MATH 211A – FINAL EXAM

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1. Consider the pendulum equation $x'' = -\sin x$.

(a) Prove it is a Hamiltonian system.

First make the standard transformation into a system of first-order equations: let $y = x'$ so that $y' = x''$ and the system becomes

$$\begin{cases} x' = y \\ y' = -\sin x \end{cases}$$

Then take $H(x, y) = \frac{y^2}{2} - \cos x$. Then

$$\tfrac{\partial H}{\partial y} = y \quad \text{and} \quad -\tfrac{\partial H}{\partial x} = -\sin x,$$

so we have

$$\begin{cases} x' = \frac{\partial H}{\partial y} \\ y' = -\frac{\partial H}{\partial x} \end{cases}$$

(b) Find the general solution (in integral form) and sketch the phase plane.

We proceed by integrating each side of $x'' = -\sin x$ to obtain

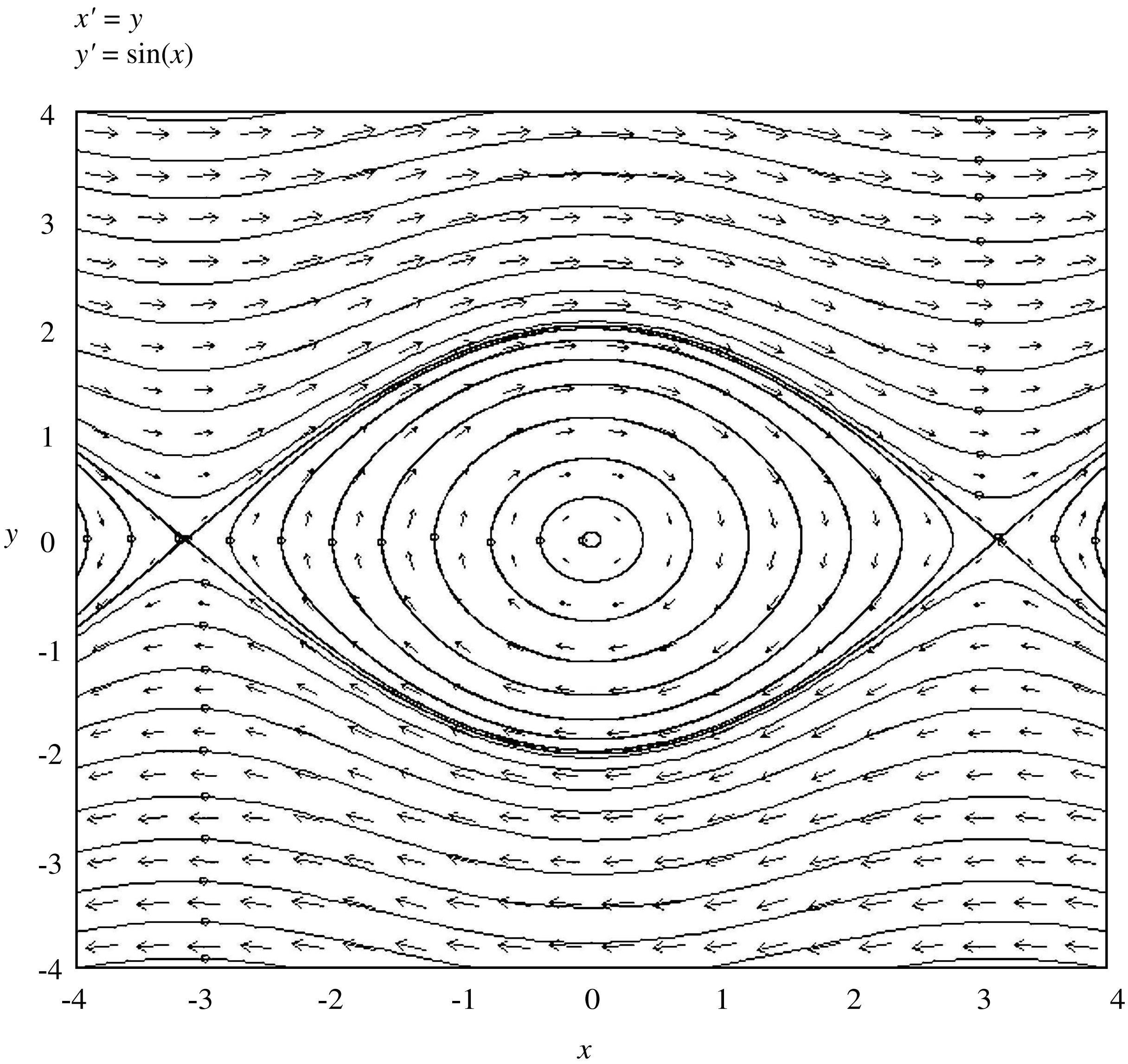
$$x'(t) = x_0 - \int_{t_0}^{t} \sin x(s)\, ds.$$

Then, considering $F(t) := x_0 - \int_{t_0}^{t} \sin x(s)\, ds$, we integrate again and obtain

$$\begin{aligned} x(t) &= x_1 + \int_{t_0}^{t} F(u)\, du \\ &= x_1 + \int_{t_0}^{t} \left(x_0 - \int_{u_0}^{u} \sin x(s)\, ds \right) du \\ &= x_1 + \int_{t_0}^{t} x_0\, du - \int_{t_0}^{t} \int_{u_0}^{u} \sin x(s)\, ds\, du \\ &= x_1 + x_0(t - t_0) - \int_{t_0}^{t} \int_{u_0}^{u} \sin x(s)\, ds\, du \end{aligned}$$

This is verified to be correct by differentiating twice with respect to t and applying the Fundamental Theorem of Calculus each time.

FIGURE 1. $x'' = \sin x$.



2. For $f \in C^1(\mathbb{R}^n)$ and each $x_0 \in \mathbb{R}^n$, prove the initial value problem

$$\begin{cases} x' = \frac{f(x)}{1+f(x)}, \\ x(0) = x_0 \end{cases}$$

has a unique solution for all $t \in \mathbb{R}$.

It would be nice to use Gronwall's inequality, but this is not possible as we have neither the continuity of $1/(1+f(x)$ nor a relation between $x(t)$ and $f(x)$ or $1/(1+f(x)$, so we will have to use the autonomy of the system instead.
Define

$$F(t,x) = F(x) := \frac{f(x)}{1+f(x)} \quad \text{and} \quad E := \{(t,x) \vdots f(x(t)) \neq -1\}$$

so that F has continuous partial derivatives

$$\frac{\partial F}{\partial x_i} = \frac{\frac{\partial f}{\partial x_i}(x)}{(1+f(x))^2}$$

throughout E. If $E = \varnothing$ then the system is degenerate, so let $x_0 \in E$. Then if the complement of E is dense in $\mathbb{R}^n$, $E = \varnothing$ by continuity, so we can find some open connected domain B such that $x_0 \in B \subseteq E$.

Note that $F(t,x)$ is defined and continuous in B. By the Existence and Uniqueness Theorem, there is a solution $x = \varphi(t)$ satisfying the system

$$\begin{cases} x' = F(t,x) \\ x(0) = x_0 \end{cases}$$

and defined in some neighbourhood of $(0, x_0)$. Now note that $x' = F(t,x) = F(x)$ is an autonomous equation! This fact may be exploited. Let $s \in \mathbb{R}$. Then

$$\frac{\partial \varphi}{\partial t}(t+s)\Big|_{t=t_0} = \frac{\partial \varphi}{\partial t}(t)\Big|_{t=t_0+s} = F(\varphi(t))\Big|_{t=t_0+s} = F(\varphi(t+s))\Big|_{t=t_0}.$$

We have just shown that if $\varphi(t)$ is a solution to the autonomous system, then $\varphi(t+s)$ will also be a solution, for any $s \in \mathbb{R}$. This makes it clear that the solution $\varphi(t)$ must exist for all time. Otherwise, if it had some bounded maximal interval of existence (t_1, t_2), we would have a contradiction: for $t_0 \in (t_1, t_2)$,

$$\frac{\partial \varphi}{\partial t}(t_0) = F(\varphi(t_0)),$$

by definition of interval of existence. But then letting $s = (t_2 - t_0)$, the previous argument shows that

$$\frac{\partial \varphi}{\partial t}(t_2) = \frac{\partial \varphi}{\partial t}(t_0 + s) = F(\varphi(t_0+s)) = F(\varphi(t_2)),$$

contradicting the maximality of the interval. Indeed, taking $s = \alpha(t_2 - t_0)$ for any $\alpha > 0$ shows that $\varphi(t)$ is a valid solution for all positive time. A symmetric argument shows $\varphi(t)$ is also a valid solution for all negative time.

3. Use Liapunov's method to determine the stability of the critical point (0,0,0) of the system

$$\begin{cases} x_1' = -2x_2 + x_2x_3 - x_1^3, \\ x_2' = x_1 - x_1x_3 - x_2^3, \\ x_3' = x_1x_2 - x_3^3. \end{cases}$$

We try the Liapunov candidate

$$V(x) = ax_1^2 + bx_2^2 + cx_3^2.$$

Then

$$\begin{aligned} \dot{V} &= 2ax_1\dot{x}_1 + 2bx_2\dot{x}_2 + 2cx_3\dot{x}_3 \\ &= -4ax_1x_2 + 2ax_1x_2x_3 - 2ax_1^4 + 2bx_1x_2 - 2bx_1x_2x_3 - 2bx_3^4 + 2cx_1x_2x_3 - 2cx_3^4 \\ &= -2\left(ax_1^4 + bx_2^4 + cx_3^4\right) + 2(a-b+c)x_1x_2x_3 - 2(2a-b)x_1x_2 \end{aligned}$$

To make things simpler, let $b = 2a$ to make the last term vanish. Then make $c = a$ to make the second term vanish. For definiteness, use $a = c = 1$, $b = 2$. Thus, the Liapunov function

$$V(x, y) = x_1^2 + 2x_2^2 + x_3^2$$

is positive definite and

$$\dot{V}(x, y) = -2\left(x_1^4 + 2x_2^4 + x_3^4\right) < 0, \quad \text{and} \quad \dot{V}(0, 0) = 0,$$

shows that $\dot{V}$ is negative definite. This is enough to show that $(0, 0, 0)$ is stable, but we can go even further and define

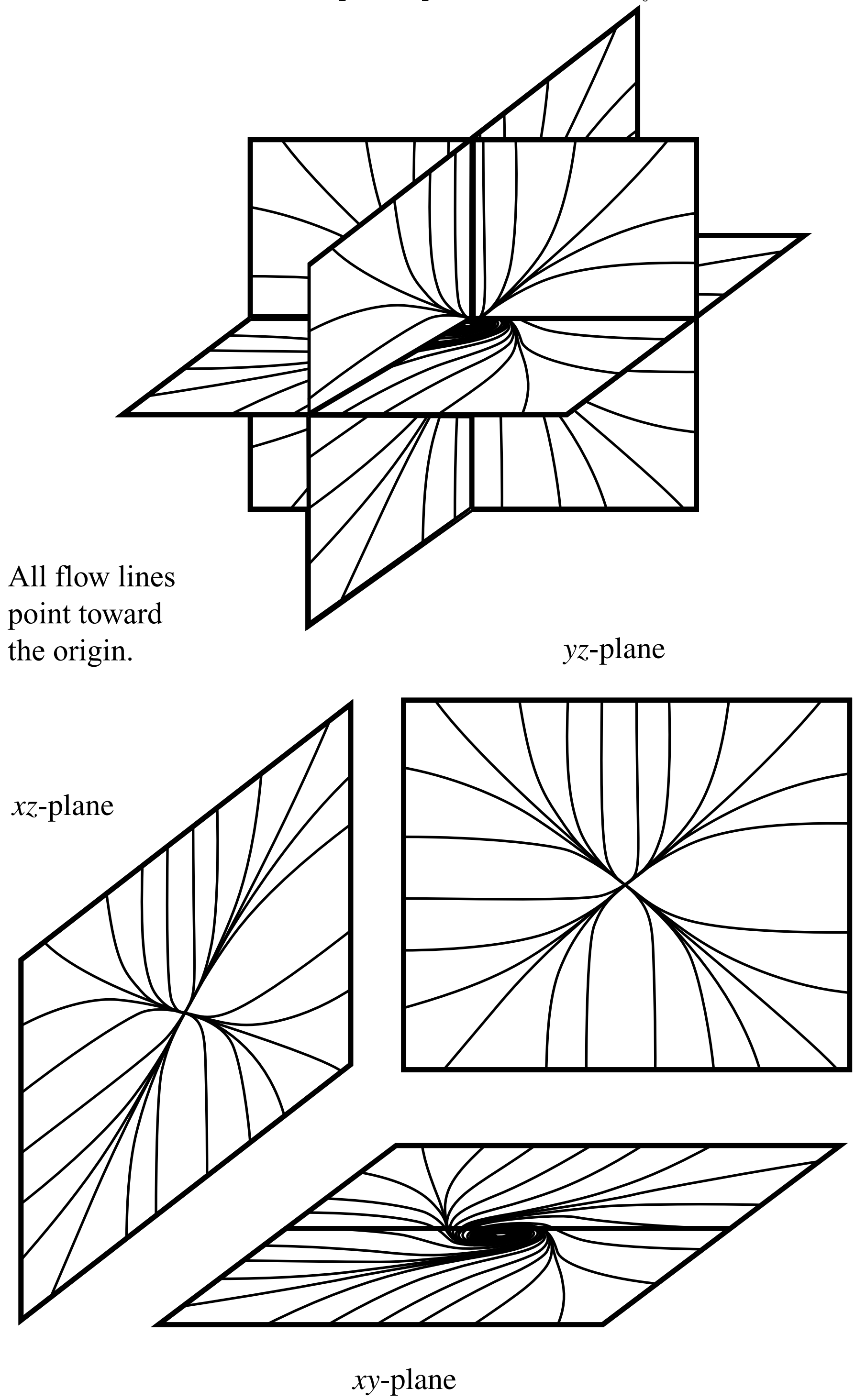
$$\psi\left(\|x\|\right) = \sqrt{2}\|x\|.$$

Then for $\|x\| < 1$ we have

$$\begin{aligned} \psi\left(\|x\|\right) &= \sqrt{2}\|x\| \\ &= \sqrt{2}\sqrt{x_1^2 + x_2^2 + x_3^2} \\ &= \sqrt{2x_1^2 + 2x_2^2 + 2x_3^2} \\ &\geq 2x_1^2 + 2x_2^2 + 2x_3^2 \\ &\geq x_1^2 + 2x_2^2 + x_3^2 \\ &= V(x), \end{aligned}$$

so that V is descrescent. Therefore, $(0, 0, 0)$ is asymptotically stable, by Thm 5.4.2.

Figure 2. A phase portrait for the system in Problem 3.



4. Consider the linear system

$$x'' + p(t)x' + q(t)x = 0,$$

where p and q are real-valued and continuous. Let y_1, y_2 be a real-valued fundamental pair of solutions. Show that y_2 must vanish between any two consecutive zeroes of y_1.

Let t_1, t_2 be consecutive zeroes of y_1. Then

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

is continuous, as a sum of products of continuous functions. Then

$$W(t_1) = -y_1'(t_1)y_2(t_1) \quad \text{and} \quad W(t_2) = -y_1'(t_2)y_2(t_2).$$

So none of $y_1'(t_1), y_2(t_1), y_1'(t_2), y_2(t_2)$ can be 0, or else W would be 0, and then y_1, y_2 couldn't be linearly independent, by Thm 2.3.2.

Suppose that $W(t_1) > 0$ and that $y_2(t_1) > 0$. Then $y_1'(t_1) < 0$, that is, $y_1(t)$ is decreasing as it passes through t_1. Then, since $y_1(t)$ is continuous and has no zeroes between t_1 and t_2, $y_1(t)$ must be increasing as it passes through t_2 by basic calculus. I.e., $y_1'(t_2) > 0$.

Note that we cannot have $W(t_2) < 0$: since $W(t)$ is continuous, the Intermediate Value Theorem would imply the existence of some $t_0 \in (t_1, t_2)$ for which $W(t_0) = 0$, which would be a contradiction as described above. Thus we have

$$W(t_2) = -y_1'(t_2)y_2(t_2) > 0.$$

Since $y_1'(t_2) > 0$, this implies $y_2(t_1) < 0$. So $y_2(t)$ has changed sign somewhere between t_1 and t_2; by the IVT again, there must be a $t_0 \in (t_1, t_2)$ for which $y_2(t_0) = 0$.

This argument has given the desired result for the case when $W(t_1) > 0$ and $y_2(t_1) > 0$, but it is clear that a similar argument works just as well if we take $y_2(t_1) < 0$, and for the two cases when $W(t_1) < 0$. So in any case, we can find a zero of y_2 between any two consecutive zeroes of y_1.