MATH 217A – HOMEWORK

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1. (Chap. 1, Problem 2).

(a) Let (Ω, Σ, P) be a probability space and $\{A_i, 1 \leq i \leq n\} \subseteq \Sigma, n \geq 2$. Prove that

$$\begin{aligned}
P\left(\bigcup_{i=1}^n A_i\right) &= \sum_{i=1}^n P(A_i) - \sum_{1\leq i<j\leq n} P(A_i \cap A_j) + \sum_{1\leq i<j<k\leq n} P(A_i \cap A_j \cap A_k) \\
&\qquad - ... + (-1)^{n-1} P\left(\bigcap_{i=1}^n A_i\right) \\
&\geq \sum_{i=1}^n P(A_i) - \sum_{1\leq i<j\leq n} P(A_i \cap A_j).
\end{aligned}$$

First, the basis case. We make a union disjoint as follows:

$$A \cup B = A \sqcup (A^c \cap B).$$

Thus

$$P(A \cup B) = P(A) + P(A^c \cap B). \tag{1}$$

Similarly, we can write

$$\begin{aligned}
B &= (A \cap B) \sqcup (A^c \cap B) \\
P(B) &= P(A \cap B) + P(A^c \cap B) \\
P(B) - P(A \cap B) &= P(A^c \cap B),
\end{aligned}$$

which we plug into (1) to get

$$\begin{aligned}
P(A \cup B) &= P(A) + P(A^c \cap B) \\
&= P(A) + P(B) - P(A \cap B).
\end{aligned}$$

Now we proceed by induction. Assume

$$\begin{aligned}
P\left(\bigcup_{i=1}^n A_i\right) &= \sum_{i=1}^n P(A_i) - \sum_{1\leq i<j\leq n} P(A_i \cap A_j) + \sum_{1\leq i<j<k\leq n} P(A_i \cap A_j \cap A_k) \\
&\qquad - ... + (-1)^{n-1} P\left(\bigcap_{i=1}^n A_i\right).
\end{aligned}$$

Using the basis case and then the inductive hypothesis gives

$$\begin{aligned} P\left(\bigcup_{i=1}^{n+1} A_i\right) &= P\left(\bigcup_{i=1}^{n} A_i \cup A_{n+1}\right) \\ &= P\left(\bigcup_{i=1}^{n} A_i\right) + P\left(A_{n+1}\right) - P\left(\bigcup_{i=1}^{n} A_i \cap A_{n+1}\right) \\ &= \sum_{i=1}^{n+1} P(A_i) - \sum_{1\le i<j\le n} P(A_i \cap A_j) + \sum_{1\le i<j<k\le n} P(A_i \cap A_j \cap A_k) \\ &\qquad - ... + (-1)^{n-1} P\left(\bigcap_{i=1}^{n} A_i\right) - P\left(\bigcup_{i=1}^{n} A_i \cap A_{n+1}\right). \end{aligned} \tag{2}$$

Using the inductive hypothesis again, the last term in (2) becomes

$$\begin{aligned} P\left(\bigcup_{i=1}^{n} A_i \cap A_{n+1}\right) &= P\left(\bigcup_{i=1}^{n} (A_i \cap A_{n+1})\right) \qquad \text{by distribution} \\ &= \sum_{i=1}^{n+1} P(A_i \cap A_{n+1}) - \sum_{1\le i<j\le n} P(A_i \cap A_j \cap A_{n+1}) \\ &\quad + \sum_{1\le i<j<k\le n} P(A_i \cap A_j \cap A_k \cap A_{n+1}) - ... + (-1)^{n-1} P\left(\bigcap_{i=1}^{n} A_i \cap A_{n+1}\right) \end{aligned}$$

We plug this back into (2), and get, for example,

$$- \sum_{1\le i<j\le n} P(A_i \cap A_j) - \sum_{i=1}^{n+1} P(A_i \cap A_{n+1}) = - \sum_{1\le i<j\le n+1} P(A_i \cap A_j).$$

Similarly, collecting like terms in the other sums (i.e., terms with the same number of A_j's getting unioned together) and rearranging gives

$$\begin{aligned} P\left(\bigcup_{i=1}^{n+1} A_i\right) &= \sum_{i=1}^{n} P(A_i) - \sum_{1\le i<j\le n} P(A_i \cap A_j) + \sum_{1\le i<j<k\le n} P(A_i \cap A_j \cap A_k) \\ &\qquad - ... + (-1)^{n-1} P\left(\bigcap_{i=1}^{n} A_i\right), \end{aligned}$$

as desired.

2. (Chap. 1, Problem 3).

(a) Let $\{X_n, n \geq 1\}$ be a sequence of random variables on a probability space (Ω, Σ, P). Show that

$$X_n \xrightarrow{\text{P}} X \qquad \iff \qquad E\left(\frac{|X_n - X|}{1 + |X_n - X|}\right) \xrightarrow{n\to\infty} 0.$$

($\Rightarrow$) Let $X_n \xrightarrow{\text{P}} X$, i.e., $P[|X_n - X| \geq \varepsilon] \xrightarrow{n\to\infty} 0$. Now if we define

$$A_{n,\varepsilon} := \{|X_n - X| < \varepsilon\},$$

we can say

$$\begin{aligned}
E\left(\frac{|X_n - X|}{1 + |X_n - X|}\right) &= \int_\Omega \frac{|X_n - X|}{1 + |X_n - X|} dP \\
&= \int_{A_{n,\varepsilon}} \frac{|X_n - X|}{1 + |X_n - X|} dP + \int_{A_{n,\varepsilon}^C} \frac{|X_n - X|}{1 + |X_n - X|} dP \\
&< \int_{A_{n,\varepsilon}} \frac{\varepsilon}{1+\varepsilon} dP + \int_{A_{n,\varepsilon}^C} 1 dP \\
&= \int_{A_{n,\varepsilon}} \frac{\varepsilon}{1+\varepsilon} dP + P[|X_n - X| \geq \varepsilon] \\
&\xrightarrow{n\to\infty} \int_{\Omega\setminus N} \frac{\varepsilon}{1+\varepsilon} dP + 0
\end{aligned}$$

where $P(N) = 0$. But then

$$\begin{aligned}
\lim_{n\to\infty} E\left(\frac{|X_n - X|}{1 + |X_n - X|}\right) &\leq \int_\Omega \frac{\varepsilon}{1+\varepsilon}\, dP \\
&\leq \int_\Omega \varepsilon\, dP \\
&\leq \varepsilon \int_\Omega dP \\
&\leq \varepsilon
\end{aligned}$$

for any $\varepsilon > 0$, which shows $\lim_{n\to\infty} E\left(\frac{|X_n - X|}{1+|X_n - X|}\right) = 0$.

($\Leftarrow$) Now assume $\lim_{n\to\infty} E\left(\frac{|X_n - X|}{1+|X_n - X|}\right) = 0$. Define

$$A_n := \{|X_n - X| \geq \varepsilon\}.$$

Now

$$\begin{aligned}
P\left[|X_n - X| \geq \varepsilon\right] &= \int_\Omega \chi_{A_n}\, dP \\
&= \int_\Omega \lim_{n\to\infty} \frac{|X_n - X|}{1 + |X_n - X|} \chi_{A_n}\, dP
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \int_\Omega \frac{|X_n - X|}{1+|X_n - X|} \chi_{A_n}\, dP \\
&\le \lim_{n\to\infty} \int_\Omega \frac{|X_n - X|}{1+|X_n - X|}\, dP \\
&= \lim_{n\to\infty} E\left(\frac{|X_n - X|}{1+|X_n - X|}\right) \\
&\xrightarrow{n\to\infty} 0.
\end{aligned}
$$

(b) Verify that

$$d(X,Y) := E\left(\frac{|X-Y|}{1+|X-Y|}\right)$$

defines a metric on the space of random variables $\mathcal{L}^0$, and that $\mathcal{L}^0$ is an algebra.

(i) Positivity. Clearly, $d(X,X) = E\left(\frac{|X-X|}{1+|X-X|}\right) = E(0) = 0$. So let $X \neq Y$. Then define

$$A = \{X \neq Y\} \quad \text{and} \quad A_n = \{|X-Y| \geq \tfrac{1}{n}\}.$$

Now we have

$$
\begin{aligned}
d(X,Y) &= \int_A \frac{|X-Y|}{1+|X-Y|}\, dP \\
&\ge \int_{A_n} \frac{|X-Y|}{1+|X-Y|}\, dP \\
&\ge \int_{A_n} \frac{1/n}{1+1/n}\, dP \\
&= \frac{n}{n+1} P(A_n) \\
&> 0 \text{ for } P(A_n) > 0.
\end{aligned}
$$

So $X \neq Y$ implies there is some n for which $P(A_n) > 0$, in which case $d(X,Y) > 0$.

(ii) Symmetry. $d(X,Y) = E\left(\frac{|X-Y|}{1+|X-Y|}\right) = E\left(\frac{|Y-X|}{1+|Y-X|}\right) = d(Y,X)$.

(iii) Triangle inequality. Consider the function $f : \mathbb{R}^+ \to [0,1]$ by $f(x) = \frac{x}{1+x}$. Taking derivatives of this function shows that it is concave increasing with slope less than 1 for all $x > 0$. Alternatively, see Lemma 1 in Problem 2. This gives $f(a+b) \leq f(a) + f(b)$ immediately, for $a, b \geq 0$. Using $a = |X-Y|$ and $b = |Y-Z|$,

$$f(a+b) = \frac{|X-Y| + |Y-Z|}{1 + |X-Y| + |Y-Z|}$$

$$\leq \frac{|X-Y|}{1+|X-Y|} + \frac{|Y-Z|}{1+|Y-Z|}$$
$$= f(a) + f(b).$$

Since
$$\frac{|X-Z|}{1+|X-Z|} \leq \frac{|X-Y|+|Y-Z|}{1+|X-Y|+|Y-Z|}$$
by the triangle inequality,
$$\begin{aligned} d(X,Z) &= \int_\Omega \frac{|X-Z|}{1+|X-Z|}\,dP \\ &\leq \int_\Omega \frac{|X-Y|}{1+|X-Y|}\,dP + \int_\Omega \frac{|Y-Z|}{1+|Y-Z|}\,dP \\ &= d(X,Y) + d(Y,Z). \end{aligned}$$

To see that $\mathcal{L}^0$ is an algebra, we make some basic observations, namely:

(i) A sum of measurable functions is again measurable.

(ii) The pointwise product of measurable functions is again measurable.

(iii) Any scalar multiple of a measurable function is again measurable.

Pointwise multiplication is associative, even commutative. Also, we have the identity $f(x) \equiv 0$ and unit $g(x) \equiv 1$.

3. (Chap. 2, Problem 2).

(a) Let $\phi : \mathbb{R} \to \mathbb{R}^+$ be a continuous function such that ϕ is increasing and convex on $\mathbb{R}^+$, and with $\phi(0) = 0$ and $\phi(-x) = x$. Also, assume ϕ satisfies $\phi(2x) \leq c\phi(x)$ for $x \geq 0$, for some $0 < c < \infty$. Let $X_i : \Omega \to \mathbb{R}, i = 1,2$ be two random variables on (Ω, Σ, P). If $E(\phi(X_i)) < \infty, i = 1,2$, then verify $E(\phi(X_1 + X_2)) < \infty$. Show the converse is also true if the X_i are independent.

($\Rightarrow$) Since ϕ increasing implies ϕ is order-preserving, we bound $E(\phi(X_1 + X_2))$ as follows:

$$\begin{aligned} E(\phi(X_1+X_2)) &= \int_\Omega \phi(X_1+X_2)\,dP \\ &\leq \int_{\{X_1 \geq X_2\}} \phi(2X_1)\,dP + \int_{\{X_2 > X_1\}} \phi(2X_2)\,dP && \phi \text{ increasing} \\ &\leq E(\phi(2X_1)) + E(\phi(2X_1)) && P \text{ is monotone} \\ &\leq cE(\phi(2X_1)) + cE(\phi(X_1)) && \text{given} \\ &< \infty \end{aligned}$$

($\Leftarrow$) Now we take X_1, X_2 to be independent. Then with $A_n = \{|X_2| \leq n\}$,

$$\begin{aligned}
E(\phi(X_1+X_2)) &= E(\phi(|X_1+X_2|)) && \phi(-x)=\phi(x) \\
&\geq E\left(\phi\left(||X_1|-|X_2||\right)\right) && |a+b| \geq ||a|-|b|| \\
&= E\left(\phi\left(|X_1|-|X_2|\right)\right) && \phi(-x)=\phi(x) \\
&= \int_\Omega \phi\left(|X_1|-|X_2|\right)\,dP && \text{def of } E \\
&= \int_{A_n} \phi\left(|X_1|-|X_2|\right)\,dP \\
&\qquad + \int_{A_n^c} \phi\left(|X_1|-|X_2|\right)\,dP && \Omega = A_n \sqcup A_n^c \\
&\geq \int_{A_n} \phi\left(|X_1|-n\right)\,dP + 0 && \text{def of } A_n \\
&= E\left(\phi\left(|X_1|-n\right)\chi_{A_n}\right) && \text{def of } E \\
&= E\left(\phi\left(|X_1|-n\right)\right)P\left(A_n\right). && \text{independence}
\end{aligned}$$

Now we take note of two things. First,

$$A_n \nearrow \Omega \quad \Longrightarrow \quad P(A_n) \nearrow 1,$$

so we may assume $0 < P(A_n) \leq 1$ and concern ourselves just with the other factor. Second,

$$\begin{aligned}
E\left(\phi\left(|X_1|-n\right)\right) &= \int_{\mathbb{R}} \phi\left(|x_1|-n\right)\,dF_X(x) \\
&= \int_{\mathbb{R}} \phi\left(|x_1|\right)\,dF_X(x) \\
&= E\left(\phi\left(|X_1|\right)\right)
\end{aligned}$$

by FLoP. (Translation doesn't matter when we integrate over all of $\mathbb{R}$.) Thus

$$\begin{aligned}
E(\phi(X_1+X_2)) &\geq E\left(\phi\left(|X_1|\right)\right)P\left(A_n\right) \\
&\xrightarrow{n\to\infty} E(\phi(|X_1|)) = E(\phi(X_1))
\end{aligned}$$

Since a similar procedure may be used to bound $E(\phi(X_2))$, we have

$$E(\phi(X_1)), E(\phi(X_2)) < \infty.$$

(b) Let $\phi : \mathbb{R} \to \mathbb{R}^+$ be a continuous function such that ϕ is increasing and concave on $\mathbb{R}^+$, and with $\phi(0) = 0$ and $\phi(-x) = x$. Let $X_i : \Omega \to \mathbb{R}, i = 1, 2$ be two random variables on (Ω, Σ, P). If $E(\phi(X_i)) < \infty, i = 1, 2$, then verify $E(\phi(X_1 + X_2)) < \infty$. Show the converse is also true if the X_i are independent.

($\Rightarrow$) First we prove the following lemma.

Lemma 1. *If ϕ is concave on $\mathbb{R}^+$, then for any $x, y > 0$ we have $\phi(x + y) \leq \phi(x) + \phi(y)$.*

Proof. Method 1. Since ϕ is concave, it is absolutely continuous on any open interval and hence may be represented as the integral of its derivative. Thus we may write

$$\begin{aligned}
\phi(x+y) &= \int_0^{x+y} \phi'(t)\,dt \\
&= \int_0^{x} \phi'(t)\,dt + \int_x^{x+y} \phi'(t)\,dt \\
&= \phi(x) + \int_0^{y} \phi'(t+x)\,dt && \text{CoV} \\
&\leq \phi(x) + \int_0^{y} \phi'(t)\,dt && \phi' \text{ decreasing} \\
&= \phi(x) + \phi(y), && \text{FToC}
\end{aligned}$$

where the inequality is due to the fact that ϕ' is decreasing whenever ϕ is concave. □

Proof. Method 2. Wlog, take $0 < x < y$. Then $x < y < x + y$, so

$$y = \alpha x + (1 - \alpha)(x + y) \qquad \text{for } \alpha = \tfrac{x}{y} \in (0, 1).$$

Then concavity means

$$\begin{aligned}
\phi(y) &= \phi(\alpha x + (1-\alpha)(x+y)) \\
&\geq \alpha\phi(x) + (1-\alpha)\phi(x+y) \\
\phi(x) + \phi(y) &\geq \phi(x) + \alpha\phi(x) + \phi(x+y) - \alpha\phi(x+y).
\end{aligned}$$

So it remains to show

$$\phi(x) + \alpha\phi(x) - \alpha\phi(x+y) \geq 0.$$

But this is just equivalent to

$$\begin{aligned}
\phi(x) + \alpha\phi(x) &\geq \alpha\phi(x+y) \\
\phi(x) + \tfrac{x}{y}\phi(x) &\geq \tfrac{x}{y}\phi(x+y) \\
(x+y)\phi(x) &\geq x\phi(x+y) \\
\frac{\phi(x)}{x} &\geq \frac{\phi(x+y)}{(x+y)},
\end{aligned}$$

which is another form of the definition of concavity; the decreasing secants:

$$s < t < u \quad \Longrightarrow \quad \frac{f(t)-f(s)}{t-s} \geq \frac{f(u)-f(s)}{u-s} \geq \frac{f(u)-f(t)}{u-t},$$

with $s = 0, t = x, u = x + y$. □

Hence,

$$\begin{aligned}
E\left(\phi\left(X_1+X_2\right)\right) &= E\left(\phi\left(\left|X_1+X_2\right|\right)\right) && \phi(-x)=\phi(x)\\
&\leq E\left(\phi\left(\left|X_1\right|+\left|X_2\right|\right)\right) && \triangle\text{-ineq}\\
&\leq E\left(\phi\left(\left|X_1\right|\right)+\phi\left(\left|X_2\right|\right)\right) && \text{by Lemma}\\
&= E\left(\phi\left(\left|X_1\right|\right)\right)+E\left(\phi\left(\left|X_2\right|\right)\right) && \text{linearity}\\
&= E\left(\phi(X_1)\right)+E\left(\phi(X_2)\right) && \phi(-x)=\phi(x)\\
&< \infty
\end{aligned}$$

($\Leftarrow$) The converse here goes through exactly as it did in the previous case.

4. (Chap. 2, Problem 3).
Let $X_1, X_2 : \Omega \to \mathbb{R}$ be independent with $E(X_1) = 0$. Again, take $\phi : \mathbb{R} \to \mathbb{R}^+$ to be a continuous function which is increasing and convex on $\mathbb{R}^+$, and satisfies $\phi(0) = 0$ and $\phi(-x) = x$. Prove that $E(\phi(X_1+X_2)) < \infty$ implies $E(\phi(X_2)) \leq E(\phi(X_1+X_2))$. If $E(X_2) = 0$ is also assumed, prove $E(\phi(X_i)) \leq E(\phi(X_1 + X_2))$.

We write

$$\begin{aligned}
\phi(x_2) = \phi(0+x_2) &= \phi\left(E(X_1)+x_2\right) && E(X_1)=0\\
&= \phi\left(E(X_1+x_2)\right) && \text{linearity}\\
&\leq E\left(\phi(X_1+x_2)\right) && \text{Jensen's ineq}\\
&= \int_{\mathbb{R}} \phi(x_1+x_2)\, dF_{X_1}. && \text{FLoP}
\end{aligned}$$

Now we integrate both side with respect to dF_{X_2}, as follows:

$$\begin{aligned}
\int_{\mathbb{R}} \phi(x_2)\, dF_{X_2} &\leq \int_{\mathbb{R}}\int_{\mathbb{R}} \phi(x_1+x_2)\, dF_{X_1} dF_{X_2}\\
E(\phi(X_2)) &\leq \int_{\mathbb{R}}\int_{\mathbb{R}} \phi(x_1+x_2)\, dF_{X_1+X_2} && \text{independence}\\
E(\phi(X_2)) &\leq E(\phi(X_1+X_2)).
\end{aligned}$$

For the case $E(X_2) = 0$, the identical technique may be applied.

If $X_1, X_2 \geq 0$, the problem becomes easy. first we note that the Lemma proven in the previous problem will also work for convex functions, with the inequality reversed: $\phi(x+y) \geq \phi(x) + \phi(y)$. Then

$$
\begin{aligned}
E(\phi(X_1 + X_2) &\geq E(\phi(X_1) + \phi(X_2)) && \text{Lemma} \\
&= E(\phi(X_1)) + E(\phi(X_2)) && \text{linearity} \\
&\geq \phi(E(X_1)) + E(\phi(X_2)) && \text{Jensen's} \\
&= \phi(0) + E(\phi(X_2)) && E(X_1) = 0 \\
&= E(\phi(X_2)). && \phi(0) = 0
\end{aligned}
$$

The other case follows similarly.