#### Abstract of the Dissertation Complex Dimensions of Self-Similar Systems by

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The theory of complex dimensions of self-similar subsets of  $\mathbb{R}$  is studied in some detail in [La-vF4]. This dissertation extends that theory to self-similar sets of  $\mathbb{R}^d$ . The major themes of this work (as reflected in the title) are: (i) the unification of different aspects of mathematics via the theory of *complex dimensions*, and (ii) that when studying self-similarity, it is necessary to take the defining system of mappings as the primary object of study.

A self-similar set F (or *attractor*) is one satisfying

$$F = \bigcup_{j=1}^{J} \Phi_j(F)$$

for some family of contraction similarities  $\Phi = {\Phi_j}$ , henceforth called a *self-similar* system. In order to define the zeta functions which lie at the heart of this study, we first define a *self-similar tiling*; a canonical decomposition of the complement of the attractor F of the self-similar system  $\Phi$ , within its convex hull. The tiling shares key properties of the system itself and allows for the extension of the theory of complex dimensions to higher-dimensional fractal sets. The tiles lie in a neighbourhood of the fixed point F; by examining the dynamics of  $\Phi$  on them, we study more than just the fixed point of the system.

A zeta function, a generating function for the geometry of the object, is defined in terms of the action of  $\Phi$  on the tiling. The *complex dimensions* of the system (or tiling) are the poles of this zeta function.

A key result of [La-vF4] is the (explicit) tube formula for fractal strings (i.e., 1dimensional tilings). This dissertation obtains a higher-dimensional analogue of this result and exposes connections to Geometric Measure Theory in the process. It turns out that the tube formula for tilings is also a fractal extension of the classical Steiner formula. Instead of being a polynomial in  $\varepsilon$  summed over the integers  $\{0, \ldots, d\}$ , however, the tube formula for tilings is a power series which also includes one term for each complex dimension. This further justifies the term 'complex dimensions' and takes a step toward defining curvature for a fractal.

This dissertation should have applications to spectral asymptotics on domains which are fractal or have fractal boundaries. It may also lead to a robust notion of curvature (measures) for self-similar sets.

## Chapter 1

# Introduction

### 1.1 Background

In [La-vF4], Lapidus and van Frankenhuijsen lay the foundations for a theory of complex dimensions with a rather thorough investigation of the theory of fractal strings; that is, fractal subsets of  $\mathbb{R}$ .<sup>1</sup> Such an object may be represented by a sequence of bounded open intervals  $L = \{L_n\}_{n=1}^{\infty}$  with lengths

$$\mathcal{L} := \{\ell_n\}_{n=1}^{\infty}, \quad \text{with } \sum_{n=1}^{\infty} \ell_n < \infty.$$
(1.1)

The positive numbers  $\ell_n$  are the lengths of the connected components (open intervals) of L, written in nonincreasing order.

The authors of [La-vF4] are able to relate geometric and physical properties of such objects through the use of zeta functions which contain geometric and spectral information about the given string. This information includes the dimension and measurability of the fractal under consideration, which we now recall.

For a nonempty bounded open set  $L \subseteq \mathbb{R}$ ,  $V_L(\varepsilon)$  is defined to be the volume of the inner  $\varepsilon$ -neighborhood of L:

$$V_L(\varepsilon) := \operatorname{vol}_1\{x \in L : dist(x, \partial L) < \varepsilon\},$$
(1.2)

where vol<sub>1</sub> denotes 1-dimensional Lebesgue measure. In general, a *tube formula* is an explicit expression for  $V_L(\varepsilon)$  as a function of  $\varepsilon$ . As it is shown in [LaPo1] that  $V_L$  depends exclusively on  $V_{\mathcal{L}}$ , the tube formula for a fractal string is defined to be

$$V_{\mathcal{L}}(\varepsilon) := V_L(\varepsilon). \tag{1.3}$$

<sup>&</sup>lt;sup>1</sup>For supplementary references on fractal strings, see [LaPo1–2,LaMa,La1–3, HaLa, HeLa,La-vF2–3].

The Minkowski dimension of the boundary  $\partial L$  (i.e., of the fractal string  $\mathcal{L}$ ) is

$$D = D_{\partial L} = \inf\{t \ge 0 : V(\varepsilon) = O(\varepsilon^{1-t}) \text{ as } \varepsilon \to 0^+\}.$$
 (1.4)

The set  $\partial L$  is *Minkowski measurable* if and only if the limit

$$\mathcal{M} = \mathcal{M}(D; \partial L) = \lim_{\varepsilon \to 0+} V(\varepsilon)\varepsilon^{-(1-D)}$$
(1.5)

exists, and lies in  $(0, \infty)$ . In this case,  $\mathcal{M}$  is called the Minkowski content of  $\partial L$ .  $\mathcal{M}$  is not a measure as it is not countably additive. Minkowski–Bouligand dimension (also called 'box dimension' and other names) and Minkowski content are discussed extensively in the literature. See, e.g., [Man,Tr,La1, LaPo1–2,Mat,La-vF4] and the relevant references therein for further information.

A primary goal of this dissertation is to extend much of the 1-dimensional theory of fractal strings to fractal subsets of higher-dimensional Euclidean spaces. Reconsidering the above definitions, if L is an open subset of  $\mathbb{R}^d$ , then analogous statements hold if 1 is replaced by d in (1.2)–(1.5). In this case, vol<sub>d</sub> denotes the d-dimensional volume (which is area for d = 2) in the counterpart of (1.2).

Much of the geometric information about a fractal string is encoded in its *geomet*ric zeta function, defined to be the meromorphic extension of

$$\zeta_{\mathcal{L}}(s) = \sum_{n=1}^{\infty} \ell_n^s. \tag{1.6}$$

It is shown in [La-vF4, Thm. 1.11] that

$$D = \inf\{\sigma \ge 0 : \sum_{n=1}^{\infty} \ell_n^{\sigma} < \infty\},\tag{1.7}$$

i.e., that the Minkowski dimension of a fractal string is the abscissa of convergence of its geometric zeta function [La2]. In accordance with this result, the *complex dimensions of*  $\mathcal{L}$  are defined to be the set

$$\mathcal{D}_{\mathcal{L}} = \{ \omega \in \mathbb{C} : \zeta_{\mathcal{L}} \text{ has a pole at } \omega \}.$$
(1.8)

One reason why these complex dimensions are important is the explicit tubular formula for fractal strings, a key result of [La-vF4]. Namely, under suitable conditions on the string  $\mathcal{L}$ , one has the following *tube formula*:

$$V_{\mathcal{L}}(\varepsilon) = \sum_{\omega \in \mathcal{D}_{\mathcal{L}}} \operatorname{res}\left(\zeta_{\mathcal{L}}(s);\omega\right) \frac{(2\varepsilon)^{1-\omega}}{\omega(1-\omega)} + \mathcal{R}(\varepsilon), \tag{1.9}$$

where the sum is taken over the complex dimensions  $\omega$  of  $\mathcal{L}$ , and the error term  $\mathcal{R}(\varepsilon)$  is of lower order than the sum as  $\varepsilon \to 0^+$ . (See [La-vF4, Thm. 6.1, p. 144].) For the present discussion, the important thing about this formula is its general form:

$$V_{\mathcal{L}}(\varepsilon) = \sum_{\omega \in \mathcal{D}_{\mathcal{L}}} c_{\omega} \varepsilon^{1-\omega} + \text{ error}, \qquad (1.10)$$

In the case when  $\mathcal{L}$  is a self-similar string i.e., when  $\partial \mathcal{L}$  is a self-similar subset of  $\mathbb{R}$ , one distinguishes two complementary cases:

- (1) In the *lattice case*, i.e., when the underlying scaling ratios have rationally dependent logarithms, the error term is shown to vanish identically and the complex dimensions lie periodically on finitely many vertical lines (including the line  $\operatorname{Re} s = D$ ). In this case, there are infinitely many complex dimensions with real part D.
- (2) In the *nonlattice case*, the complex dimensions are quasiperiodically distributed and s = D is the only complex dimension with real part D. Estimates for  $R(\varepsilon)$ are given in [La-vF4, Thm. 6.20, p. 154] and more precisely in [La-vF2–3]. Also,  $\mathcal{L}$  is Minkowski measurable if and only if it is nonlattice.

See [La-vF4, Chap. 2–3] for a discussion of quasiperiodicity, and see [L,La3] and [La-vF4, §2.3] for further discussion of the lattice/nonlattice dichotomy.

The results described above hold only for fractal subsets of  $\mathbb{R}$ . It is the goal of this dissertation to develop the higher-dimensional analogues of these results and ideas. The primary goal is the tube formula  $V_{\mathcal{T}}(\varepsilon)$  for self-similar tilings (1.12) obtained in Chapter 5. This result is central to the others (as it incorporates aspects of all the other notions discussed), and has the advantage of being independently verifiable; this is carried out for the Koch tiling in Rem. 6.1.3 of Example 6.1.2.

#### 1.2 Overview

Chapter 2 gives a computation of the tube formula for the Koch snowflake curve by hand, i.e., via basic geometric considerations and much computation. The result is a formula of the form

$$V(\varepsilon) = \sum_{n \in \mathbb{Z}} \varphi_n \varepsilon^{2 - D - in\mathbf{p}} + \sum_{n \in \mathbb{Z}} \psi_n \varepsilon^{2 - in\mathbf{p}},$$

given in full detail in §2.5. This preliminary result serves as a guide and a way to check the theory of the ensuing chapters; the general tube formula should match this one when applied to the Koch curve.

Additionally, the investigations leading to the results obtained in this chapter suggest a different approach. As described in §2.7, it becomes apparent that one ought to consider the function system which generates the Koch curve as the primary object, rather than the curve itself.

Chapter 3 concerns the development of a framework suitable for the general analysis of self-similar objects, the self-similar tiling constructed in [Pe]. The self-similar tiling  $\mathcal{T}$  is the the natural higher-dimensional counterpart of the *self-similar* fractal strings studied in [La-vF4].

To get the flavour of the self-similar tilings, it may be helpful to preview some of the examples given in §3.2.3; especially Fig. 6.1–3.2. Roughly speaking, the tiling  $\mathcal{T}$ is obtained in 4 steps. (1) Begin with an iterated function system (IFS) where the functions are contraction similitudes  $\{\Phi_j\}$ . (2) Take the convex hull of the attractor of this system. (3) The image of the hull under these mappings is a subset of the hull itself. The components of the complement of this subset will be the generators of the tiling. (4) The successive iteration of the mappings  $\{\Phi_j\}$  on the generators produces a tiling of the original attractor. The details of the construction are given in §3.2.

Chapter 4 develops the notion of inradius, the higher-dimensional analogue of the length  $\ell_n$ . The scaling measure and geometric measure are then defined using the inradius. More precisely, the *scaling measure* encodes all the scaling ratios that occur under iteration of the self-similar system (a type of iterated function system defined in Def. 3.2.1); and the *geometric measure* comes from the scaling measure and encodes the 'sizes' of all the tiles. The geometric measure gives the density of geometric states of the tiling; it records the size and type of tiles occurring in the tiling  $\mathcal{T}$ .

Also, a given tiling has a scaling zeta function  $\zeta_{\mathfrak{s}}$  which is defined as the Mellin transform of the scaling measure; and a tiling zeta function  $\zeta_{\mathcal{T}}$  which is defined in terms of the scaling zeta function. The scaling zeta function is similar in form to the geometric zeta function of a string (1.6), but is defined entirely in terms of the scaling ratios of the similarity transformations  $\{\Phi_j\}$ . The tiling zeta function is closely related, but is actually a generating function for the geometry of the tiling; it incorporates geometric data in the form of the inradii of the generators of the tiling. Accordingly, the tiling zeta function is vector-valued, with one component for each generator  $G_q$ . It is possible for two different self-similar tilings to have the same scaling zeta function, but the tiling zeta function characterizes it uniquely.

The *complex dimensions* of the tiling are defined to be the poles of the scaling zeta function. The structure of the complex dimensions of a self-similar tiling turns out to be identical to the complex dimensions of a fractal string, as studied in [La-vF4].

The second section of Chapter 4 describes how the pieces of the puzzle fit together, i.e., how the tube formula  $V_{\mathcal{T}}(\varepsilon)$  is understood as a distribution, and the general strategy for assembling the tube formula from the various ingredients (measures, zeta functions, etc.).

Chapter 5 uses the self-similar tiling and associated zeta functions to develop a tube formula for self-similar tilings. The generality of the theory of distributional

explicit formulas developed in [La-vF4] makes it perfectly suitable for self-similar tilings. Although the main result of this chapter is obtained for fractal sprays (a slightly more general object), we are primarily interested in its applications to self-similar tilings.

The extended distributional formula [La-vF4, Thm. 5.26] is used to obtain an expression for the distributional action of the geometric measure  $\eta_{\mathfrak{g}}$  on a specific test function  $v_{\varepsilon}(x)$ :

$$V_{\mathcal{T}}(\varepsilon) = \langle \eta_{\mathfrak{g}}, v_{\varepsilon} \rangle. \tag{1.11}$$

This function  $v_{\varepsilon}(x)$  is a polynomial in  $\varepsilon$  which gives the tube formula for a tile of  $\mathcal{T}$  that has inradius 1/x. Development of this expression produces the higherdimensional tube formula given in Theorem 5.4.2:

$$V_{\mathcal{T}}(\varepsilon) = \sum_{\omega \in \mathcal{D}_{\mathfrak{s}} \cup \{0, 1, \dots, d\}} c_{\omega} \varepsilon^{d-\omega} + \mathcal{R}(\varepsilon), \qquad (1.12)$$

where  $\mathcal{D}_{\mathfrak{s}}$  is the set of poles  $\omega$  of the scaling zeta function  $\zeta_{\mathfrak{s}}$ , and each coefficient  $c_{\omega}$  in (1.12) is defined in terms of the residue of the tiling zeta function at the complex dimension  $\omega$ . Thm. 5.4.6 shows how the error term  $\mathcal{R}$  vanishes for self-similar tilings.

It is clear that (1.12) is an extension of (1.10), it is shown in §5.4.1 precisely how the two coincide for d = 1. However, much more is true. When the tileset condition (defined in Def. 3.2.4) is not satisfied, the self-similar system has a convex or 'trivially self-similar' attractor, and the tube formula (1.12) devolves into a close relative of the classical Steiner tube formula for convex or polyconvex sets; this is discussed in Rem. 5.4.4. Steiner's formula is discussed in more detail in Rem. 5.2.4 and Rem. 5.2.5 of §5.2, but it is essentially a polynomial in  $\varepsilon$  of the form

$$V_A(\varepsilon) = \sum_{i \in \{0,1,\dots,d-1\}} c_i \varepsilon^{d-i}.$$
 (1.13)

Thus, the investigations described in this dissertation have uncovered connections between fractal geometry and geometric measure theory! While these two fields are thematically intimate, there has not previously been much overlap in the way of formulas or specific results. The relationships discovered in the course of the research leading to this dissertation yield new insights into the 1-dimensional theory of fractal strings and provide new geometric interpretations for previous results. Additionally, some of the ideas developed herein may allow for the development of a rigourous notion of fractal curvature in the near future. In particular the coefficients  $c_{\omega}$  appearing in (1.12) are almost exactly equal to those in (1.13) for  $\omega = \{0, 1, \ldots, d-1\}$ , as one might expect by comparison with the tube formulas of Steiner, Weyl, and Federer; see [Schn2, We,Fed]. This leads one to believe that the other coefficients  $c_{\omega}$  (which are defined in terms of residues of the tiling zeta function  $\zeta_{\mathcal{T}}$  at  $\omega$ ) may also have an interpretation in terms of curvature.