measure encodes the density of geometric states of the tiling by recording the size and type of components which contribute volume to the tube formula. The *scaling zeta function* of a given tiling is defined as the Mellin transform of the measure, and the *scaling (complex) dimensions* of the tiling are defined to be poles of the scaling zeta function. The structure of this set of complex dimensions turns out to be identical to that of the 1-dimensional case, as described in [La-vF1]. The dissertation includes several examples with illustrated tilings, and the associated measures and zeta functions for each.

For each generator, there is a *generator tube formula*, and these are essential ingredients in the tube formula for the tiling. For the moment, such a formula for the generators is assumed to be of a certain form consistent with constructive geometric methods. Explicit expressions are readily obtained for most well-known examples and a general formula is currently in development. The generator tube formula allows one to construct a certain polynomial in x^{-1}, which constitutes a suitable test function with regard to the distributional theory developed in [La-vF1].

From the self-similar tiling, scaling zeta function, and generator tube formula, one can finally construct the *geometric zeta function of the self-similar tiling*. This turns out to be a distribution-valued meromorphic function and it acts as a generating function for the geometric properties of the entire tiling. Roughly, the geometric zeta function is the inner product of the vector of inradii of the generators with a vector related to the *integral dimensions* of the generators. (The integer dimensions of a convex compact n-dimensional subset of Euclidean space are $\{0,1,\ldots,n\}$.) The inner product is defined by a matrix whose entries appear to play the role of *curvature measures*, in the sense of convex geometry.

Distributional arguments and various results of [La-vF1] then yield an explicit *tube formula for the self-similar tiling* which is given as a sum over the complex dimensions (that is, the scaling dimensions and the integral dimensions) and a precise error term. The sum in this formula has roughly the form of a power series expansion in ε taken over *all* the complex dimensions, with coefficients given by the residues of the geometric zeta function. There is an immediate analogy with the Steiner formula for convex compact sets, or with Weyl's tube formula for smooth submanifolds. In both of these classical cases, the tube formula takes the form of a polynomial in ε (i.e., a sum over the integral dimensions), where the coefficients are given by curvature measures. In the 1-dimensional case, the tube formula for self-similar tilings is shown to devolve into the tube formula for fractal strings as given in [La-vF1]. Further, in the case of trivial self-similarity (e.g., the square as the attractor of four maps with contraction ratio ½), the scaling zeta function is holomorphic. Consequently, the set of complex dimensions is just the (finite) set of integral dimensions, and the entire formula collapses into a Steiner-type formula for interior ε-neighbourhoods. Thus, complex dimensions provide a bridge between two approaches to the study of fractals (self-similarity theory and geometric measure theory) via the tube formula.

As a conclusion to the dissertation, the general results are applied to several examples, and some directions for future research are given.

RESEARCH IN PROGRESS

I am currently developing a general explicit tube formula for polyhedral generators, using techniques from geometric measure theory. It should not be hard to provide some sufficient conditions for the generators to be polyhedral. To extend to the general case, one has the option to (i) approximate a given tiling (in a suitable sense) by a sequence of tilings with polyhedral generators, or (ii) apply recent results of [HLW] and [Sc] to obtain directly a tube formula for the generators of the tiling. Again, these approaches show the volume of the ε-neighbourhood of the tiling to be (roughly) a polynomial in ε, similar to the expressions studied by Steiner and Weyl, and generalized by Federer; see [LaPe3]. The latter option (ii) is applicable very broadly, but is not well adapted to explicit computation, due to its abstract nature (it is defined in terms of the normal bundle of the set under scrutiny, and certain "reach measures").

An exciting aspect of the results of the dissertation is the appearance of something which seems to be a fractal analogue of curvature measure. This is somewhat speculative and is currently being investigated further in another work [LaPe3]. One hopes to establish a precise description of a "fractal Steiner formula"; presumably by connecting the results of my dissertation with recent developments by Winter concerning fractal curvature measures [Wi] and fractal Euler characteristic [LW] via a different approach. Eventually, the coefficients of this fractal tube formula should be understood as the appropriate fractal analogue of curvature measure, and should extend the classical results of Federer. Additionally, in the long term, I would like to pursue possible connections of this new material to fractal (co)homology theory, as introduced by Lapidus and R. Nest in [LaNe].

Although my dissertation extends many results of [La-vF1] to higher-dimensional spaces and provides the framework for further study, it does not completely explore the higher-dimensional case. In particular, I should like to leverage the self-similar tilings developed in my dissertation to extend the results of [La-vF1] concerning the spectral asymptotics of fractal domains (or domains with fractal boundary) insofar as this is possible. To this end, I am also currently studying recent results pertaining to analysis and probability on fractals, with an emphasis on differential equations on fractal domains (and domains with fractal boundaries). The Laplacian is now relatively well understood on a broad class of self-similar sets via the probabilistic methods of Barlow and others [BaPe, Ba], and the more recent constructive techniques introduced by Kigami [Ki]. Recent work by Strichartz, Teplyaev, Sabot, and others is building an extensive theory of fractal harmonic analysis (see especially [St]). In particular, Strichartz and Rogers have recently constructed smooth bump functions on certain post-critically finite (pcf) and non-pcf fractals, and are using them to develop a theory of distributions on fractals.

At some point, I also hope to learn more about Noncommutative Geometry so that I may explore the burgeoning connections between fractal geometry and Noncommutative Geometry, as described in [La2] and [La3], and in the more recent work [CIL] on the noncommutative Hausdorff and Minkowski dimension.

RESEARCH PLAN

1. Develop the inner tube formula for generators, as mentioned above, first for the polyhedral case. Use these results to study the coefficients of the tiling tube formula as fractal curvatures and connect this approach with recent results of Winter [Wi, LlWi].

2. Study the precise relationship between the epsilon-neighbourhood of the tiling developed in [Pe2] and the epsilon-neighbourhood of the attractor (self-similar set) itself. Simple conditions suffice to show equality for some cases. More generally, estimates will be needed. It should be possible to use this connection to characterize completely the Minkowski measurability of an arbitrary self-similar set.

3. Determine under what conditions (if any) the self-similar tiling can provide insight into the spectral asymptotics of an arbitrary self-similar set, and relate the spectral and geometric zeta functions, extending the results of [La-vF3].

4. Generalize the results of the self-similar case by allowing contractions which are not similarity transformations, i.e., determine what may be said when similitudes are replaced by self-affine and more general injective maps. One strategy for this may be to exploit recent developments in the study of self-homeomorphic fractals as introduced in [Hv].

5. Determine what progress may be made towards the completely general case when there is no self-similarity. In the 1-dimensional case, there is a fair body of results which hold for any bounded open subset of the real line, devoid of any self-similar or fractal structure; some analogous results should also exist for arbitrary bounded open subsets of $\mathbf{R}^d$. Teplyaev has made recent advances in this area by studying finitely ramified cell structures and using harmonic coordinates [Te].

6. Repeat key investigations with a 2-sided neighbourhood instead of the 1-sided neighbourhood, as is currently being done, so as to settle a conjecture of Lapidus: the 2-sided approach should be a subset of the complex dimensions of the 1-sided approach. This has already been seen in some examples. I would like to understand these cancellations, and compare this phenomenon to Weyl's results in the smooth case.

7. Study quasiperiodic and multiplicatively almost-periodic functions in general. While investigating the Koch snowflake curve in [LaPe1], we encountered a multiplicatively periodic function closely tied to the "approximation level" of the curve. Such functions likely exist for all sets with self-similar structure and need to be understood more clearly.

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