MATH 211B – HOMEWORK

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1. (10/7) Find the eigenvalues and eigenfunctions of

$$L[u] = u'' + \omega^2 u, \qquad \omega \in \mathbb{R}^+ \tag{$\star_1$}$$

with $u(a) = u(b) = 0$ on $I = (a, b)$.

This equation is regular and already in normal form. We obtain the eigenvalues from the characteristic equation:

$$\begin{aligned} \alpha^2 + \omega^2 &= 0 \\ \alpha^2 &= -\omega^2 \\ \alpha &= \pm i\omega \end{aligned}$$

This gives $\{e^{\pm i\omega x}\}$ as eigenfunctions. However, since the original equation has real coefficients, we would like a basis of real-valued eigenfunctions. Since $\mathrm{Re}(e^{i\omega}) = \cos \omega x$ and $\mathrm{Im}(e^{i\omega}) = \sin \omega x$, we take $\{\cos \omega x, \sin \omega x\}$ as a basis. Thus the eigenfunctions of ($\star_1$) are all of the form

$$u(x) = c_1 \cos \omega x + c_2 \sin \omega x.$$

Now we use the initial conditions:

$$\begin{aligned} u(a) &= c_1 \cos \omega a + c_2 \sin \omega a = 0 \\ u(b) &= c_1 \cos \omega b + c_2 \sin \omega b = 0 \end{aligned}$$

From the first equation above,

$$c_1 = -c_2 \tfrac{\sin \omega a}{\cos \omega a}.$$

Substituting into the second,

$$\begin{aligned} -c_2 \tfrac{\sin \omega a}{\cos \omega a} \cos \omega b + c_2 \sin \omega b &= 0 \\ -c_2 \sin \omega a \cos \omega b + c_2 \cos \omega a \sin \omega b &= 0 \\ c_2 \sin \omega(b - a) &= 0. \end{aligned}$$

Thus we must have $\omega(b - a) = 2\pi k$ for some $k \in \mathbb{Z}$, and the eigenfunctions must look like

$$\{\cos \tfrac{2\pi k}{b-a} x, \sin \tfrac{2\pi k}{b-a} x\}.$$

2. (10/12) The DE with constant coefficients

$$L[u] = u^{(n)} + a_1 u^{(n-1)} + a_2 u^{(n-2)} + \cdots + a_n u, \qquad (a_i \in \mathbb{R}) \tag{$\star_2$}$$

is stable iff no root α of its characteristic polynomial has positive real part and all multiple roots have strictly negative real part.

We know from 146 that we can find a basis for the solution space which consists of functions of the form $\{t^k e^{\alpha t}\}$ where $k \in \mathbb{N} = \{0, 1, 2, \dots\}$. So every solution to $(\star_2)$ can be written

$$u(t) = \sum_{j=0}^{n} c_j t^{k_j} e^{\alpha_j t}. \tag{1}$$

($\Rightarrow$) Assume that $(\star_2)$ is stable, i.e., all solutions remain bounded as $t \to \infty$. If $\mathrm{Re}(\alpha)$ were strictly positive, $t^k e^{\alpha t}$ would blow up as $t \to \infty$, for any k, and hence so would u. This contradiction shows $\mathrm{Re}(\alpha) < 0$ for every root α of the characteristic polynomial. Consider the case when α is a repeated root. Then there is a term in (1) of the form $t^k e^{\alpha t}$ for $k \geq 1$. In this situation, $\mathrm{Re}(\alpha) = 0$ would imply that this term is of the form $t^k e^{i\nu}$. Since $|t^k e^{i\nu}| \xrightarrow{t\to\infty} \infty$ but u does not blow up, it must not be the case that $\mathrm{Re}(\alpha) = 0$.

($\Leftarrow$) Assume that no roots α of the characteristic polynomial of $(\star_2)$ have positive real part and all multiple roots have strictly negative real part. Then we have

$$|u(t)| = \left| \sum_{j=0}^{n} c_j t^{k_j} e^{\alpha_j t} \right| \leq \sum_{j=0}^{n} \left| c_j t^{k_j} e^{\alpha_j t} \right| \tag{2}$$

by the triangle inequality. If α_j is not a multiple root, then $k_j = 0$ and we have

$$\left| c_j t^{k_j} e^{\alpha_j t} \right| = \left| c_j e^{\alpha_j t} \right| \leq |c_j|.$$

Equality holds iff $\mathrm{Re}(\alpha_j) = 0$. Otherwise,

$$\mathrm{Re}(\alpha_j) < 0 \quad \Longrightarrow \quad \left| c_j e^{\alpha_j t} \right| \xrightarrow{t\to\infty} 0.$$

If α_j is a multiple root, then $k_j > 1$ and the hypothesis gives $\mathrm{Re}(\alpha_j) < 0$. In this case we still have

$$\left| c_j t^{k_j} e^{\alpha_j t} \right| \xrightarrow{t\to\infty} 0$$

because $e^{-\mu t}$ goes to zero faster than any polynomial for $\mu = -\alpha_j > 0$. Returning to (2), we see that the worst case scenario is when there are no multiple roots and $\mathrm{Re}(\alpha_j) = 0$. In this case, the largest u can get is

$$|u(t)| \leq \sum_{j=0}^{n} \left| c_j t^{k_j} e^{\alpha_j t} \right| = \sum_{j=0}^{n} |c_j|,$$

which is clearly bounded. Hence u is stable.

3. (10/14) For solutions u, v of

$$L[u] = u'' + p(x)u' + q(x)u = 0 \qquad p, q \text{ continuous on } I, \tag{$\star_3$}$$

$\{u, v\}$ are linearly independent solutions of $(\star_3)$ iff $\{u, v\}$ is a fundamental set of solutions.

($\Rightarrow$) Assume $\{u, v\}$ are linearly independent solutions. The solution space of a 2nd order DE will be 2-dimensional, so two linearly independent solutions will span the entire space. Hence $\{u, v\}$ is a basis and thus also a fundamental set of solutions.

($\Leftarrow$) Assume $\{u, v\}$ is a fundamental set of solutions. Then it is a basis, and hence its elements are linearly independent by definition.

4. (10/14) Consider the equation

$$y'' + q(x)y = 0 \qquad q \text{ is piecewise continuous on } \mathbb{R}. \tag{$\star_4$}$$

Define a "soln" of $(\star_4)$ to be a function $y = f(x)$ which is C^1 (but not C^2) and satisfies the DE $(\star_4)$ at all points where q is continuous.

(a) Describe explicitly a basis of solutions of $(\star_4)$ where

$$q(x) = \begin{cases} 1, & x > 0 \\ -1 & x < 0 \end{cases}.$$

For $x > 0$, a basis of solutions would be $\{\cos x, \sin x\}$, and for $x < 0$ a basis of solutions would be $\{e^x, e^{-x}\}$. Thus, a general solution would be

$$u(x) = \begin{cases} a\cos x + b\sin x, & x > 0 \\ \alpha e^x + \beta e^{-x}, x < 0 \end{cases}.$$

This is clearly C^1 on $(-\infty, 0)$ and $(0, \infty)$; we only need to worry about 0. To be C^1 at 0, we require

$$(a\cos x + b\sin x)|_{x=0} = (\alpha e^x + \beta e^{-x})|_{x=0} \qquad \text{and}$$
$$(a\cos x + b\sin x)'|_{x=0} = (\alpha e^x + \beta e^{-x})'|_{x=0}.$$

In other words,

$$(a\cos 0 + b\sin 0) = a = \alpha + \beta = (\alpha e^0 + \beta e^{-0}) \qquad \text{and}$$
$$(a\sin 0 + b\cos 0) = b = \alpha - \beta = (\alpha e^0 - \beta e^{-0}).$$

Thus, solve this system for α and β and get

$$\alpha = \tfrac{a+b}{2} \quad \text{and} \quad \beta = \tfrac{a-b}{2},$$

and note that

$$\alpha e^x + \beta e^{-x} = \tfrac{a+b}{2}e^x + \tfrac{a-b}{2}e^{-x} = \tfrac{a}{2}(e^x + e^{-x}) + \tfrac{a}{2}(e^x - e^{-x}).$$

We can write the basis of solutions as $\{u_1, u_2\}$ where

$$u_1(x) = \begin{cases} \frac{e^x+e^{-x}}{2}, & x < 0 \\ \cos x, & x > 0 \end{cases} \quad \text{and} \quad u_2(x) = \begin{cases} \frac{e^x-e^{-x}}{2}, & x < 0 \\ \sin x, & x > 0 \end{cases}.$$

(b) State & prove an existence and uniqueness theorem for the corresponding IVP and examine the DE satisfied by the Wronskian, if any.

Given any initial conditions $y(x_0) = y_0, y'(x_0) = v_0$, there exists a unique solution to equation $(\star_4)$. This follows essentially from the more basic case when q is continuous. For example, take $x_0 < 0$. Then

$$y'' - y = 0, \quad y(x_0) = y_0, y'(x_0) = v_0$$

has a unique solution $u(x)$ on $(-\infty, 0)$ by the basic theory. However, this determines $u(0), u'(0)$ by continuity, which may then be used as the initial conds for the other half interval. Now

$$y'' + y = 0, \quad y(0) = u(0), y'(0) = u(0)$$

has a unique solution on $\mathbb{R}^+$. Combining, we obtain a unique solution on $\mathbb{R}$. Examining the Wronskian of this system, we have

$$W(x) = u_1 u_2' - u_1' u_2.$$

Differentiating gives

$$W'(x) = u_1 u_2'' - u_1'' u_2 = \begin{cases} u_1(-u_2) - (-u_1)u_2, & x < 0 \\ u_1(u_2) - (u_1)u_2, & x > 0 \end{cases},$$

which is 0 in either case. The central equality follows because the given q makes $(\star_4)$ into

$$y'' = \begin{cases} -y, & x < 0 \\ y, & x > 0 \end{cases}.$$

Hence, the Wronskian satisfies $W' = 0$, i.e., W is constant.

(c) Relate this to

(i) the Laplace transform method for solving such equations, and

(ii) the distributional approach to ODEs.

5. (10/19)

$(\star_5)$
$$L[u] = u'' + u = f \qquad (f \in \mathrm{L}^1_{\mathrm{loc}})$$

Show that if u is a classical solution (i.e., $u \in C^2$) of $L[u] = f$, then u is a distributional solution also.

Take a solution $u \in C^2$ of $(\star_5)$ and fix $\varphi \in C_c^\infty(\mathbb{R})$. Then

$$\begin{aligned}
\int L[\varphi]u &= \int (\varphi'' + \varphi)u \\
&= \int \varphi'' u + \int \varphi u \\
&= (-1)^2 \int \varphi u'' + \int \varphi u && \text{ibp twice} \\
&= \int \varphi(u'' + u) \\
&= \int \varphi f && u \text{ is a soln}
\end{aligned}$$

shows $\int L[\varphi]u = \int f\varphi$, i.e., u is also a distributional solution.

6. (10/29) The Bessel DE:

$$L[u] = x^2u'' + xu' + (x^2 - n^2)u = 0 \tag{$\star_6$}$$

Show that the self-adjoint form of this equation is

$$\tfrac{d}{dx}\left[xu'\right] + \left(x - \tfrac{n^2}{x}\right)u = 0. \tag{3}$$

We obtain $(\star_6)$ from (3):

$$\begin{aligned}
\tfrac{d}{dx}\left[xu'\right] + \left(x - \tfrac{n^2}{x}\right)u &= 0 \\
xu'' + u' + \left(x - \tfrac{n^2}{x}\right)u &= 0 && \text{product rule} \\
x^2u'' + xu' + \left(x^2 - n^2\right)u &= 0. && \text{mult through by } x
\end{aligned}$$

Then note that (3) is self-adjoint by the theorem which states that the 2nd order DE

$$L[u] = p_0(x)u'' + p_1(x)u' + p_2(x)u = 0$$

is self-adjoint iff it is of the form

$$\tfrac{d}{dx}\left[p(x)\tfrac{du}{dx}\right] + q(x)u = 0.$$

7. (10/29) The Legendre DE of order n, self-adjoint form:

$$\tfrac{d}{dx}\left[(1 - x^2)\tfrac{du}{dx}\right] + \lambda u = 0 \qquad x \in \mathbb{R}. \tag{$\star_7$}$$

Show that for $\lambda = n(n+1), n \in \mathbb{N}$, this DE has polynomial solutions.

By differentiating the first term and plugging in λ, we have the equation

$$(1 - x^2)u'' - 2xu' + n(n+1)u = 0.$$

Two linearly independent solutions of this equation are

$$u_1(x) = 1 + \sum_{m=1}^{\infty} (-1)^m \left(\prod_{k=0}^{m-1} (n-2k) \right) \left(\prod_{k=0}^{m-1} (n+1+2k) \right) \frac{x^{2m}}{(2m)!}$$

and

$$u_2(x) = x + \sum_{m=1}^{\infty} (-1)^m \left(\prod_{k=0}^{m-1} (n-1-2k) \right) \left(\prod_{k=0}^{m} (n+2k) \right) \frac{x^{2m+1}}{(2m+1)!}$$

While these are ostensibly infinite series, $n \in \mathbb{N}$ may result in the disappearance of many terms. For example, suppose n is even and consider u_1. For $m \geq \frac{n}{2}+1$, the coefficient will contain a factor $\left(n - 2(\frac{n}{2})\right) = 0$, and will hence vanish. So for n even, $u_1(x)$ is a polynomial solution of degree n.

Alternatively, suppose n is odd and consider u_2. For $m \geq \frac{n-1}{2}+1$, the coefficient will contain a factor $\left(n - 1 - 2(\frac{n-1}{2})\right) = 0$, and will hence vanish. So for n odd, $u_2(x)$ is a polynomial solution of degree n.

Either way, $(\star_7)$ has a polynomial solution for $n \in \mathbb{N}$.

8. (10/29) Show that the third order linear homogeneous DE

$$p_0(x)u''' + p_1(x)u'' + p_2(x)u' + p_3(x)u = 0 \tag{$\star_8$}$$

is "exact" iff its coefficients satisfy $p_0''' - p_1'' + p_2' - p_3 = 0$.

We define a DE of the form $(\star_8)$ to be *exact* iff

$$p_0(x)u''' + p_1(x)u'' + p_2(x)u' + p_3(x)u = \tfrac{d}{dx}\left[A(x)u'' + B(x)u' + C(x)u\right]$$

for some $A, B, C \in C^1$. Differentiating the right-hand side above,

$$p_0u''' + p_1u'' + p_2u' + p_3u = Au''' + A'u'' + Bu'' + B'u' + Cu' + C'u.$$

Matching coefficients,

$$p_0 = A, \quad p_1 = A' + B, \quad p_2 = B' + C, \quad \text{and } p_3 = C'.$$

Then using these equations we expand p_3 as

$$\begin{aligned} p_3 = C' &= (p_2 - B')' \\ &= p_2' - (p_1 - A')'' \\ &= p_2' - p_1'' + p_0'''. \end{aligned}$$

9. (10/29)

$$d_n(x) = \frac{n}{\pi(1+n^2x^2)}$$

Show that $d_n \geq 0$, $\int_{\mathbb{R}} d_n(x)dx = 1$. Sketch the graph of $d_n, n = 1, 2, \ldots$ and argue that d_n is a δ-sequence, i.e., $d_n \to \delta$.

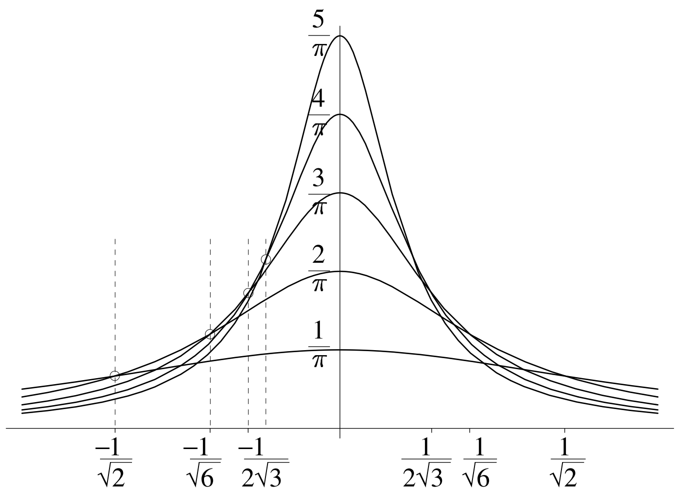

FIGURE 1. An sketch of d_n for $n = 1, 2, 3, 4, 5$.

Clearly, $d_n \geq 0$ as both the numerator and denominator are positive for any real x and $n = 1, 2, \ldots$. We compute the integral and find

$$\begin{aligned}\int_{\mathbb{R}} d_n(x)dx &= \int_{\mathbb{R}} \frac{n}{\pi(1+n^2x^2)}dx \\ &= \frac{n}{\pi}\int_{\mathbb{R}} \frac{1}{1+(nx)^2}dx \\ &= \frac{n}{\pi}\left[\frac{1}{n}\arctan(nx)\right]_{-\infty}^{\infty} \\ &= \frac{1}{\pi}\left(\arctan(\infty) - \arctan(-\infty)\right) \\ &= \frac{1}{\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) \\ &= 1 \end{aligned} \tag{4}$$

for any $n = 1, 2, \ldots$. A sketch of d_n is depicted in Figure 1.

Note that

$$d_n(x) = \frac{n}{\pi(1+n^2x^2)} = nd(xn) \quad \text{for} \quad d(x) := \frac{1}{\pi(1+x^2)}. \tag{5}$$

Thus,

$$\begin{aligned}\lim_{n\to\infty}\langle d_n, \varphi\rangle &= \lim_{n\to\infty}\int d_n(x)\varphi(x)\,dx \\ &= \varphi(0) + \lim_{n\to\infty}\int d_n(x)\left[\varphi(x) - \varphi(0)\right]\,dx,\end{aligned}$$

and we just need to show that the latter integral goes to 0. Fix $\varepsilon > 0$. Then

$$\begin{aligned}
&\left|\int d_n(x)\,[\varphi(x)-\varphi(0)]\;dx\right| \\
&\qquad \le \int_{|x|<r} d_n(x)\,|\varphi(x)-\varphi(0)|\;dx + \int_{|x|\ge r} d_n(x)\,|\varphi(x)-\varphi(0)|\;dx \\
&\qquad \le \max_{|x|<r}\{|\varphi(x)-\varphi(0)|\}\cdot\int_{|x|<r} d_n(x)\,dx \\
&\qquad\qquad\qquad + \max_{|x|\ge r}\{|\varphi(x)-\varphi(0)|\}\cdot\int_{|x|\ge r} d_n(x)\,dx \\
&\qquad \le \max_{|x|<r}\{|\varphi(x)-\varphi(0)|\}\cdot 1 + M\cdot\int_{|x|\ge r} d_n(x)\,dx,
\end{aligned}$$

where $M := \max_{|x|\ge r}\{|\varphi(x)-\varphi(0)|\}$, and the last equality follows by (4). Now for sufficiently small r, the continuity of φ gives

$$\max_{|x|<r}\{|\varphi(x)-\varphi(0)|\} < \tfrac{\varepsilon}{2}.$$

Note that we don't let r go to 0, just pick $r > 0$ small enough that the inequality will hold. Then deal with the second term as follows:

$$\int_{|x|\ge r} d_n(x)\,dx = \int_{|x|\ge r} nd(nx)\,dx = \int_{|x|\ge rn} d(u)\,du$$

where the first equality comes by (5) and the second comes by the change of variables $u = nx$. Then since

$$\int_{\mathbb{R}} d(u)\,du = 1$$

by (4) again, it must be possible to pick n so large that

$$\int_{|x|\ge rn} d(u)\,du < \frac{\varepsilon}{2M}.$$

This is sufficient to get

$$\left|\int d_n(x)\,[\varphi(x)-\varphi(0)]\;dx\right| < \varepsilon.$$

10. (11/02) We saw for $f(x) = \frac{1}{r}, r := \|x\|$, that $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^2)$. Extend this to $d \ge 2$.

We want to show

$$f(x) := \|x\|^{1-d} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d) \quad \text{for } d \ge 2.$$

Since this function is clearly in $\mathrm{L}^1_{\mathrm{loc}}\left(\mathbb{R}^d \backslash \{0\}\right)$, we only really need to check f in a neighborhood of the origin. Let $B = B(0,1)$ be the ball of radius 1 centered at the origin of $\mathbb{R}^d$.

We convert to spherical coordinates with

$$r = \|x\| = \left(\sum_{k=1}^{d} x_k^2 \right)^{1/2}$$

so that

$$f(x) = r^{1-d}.$$

The rectangular coordinates $x_1, x_2, \ldots, x_d$ are related to the spherical coordinates $r, \varphi_1, \ldots, \varphi_{d-1}$ by the equations:[1]

$$\begin{aligned} x_k &= rS^{k-1}c_k \\ x_d &= rS^{d-1} \\ r &= (x_1^2 + x_2^2 + \cdots + x_d^2)^{1/2} \\ \varphi_k &= \arctan\left(\frac{R_k}{x_k}\right) \end{aligned}$$

where

$$\begin{aligned} S_m^k &= \prod_{j=m}^{k} \sin\varphi_j \\ c_m &= \cos\varphi_m \\ s_m &= \sin\varphi_m \\ R_k &= (r^2 - x_1^2 - \cdots - x_k^2)^{1/2} = rS_1^k \end{aligned}$$

Although we do not wish to work it out explicitly, the Jacobian for this change of coordinates will be of the form

$$|J| = \begin{vmatrix} \frac{\partial}{\partial r}(rS^0c_1) & \frac{\partial}{\partial \varphi_1}(rS^0c_1) & \ldots & \frac{\partial}{\partial \varphi_{d-1}}(rS^0c_1) \\ \frac{\partial}{\partial r}(rS^1c_2) & \frac{\partial}{\partial \varphi_1}(rS^1c_2) & \ldots & \frac{\partial}{\partial \varphi_{d-1}}(rS^1c_2) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial r}(rS^{d-1}c_2) & \frac{\partial}{\partial \varphi_1}(rS^{d-1}c_d) & \ldots & \frac{\partial}{\partial \varphi_{d-1}}(rS^{d-1}c_d) \end{vmatrix}.$$

As the determinant is expanded by cofactors and each cofactor is evaluated recursively, a factor of r will emerge for each term that does not stem from a $\frac{\partial}{\partial r}$. Thus, there will be a common factor of r^{d-1} in the final computed Jacobian $|J| = r^{d-1}|K|$.

[1]Thanks to Andrew Snowden of the University of Maryland for these conversion formulas.

Thus,

$$\begin{aligned}\int_B f\,dx &= \int_B r^{1-d}|J|\,d\varphi_{d-1}\dots d\varphi_1 dr\\ &= \int_B r^{1-d}r^{d-1}|K|\,d\varphi_{d-1}\dots d\varphi_1 dr\\ &= \int_B |K|\,d\varphi_{d-1}\dots d\varphi_1 dr\\ &= \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)},\end{aligned}$$

the volume of the unit ball in $\mathbb{R}^d$. Since this is finite, $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$.

11. (11/02) Prove that for any regular distributions T_f, T_g,

$$T_f = T_g \implies f =_{\mathrm{ae}} g.$$

First we note that $T : \mathcal{D} \to \mathcal{D}'$ by $f \mapsto T_f$ is a linear map. Since

$$\begin{aligned}\langle T_f + T_g, \varphi\rangle &= \langle T_f, \varphi\rangle + \langle T_g, \varphi\rangle\\ &= \int f\varphi\,dx + \int g\varphi\,dx\\ &= \int (f+g)\varphi\,dx\\ &= \langle T_{f+g}, \varphi\rangle\end{aligned}$$

for every $\varphi \in \mathcal{D}$, we have $T_f + T_g = T_{f+g}$. Thus it suffices to show that

$$T_f = 0 \implies f =_{\mathrm{ae}} 0.$$

Let $\rho_c(x)$ be the standard mollifier, i.e.,

(i) $\rho_c \in C^\infty$,

(ii) $\mathrm{spt}(\rho_c) = [-\frac{1}{c}, \frac{1}{c}]$

(iii) $\int \rho_c\,dx = 1$, $\forall c > 0$.

Then

$$\begin{aligned}(f * \rho_n)(x) &= \int f(y)\rho_n(x-y)dy\\ &= T_f(\rho_n \circ \nu),\end{aligned}$$

where $\nu(y) = x - y$ for any fixed x. Since $T_f = 0$, this shows $f * \rho_n = 0$ for any n. Then by Proposition 1,

$$f * \rho_n \xrightarrow{n\to\infty} 0 \quad \implies \quad f \equiv 0, \mathrm{ae}.$$

Proposition 1. *If ρ_n is the standard mollifer and $f \in L^p$, then*

$$\lim_{n\to\infty} (f * \rho_n) = f, \text{ in } L^p.$$

Proof. [Al-G, Example 2.23]
C_c^0 is dense in L^p, so we can choose $\varphi \in C_c^0$ such that $\|f - \varphi\|_p < \varepsilon$. Then

$$\|f * \rho_n - \varphi * \rho_n\|_p = \|(f - \varphi) * \rho_n\|_p \leq \|f - \varphi\|_p < \varepsilon,$$

where the central inequality comes by Lemma 2. Hence it suffices to prove

$$\varphi * \rho_n \xrightarrow{\mathrm{L}^p} \varphi.$$

Since $\varphi * \rho_n \xrightarrow{\text{unif}} \varphi$ on $K := \operatorname{spt}\varphi$ by Lemma 3, we can write

$$\begin{aligned}
\|\varphi * \rho_n - \varphi\|_p &= \left[\int_K |(\varphi * \rho_n)(x) - \varphi(x)|^p \, dx\right]^{1/p} \\
&\leq \sup_{x\in K}\{|(\varphi * \rho_n)(x) - \varphi(x)|\} \left[\int_K dx\right]^{1/p} \\
&< \varepsilon
\end{aligned}$$

if n is large enough. Thus

$$\|f * \rho_n - f\|_p \leq \|f * \rho_n - \varphi * \rho_n\|_p + \|\varphi * \rho_n - \varphi\|_p + \|\varphi - f\|_p < 3\varepsilon.$$

□

Lemma 2. *For $f \in L^p$ and $1 \leq p < \infty$, we have $\|f * \rho_n\|_p \leq \|f\|_p$.*

Proof. [Al-G, Example 2.23]
For $p = 1$,

$$\begin{aligned}
\|f * \rho_n\|_1 &\leq \int\int \rho_n(y)\,|f(x-y)| \; dy \; dx \\
&= \int \rho_n(y) \left[\int |f(x-y)| \; dx\right] \; dy \\
&= \|f\|_1.
\end{aligned}$$

Otherwise, for $1 < p < \infty$,

$$\|f * \rho_n\|_p^p = \int \left|\int \rho_n(y) f(x-y)\, dy\right|^p dx.$$

To use Hölder's inequality, we split the mollifier as

$$f\rho_n = (f\rho_n^{1/p})(\rho_n^{1/q}),$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Then the inequality gives

$$\int \rho_n(y)\,|f(x-y)|\, dy \leq \left[\int \rho_n(y)\,|f(x-y)|^p\, dy\right]^{1/p} \left[\int \rho_n(y)\, dy\right]^{1/q}.$$

Since $\int \rho_n(y)\,dy = 1$, we have

$$\begin{aligned}\|f*\rho_n\|_p^p &\leq \int\int \rho_n(y)\,|f(x-y)|^p\;dy\;dx\\ &= \int \rho_n(y)\left[\int |f(x-y)|^p\;dx\right]\;dy\\ &= \int \rho_n(y)\|f\|_p^p\,dy\\ &= \|u\|_p^p.\end{aligned}$$

Thus $\|f*\rho_n\|_p \leq \|f\|_p$ for all $1 \leq p < \infty$. ☐

Lemma 3. *If $f \in C^0$, then $f*\rho_n \xrightarrow{n\to\infty} f$ uniformly on every compact subset.*

Proof. [Al-G, Theorem 2.28(iii)]
Since f is continuous, it is uniformly continuous on any compact set E. I.e., given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|f(x-y) - f(x)| < \varepsilon$$

for all $x \in E$ and for all $y \in B(0,\delta)$. Then

$$\begin{aligned}|(f*\rho_n)(x) - f(x)| &= \left|\int [f(x-y) - f(x)]\,\rho_n(y)\,dy\right|\\ &\leq \int_{B(0,1/n)} |f(x-y) - f(x)|\,\rho_n(y)\,dy\\ &< \varepsilon\end{aligned}$$

if we take n large enough that $\frac{1}{n} < \delta$. ☐

12. (11/04) Define $T \in \mathcal{D}'(\mathbb{R})$ by $T(\varphi) = \varphi^{(n)}(0)$. Prove T is a distribution.

(i) well-defined. Since φ is C^∞ and has compact support, φ and all its derivatives are bounded.

(ii) linearity. T is just composition of the evaluation operator and the differentiation operator, both of which are linear. Since evaluation is a functional, this shows T is a linear functional.

(iii) continuity. Let $\varphi_k \xrightarrow{\mathcal{D}} 0$. Part of the definition of convergence in $\mathcal{D}$ is that all derivatives of the φ_k also converge to 0. In particular,

$$\varphi_k^{(n)} \xrightarrow{k\to\infty} 0,$$

and hence

$$T(\varphi_k) = \varphi_k^{(n)}(0) \xrightarrow{k\to\infty} 0.$$

13. (11/04) Verify, via abstract nonsense, that if $T : \mathcal{D}(\Omega) \to \mathbb{C}$ is a distribution, then $\forall K \subseteq \Omega$ compact, the restriction of T to $\mathcal{D}_k(\Omega) = \{\varphi \in \mathcal{D}(\Omega) \vdots \operatorname{spt}(\varphi) \subseteq K\}$ is continuous.

Define $T|_K$ to be the restriction of T to a compact set $K \subseteq \Omega$, i.e.,

$$T|_K(\varphi) := T(\varphi),\ \forall \varphi \in \mathcal{D}_K(\Omega).$$

Abstract nonsense approach: $T|_K$ is the composition of T with the inclusion map $\iota : \mathcal{D}_K(\Omega) \to \mathcal{D}(\Omega)$, both of which are continuous.

Definitional approach: let $\{\varphi_k\} \subseteq \mathcal{D}_K(\Omega)$ with $\varphi_k \xrightarrow{k\to\infty} 0$. Then

$$T|_K(\varphi_k) = T(\varphi_k) \xrightarrow{k\to\infty} 0.$$

14. (11/09)

(a) Show $D^{\alpha+\beta}T = D^\alpha(D^\beta T)$ for $\alpha, \beta \in \mathbb{N}^d$, $\mathbb{N} = \{0, 1, 2, \dots\}$.

The equality of mixed partials gives the central equality in the following:

$$D^{\alpha+\beta}T = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1+\beta_1}x_1 \dots \partial^{\alpha_d+\beta_d}x_d} = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1}x_1 \dots \partial^{\alpha_d}x_d \partial^{\beta_1}x_1 \dots \partial^{\beta_d}x_d} = D^\alpha(D^\beta T).$$

(b) For an open set $\Omega \subseteq \mathbb{R}^d$ and $T \in \mathcal{D}'(\Omega)$, show that $D^\alpha T(\varphi) = (-1)^{|\alpha|}T(D^\alpha\varphi)$.

Define $D_j := D^{e_j} = \frac{\partial}{\partial x_j}$. Having already established the basis case

$$D_jT(\varphi) = -T(D_j\varphi)$$

in the notes, we induct on $|\alpha|$ by assuming $D^\alpha T(\varphi) = (-1)^{|\alpha|}T(D^\alpha\varphi)$ for $|\alpha| = n$. Now consider $|\alpha| = n+1$ by taking $\beta = \alpha + e_j$ where $e_j = [\delta^j_k]$ has a 1 in the j^{th} spot and 0s elsewhere. Then

$$\begin{aligned}
D^\beta T(\varphi) &= D_jD^\alpha\, T(\varphi) && \text{by (a)}\\
&= D_j(-1)^{|\alpha|}\, T(D^\alpha\varphi) && \text{by inductive hypothesis}\\
&= (-1)^{|\alpha|}(-1)\, T(D_jD^\alpha\varphi) && \text{by basis case}\\
&= (-1)^{|\beta|}\, T(D^\beta\varphi). && |\beta| = |\alpha| + 1
\end{aligned}$$

15. (11/09)

(a) Let $f \in C^\infty(\Omega)$ and $T \in \mathcal{D}'(\Omega)$. Then show $fT \in \mathcal{D}'(\Omega)$ is well-defined by

$$\langle fT, \varphi\rangle = \langle T, f\varphi\rangle, \forall \varphi \in \mathcal{D}(\Omega).$$

(i) The right-hand side shows $\langle fT, \varphi\rangle$ is well-defined, since $f\varphi \in \mathcal{D}$.

(ii) Also, the right-hand side shows that $\langle fT, \varphi\rangle$ is a linear functional.

(iii) Let $\varphi_k \xrightarrow{\mathcal{D}} 0$. Then clearly

$$f\varphi_k \xrightarrow{\mathcal{D}} 0,$$

since $f \in C^\infty(\Omega)$ and $\operatorname{spt}(f\varphi_k) \subseteq \operatorname{spt}(\varphi_k)$. By the continuity of T (which was given), this gives

$$\langle fT, \varphi_k\rangle = \langle T, f\varphi_k\rangle \xrightarrow{k\to\infty} 0.$$

(b) Moreover, state and prove an analogue of Leibnitz' rule for $D^\alpha(fT)$.

We will use induction to show that

$$D^\alpha(fT) = \sum_{\gamma\le\alpha} \binom{\alpha}{\gamma} (D^\gamma f) \left(D^{\alpha-\gamma}T\right), \tag{6}$$

where $\binom{\alpha}{\gamma} := \frac{\alpha!}{\gamma!(\alpha-\gamma)!}$, and the factorial of a multiindex is defined by $\alpha! = \alpha_1! \ldots \alpha_d!$. Also, $\gamma \le \alpha$ means that $\gamma_j \le \alpha_j$ for $j = 1, \ldots, d$.
The basis case is a straightforward calculation:

$$\begin{aligned}
D_j(fT)(\varphi) &= -fT(D_j\varphi) && \text{def of } T' \\
&= -T(fD_j\varphi) && \text{def of } fT \\
&= -T\left(D_j(f\varphi) - \varphi D_j f\right) && \text{product rule} \\
&= -T(D_j(f\varphi)) + T(\varphi D_j f) && \text{linearity} \\
&= D_jT(f\varphi) + T(\varphi D_j f) && \text{def of } T' \\
&= fD_jT(\varphi) + (D_j f)T(\varphi) && \text{def of } fT
\end{aligned}$$

shows $D_j(fT) = fD_jT + (D_jf)T$.
Now we assume that (6) holds for $|\alpha| = n$ and let $\beta = \alpha + e_j$ as in Problem 14b.

$$\begin{aligned}
D^\beta(fT) &= D_jD^\alpha(fT) \\
&= D_j \sum_{\gamma\le\alpha} \binom{\alpha}{\gamma} (D^\gamma f) \left(D^{\alpha-\gamma}T\right) && \text{inductive hypothesis} \\
&= \sum_{\gamma\le\alpha} \binom{\alpha}{\gamma} D_j \Big((D^\gamma f) \left(D^{\alpha-\gamma}T\right)\Big) && \text{linearity} \\
&= \sum_{\gamma\le\alpha} \binom{\alpha}{\gamma} (D^\gamma f)\, D_j \left(D^{\alpha-\gamma}T\right) + \left(D^{\alpha-\gamma}T\right) D_j\, (D^\gamma f) && \text{basis case} \\
&= \sum_{\gamma\le\alpha} \binom{\alpha}{\gamma} (D^\gamma f) \left(D^{\alpha-\gamma+e_j}T\right) + \left(D^{\alpha-\gamma}T\right) D_j \left(D^{\gamma+e_j} f\right) && \text{Problem 14a} \\
&= \sum_{\gamma\le\beta} \binom{\beta}{\gamma} (D^\gamma f)(D^{\beta-\gamma}T) && \text{reordering}
\end{aligned}$$

16. (11/09) With $\{a_n\} \subseteq \mathbb{R}$ and

$$H_a(x) := H(x-a) = \begin{cases} 1, & x \geq 0 \\ 0, & x < a \end{cases},$$

we have seen that the step function

$$f_k(x) = \sum_{n=1}^{k} w_n H(x - a_n)$$

is in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ by the vector space properties of $\mathrm{L}^1_{\mathrm{loc}}\,(\mathbb{R})$. Thus it induces a regular distribution T_f which has derivative

$$\tfrac{d}{dx} T_{f_k} = \sum_{n=1}^{k} w_n \tfrac{d}{dx} H(x - a_n) = \sum_{n=1}^{k} w_n \delta(x - a_n).$$

Hence, the distributional derivative of f_k is

$$\frac{df_k}{dx} = \sum_{n=1}^{k} w_n \delta_{a_n}.$$

Assuming that any bounded interval contains finitely many a_n's, extend this to the case when $f(x) = \sum_{n=1}^{\infty} w_n H(x - a_n)$.

Denote the regular distribution associated with f by T_f. Since $f_k \xrightarrow{k\to\infty} f$ in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$, the continuity of the derivative allows us to say

$$T'_{f_k} = \sum_{n=1}^{k} w_n \delta(x - a_n) \xrightarrow{k\to\infty} \sum_{n=1}^{\infty} w_n \delta(x - a_n) = T'_f.$$

Note that the right-hand side makes sense, since for any $\varphi \in \mathcal{D}$, there can only be finitely many a_n in $\mathrm{spt}(\varphi)$. If we denote them by $\{a_{n_j}\}$, we have

$$\begin{aligned} T'_f(\varphi) &= \int_{\mathbb{R}} \left(\sum_{j=1}^{m} w_{n_j} \delta(x - a_{n_j}) \right) \varphi(x)\, dx \\ &= \sum_{j=1}^{m} \int_{\mathbb{R}} w_{n_j} \delta(x - a_{n_j})\, \varphi(x)\, dx \\ &= \sum_{j=1}^{m} w_{n_j} \varphi(a_{n_j}) < \infty. \end{aligned}$$

This makes it easy to see that T'_f is continuous. If $\varphi_q \xrightarrow{\mathcal{D}} 0$, then

$$T'_f(\varphi_q) = \sum_{j=1}^{m} w_{n_j} \varphi_q(a_{n_j}) \xrightarrow{q\to\infty} 0,$$

because $\varphi_q(a_{n_j}) \xrightarrow{q\to\infty} 0$ for each j.

17. (11/16)

(a) Let $T \in \mathcal{D}'(\mathbb{R})$. Then

$$\frac{dT}{dx} = \lim_{h\to 0} \frac{\tau_{-h}T - T}{h} \qquad \text{(limit in } \mathcal{D}'\text{)},$$

where $\tau_{-h}T \in \mathcal{D}'$ is defined by $\langle \tau_{-h}T, \varphi\rangle = \langle T, \tau_h\varphi\rangle$ for $(\tau_h\varphi)(x) = \varphi(x+h)$.

The action of the right-hand side against a test function φ is given by

$$\begin{aligned}
\left\langle \lim_{h\to 0} \frac{\tau_{-h}T - T}{h}, \varphi \right\rangle &= \lim_{h\to 0} \left\langle \frac{\tau_{-h}T - T}{h}, \varphi \right\rangle && \text{continuity of } \langle\cdot,\cdot\rangle \\
&= \lim_{h\to 0} \frac{1}{h}\Big(\langle \tau_{-h}T, \varphi\rangle - \langle T, \varphi\rangle\Big) && \text{linearity of } \langle\cdot,\cdot\rangle \\
&= \lim_{h\to 0} \frac{1}{h}\Big(\langle T, \tau_h\varphi\rangle - \langle T, \varphi\rangle\Big) && \text{defn of } \tau \\
&= \lim_{h\to 0} \left\langle T, \frac{\tau_h\varphi - \varphi}{h} \right\rangle && \text{linearity of } \langle\cdot,\cdot\rangle \\
&= \langle T, -\varphi'\rangle && \text{continuity of } \langle\cdot,\cdot\rangle \\
&= -\langle T, \varphi'\rangle \\
&= \langle T', \varphi\rangle
\end{aligned}$$

(b) Extend this to higher dimensions.

Let e_j be the standard basis vector with 1 in the j^{th} slot and 0 elsewhere. Define $\tau_{-h_j}T \in \mathcal{D}'$ by $\langle \tau_{-h_j}T, \varphi\rangle = \langle T, \tau_{h_j}\varphi\rangle$ for $(\tau_{h_j}\varphi)(x) = \varphi(x + he_j)$. Then

$$\frac{\partial T}{\partial x_j} = \lim_{h\to 0} \frac{\tau_{-h_j}T - T}{h} \qquad \text{(limit in } \mathcal{D}'\text{)}.$$

$$\begin{aligned}
\left\langle \lim_{h\to 0} \frac{\tau_{-h_j}T - T}{h}, \varphi \right\rangle &= \lim_{h\to 0} \left\langle T, \frac{\tau_{h_j}\varphi - \varphi}{h} \right\rangle && \text{linearity of } \langle\cdot,\cdot\rangle \\
&= \langle T, -D_j\varphi\rangle && \text{continuity of } \langle\cdot,\cdot\rangle \\
&= \langle D_jT, \varphi\rangle
\end{aligned}$$

via the same arguments as in (a), where we are using $D_j := D^{e_j} = \frac{\partial}{\partial x_j}$. 14(b) extends this to the more general case of $D^\alpha T$.

18. (11/16) If the series $\sum_n T_n$ converges in $\mathcal{D}'$, then it can be differentiated term by term:
$$\left(\sum_n T_n\right)' = \sum_n T_n'.$$
More generally, $D^\alpha \left(\sum_n T_n\right) = \sum_n D^\alpha T_n$, for every multiindex α.

We have

$$\begin{aligned}
\left\langle \left(\sum_{n=0}^{\infty} T_n\right)', \varphi \right\rangle &= - \left\langle \lim_{k\to\infty} \sum_{n=0}^{k} T_n, \varphi' \right\rangle && \text{defn of} T', \Sigma \\
&= - \lim_{k\to\infty} \left\langle \sum_{n=0}^{k} T_n, \varphi' \right\rangle && \text{continuity of } \langle\cdot,\cdot\rangle \\
&= - \lim_{k\to\infty} \sum_{n=0}^{k} \langle T_n, \varphi' \rangle && \text{linearity of } \langle\cdot,\cdot\rangle \\
&= - \lim_{k\to\infty} \sum_{n=0}^{k} - \langle T_n', \varphi \rangle && \text{defn of } T' \\
&= \lim_{k\to\infty} \left\langle \sum_{n=0}^{k} T_n', \varphi \right\rangle && \text{linearity of } \langle\cdot,\cdot\rangle \\
&= \left\langle \lim_{k\to\infty} \sum_{n=0}^{k} T_n', \varphi \right\rangle && \text{continuity of } \langle\cdot,\cdot\rangle \\
&= \left\langle \sum_{n=0}^{\infty} T_n', \varphi \right\rangle
\end{aligned}$$

To extend this to the general case,

$$\begin{aligned}
\left\langle D^\alpha \left(\sum_{n=0}^{\infty} T_n\right), \varphi \right\rangle &= (-1)^{|\alpha|} \left\langle \lim_{k\to\infty} \sum_{n=0}^{k} T_n, D^\alpha \varphi \right\rangle && \text{defn of } D^\alpha T \\
&= (-1)^{|\alpha|} \lim_{k\to\infty} \sum_{n=0}^{k} \langle T_n, \varphi' \rangle && \text{as in (a)} \\
&= (-1)^{|\alpha|} \lim_{k\to\infty} \sum_{n=0}^{k} (-1)^{|\alpha|} \langle D^\alpha T_n, \varphi \rangle && \text{defn of } D^\alpha T \\
&= \left\langle \lim_{k\to\infty} \sum_{n=0}^{k} D^\alpha T_n, \varphi \right\rangle && \text{as in (a)} \\
&= \left\langle \sum_{n=0}^{\infty} \mathcal{D}^\alpha T_n, \varphi \right\rangle
\end{aligned}$$

19. (11/16) Suppose $T_n \in \mathcal{D}'$ is such that
$$T(\varphi) := \lim_{n\to\infty} T_n(\varphi)$$
exists in $\mathbb{C}$, $\forall \varphi \in \mathcal{D}(\Omega)$. Show $T \in \mathcal{D}'$.

Since T is clearly a linear functional, it just remains to check the continuity of T. Recall the Principle of Uniform Boundedness (for Fréchet spaces):

Theorem. [Ru, Theorem 2.6]: If $\{T_n\}$ is a sequence of continuous linear mappings from $\mathcal{D}$ to $\mathbb{C}$ and if the sets
$$\Gamma(\varphi) := \{|T_n\varphi| \vdots n \in \mathbb{N}\}$$
are bounded in $\mathbb{C}$ for each $\varphi \in \mathcal{D}$, then Γ is equicontinuous.

This shows that $\{T_n\}$ is equicontinuous, i.e., to every neighborhood W of 0 in $\mathbb{C}$, there corresponds a neighborhood V of 0 in $\mathcal{D}$ such that
$$T_n(V) \subseteq W,$$
for all n [Ru, Definition 2.3]. Note that the concept of neighborhood makes sense here: although a locally convex space need not be metrizable in general, every Fréchet space comes with a topology induced by a complete metric.

Let $\{\varphi_k\}_{k=0}^{\infty} \subseteq \mathcal{D}$ with $\varphi_k \to 0$. To see that T is continuous, we must show $\langle T, \varphi_k\rangle \xrightarrow{k\to\infty} 0$. Fix $\varepsilon > 0$ and consider $B := B(0, \varepsilon) \subseteq \mathbb{C}$. By the above remarks, there must be some $\delta > 0$ such that $T_n(\varphi_k) \in B$ for all $\varphi_k \in A := B(0, \delta)$, $\forall n \in \mathbb{N}$. Since
$$\langle T, \varphi_k\rangle = \left\langle \lim_{n\to\infty} T_n, \varphi_k \right\rangle = \lim_{n\to\infty} \langle T_n, \varphi_k\rangle,$$
we can pick N such that $k > N \implies \varphi_k \in A$, and be sure that
$$|\langle T, \varphi_k\rangle| < \varepsilon.$$

20. (11/16) Let $I = (a, b) \subseteq \mathbb{R}$, and $T \in \mathcal{D}(I)$. Show that
$$\frac{dT}{dx} = 0 \quad \implies \quad T \text{ is constant.}$$

If $\frac{dT}{dx} = 0$, we have $\langle T', \varphi\rangle = -\langle T, \varphi'\rangle = 0$, $\forall \varphi \in \mathcal{D}$. This just means, $\psi = \varphi' \implies \langle T, \psi\rangle = 0$. Since test functions are C^∞, they are clearly absolute continuous, and may thus be represented
$$\varphi(x) = \int_a^x \varphi'(t)dt + \varphi(a).$$
In particular, for $\varphi \in \mathcal{D}$, we can define $\psi(x) = \int_a^x \varphi(t)dt + c$, so that $\varphi' = \psi$. In other words, every test function can be represented as the derivative of some other test function, and we have $\langle T, \varphi\rangle = 0$, $\forall \varphi$.

21. (11/18) Show that $\mathrm{W}^{1,p}(\Omega)$ is separable for $1 \leq p < \infty$.

Because of the isometric embedding $\mathcal{I} : \mathrm{W}^{1,p}(\Omega) \to \mathrm{L}^p(\Omega)$, we may think of $\mathrm{W}^{1,p}(\Omega)$ as a closed subspace of $\mathrm{L}^p(\Omega)$.

22. (11/18) $\Omega = I = (-1,1) \subseteq \mathbb{R}$. Show that $u(x) = \frac{1}{2}(|u| + u)$ belongs to $\mathrm{W}^{1,p}(I)$, for $1 \leq p < \infty$, and that $u' = H$. More generally, a continuous function on $\overline{I} = [-1,1]$ that is piecewise C^1 belongs to $\mathrm{W}^{1,p}(\overline{I})$, $\forall 1 \leq p < \infty$.

u is clearly in L^p:

$$\int_I u\,dx = \int_{(-1,0)} 0\,dx + \int_{(0,1)} x\,dx = \tfrac{1}{2},$$

so it just remains to show $u' \in \mathrm{L}^p(I)$.

$$\begin{aligned}\langle u', \varphi \rangle &= -\langle u, \varphi' \rangle \\ &= -\int_I u\varphi'\,dx \\ &= -M \\ &< \infty\end{aligned}$$

where $M := \sup_I \varphi'(x)$.

23. (11/18) Prove that

$$\mathrm{W}^{1,p}(I) = \left\{ u \in L^p(I) \,\vdots\, \exists g \in L^p(I) \text{ such that } \int_I u\varphi' = -\int_I g\varphi,\ \forall \varphi \in C_c^1(I) \right\}.$$

Also, for $u \in \mathrm{W}^{1,p}(I)$ we have $u' = g$. Note: g is ae-unique.

By definition,

$$u \in \mathrm{W}^{1,p}(I) \quad \Longleftrightarrow \quad u \in \mathrm{L}^p \text{ and } u' \in \mathrm{L}^p,$$

where $u' \in \mathrm{L}^p$ means that $u' = T_g$ for some unique $g \in \mathrm{L}^p$.

24. (11/18)
   (a) Define $\tilde{u}(x) = \int_0^x u'(t)\,dt$ for $u \in \mathrm{W}^{1,p}(I)$. Show $\tilde{u}$ is absolutely continuous on I.

   (b) Using (a), show

$$T_{\frac{d\tilde{u}}{dx}} = \tfrac{d}{dx} T_{\tilde{u}}.$$

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