I. Introduction & Motivation

I.1. **Physical motivation.**

What can you really measure? (Realistically)

$$T(x) \text{ vs. } \int T(x)\varphi(x)\,dx.$$

I.2. **Mathematical motivation.**

How to "differentiate" nondifferentiable functions?

IBP:

$$\int_a^b T'(x)\varphi(x)\,dx = -\int_a^b T(x)\varphi'(x)\,dx,$$

(*) provided φ is nice, and the boundary terms vanish.

→ Heaviside's operational calculus (c. 1900)
→ Sobolev (1930s): continuous linear functionals over some space of test functions
→ Schwartz (1950s): duality of certain topological vector spaces

Consider the DE

$$u' = H,$$

where $H(x)$ is the Heaviside function:

$$H(x) = \begin{cases} 1, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

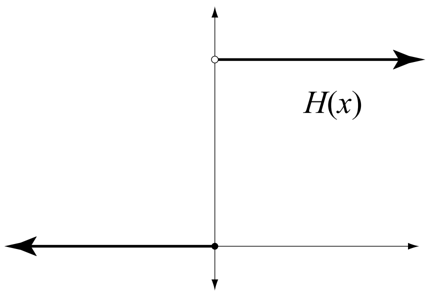

FIGURE 1. The Heaviside function $H(x)$.

We would like to say that the soln is

$$x_+ = \begin{cases} x, & x > 0 \\ 0, & x \leq 0, \end{cases}$$

but this function is not a classical soln: it is not differential at 0.

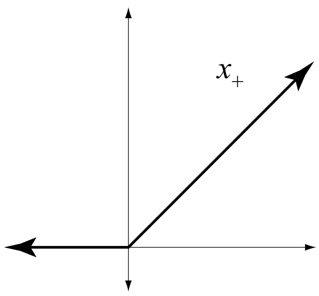

FIGURE 2. The piecewise continuous function $x+$.

II. Basics of the theory

Let us not require T to have a specific value at x. T is no longer a function— call it a *generalized function.*

Think of T as acting on a "weighted open set" instead of on a point x.

"$T(x) \mapsto T(\varphi)$"

Formalize: T acts on "test functions" φ.

Denote: $\langle T, \varphi \rangle = \int T(x)\varphi(x)\, dx$.

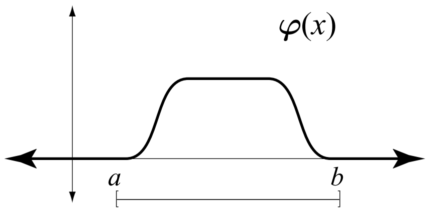

Figure 3. A test function φ.

Reqs for "nice" test functions φ :

(1) $\varphi \in C^\infty$, and

(2) boundary terms must vanish ($\varphi(-\infty) = \varphi(\infty) = 0$).

(a) $\varphi \in C_c$, i.e., φ has compact support, or

(b) $\varphi(x) \xrightarrow{|x|\to\infty} 0$ *quickly*(with derivs)

Choosing (2a) leads to the theory of distributions à la Laurent Schwartz.

Choosing (2b) leads to the theory of tempered distributions.

Definition 1. The space of *test functions* is

$$\mathcal{D}(\Omega) := C_c^\infty(\Omega)$$

Note: (2a) allows more distributions than (2b).
(larger class of test functions $\Rightarrow$ fewer distributions.

Reqs for a distribution :

(1) $\langle T, \varphi \rangle$ must exist for $\varphi \in \mathcal{D}$.
(2) Linearity: $\langle T, a_1\varphi_1 + a_2\varphi_2 \rangle = a_1\langle T, \varphi_1 \rangle + a_2\langle T, \varphi_2 \rangle$.

Taking a cue from Riesz[1]:

Definition 2. The space of *distributions* is the (topological) dual space

$$\mathcal{D}'(\Omega) = \{\text{continuous linear functionals on } \mathcal{D}(\Omega)\}.$$

[1] Actually, $\mathcal{D}'(\Omega)$ is a larger class than the class of Borel measures on Ω.

Examples

1. Any $f \in L^1_{loc}(\Omega)$.

$$\langle T_f, \varphi \rangle = \int f(x)\varphi(x)\, dx$$

"The distribution f" really means T_f.

2. Any regular Borel measure μ on Ω

$$\langle T_\mu, \varphi \rangle = \int \varphi(x)\, d\mu$$

T is *regular* iff $T = T_f$ for $f \in L^1_{loc}(\Omega)$. Otherwise, T is *singular*.

The most famous singular distribution, Dirac-δ:

$$\langle \delta, \varphi \rangle = \varphi(0) = \int \delta(x)\varphi(x)\, dx = \int \varphi(x)\, d\mu$$

so

$$“\delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases}”$$

for

$$\mu(E) = \begin{cases} 1, & 0 \in E \\ 0, & 0 \notin E \end{cases}.$$

Some distributions do not even come from a measure.

Example.

$$\langle \delta', \varphi \rangle := -\varphi'(0).$$

Note: require smooth test functions $\Rightarrow$ allow rough distributions; differentiability of a distribution relies on differentiability of test functions.
To define $\langle \delta, \varphi \rangle$, φ must be C^0.
To define $\langle \delta', \varphi \rangle$, φ must be C^1.

III. Differentiation of Distributions.

Key point: how do we understand T? Only by $\langle T, \varphi \rangle$. So what does T' mean? Must understand $\langle T', \varphi \rangle$.

Working in $\Omega = \mathbb{R}$:

$$\begin{aligned}
\langle T', \varphi \rangle &= \int_{-\infty}^{\infty} T'(x)\varphi(x)\, dx \\
&= [T(x)\varphi(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} T(x)\varphi'(x)\, dx \\
&= -\langle T, \varphi' \rangle
\end{aligned}$$

Definition 3. For any $T \in \mathcal{D}'(\Omega)$,

$$\langle D_k T, \varphi \rangle := -\langle T, D_k \varphi \rangle, \quad \varphi \in \mathcal{D}.$$

More generally (induct):

$$\langle D^\alpha T, \varphi \rangle := (-1)^{|\alpha|} \langle T, D^\alpha \varphi \rangle, \quad \varphi \in \mathcal{D}.$$

Note: this formula shows every distribution is (infinitely) differentiable.

Note: $D^{\alpha+\beta} T = D^\alpha (D^\beta T)$.

Recall that x_+ is not differentiable in the classical sense.

$$x_+ = \begin{cases} x, & x > 0 \\ 0, & x \leq 0, \end{cases}$$

But x_+ can be differentiated as a distribution:

$$\begin{aligned} \langle x_+', \varphi \rangle &= -\langle x_+, \varphi' \rangle \\ &= -\int_0^\infty x\varphi'(x)\,dx \\ &= [-x\varphi(x)]_0^\infty + \int_0^\infty \varphi(x)\,dx \\ &= 0 + \int_{-\infty}^\infty H(x)\varphi(x)\,dx \\ &= \langle H, \varphi \rangle \end{aligned}$$

So $x_+' = H$, as hoped. Similarly,

$$\begin{aligned} \langle x_+'', \varphi \rangle &= \langle H', \varphi \rangle \\ &= -\langle H, \varphi' \rangle \\ &= -\int_0^\infty \varphi'(x)\,dx \\ &= \varphi(0) \end{aligned}$$

So $x_+'' = H' = \delta$, as hoped.

(Motivation for $\varphi \in C^\infty$.)

IV. A Toolbox for Distribution Theory

Definition 4. For $\{\varphi_k\}_{k=1}^{\infty} \subseteq \mathcal{D}$, $\varphi_k \to 0$ iff

(i) $\exists K \subseteq \Omega$ compact s.t. $\operatorname{spt} \varphi_k \subseteq K, \forall k$, and

(ii) $D^\alpha \varphi_k \to 0$ uniformly on K, $\forall \alpha$.

Definition 5. For $\{T_k\}_{k=1}^{\infty} \subseteq \mathcal{D}'$,

$$T_k \to 0 \qquad \Longleftrightarrow \qquad \langle T_k, \varphi \rangle \to 0 \in \mathbb{C}, \forall \varphi \in \mathcal{D}.$$

This is *weak* or *distributional* (or "pointwise") convergence.

Theorem 6. Differentiation is linear & continuous.

Proof. a)

$$\begin{aligned} \langle (aT_1 + bT_2)', \varphi \rangle &= -\langle aT_1 + bT_2, \varphi' \rangle \\ &= -a\langle T_1, \varphi' \rangle - b\langle T_2, \varphi' \rangle \\ &= a\langle T_1', \varphi \rangle + b\langle T_2', \varphi \rangle \end{aligned}$$

b) Let $\{T_n\} \subseteq \mathcal{D}'$ converge to T in $\mathcal{D}'$.

$$\begin{aligned} \langle T_n', \varphi \rangle &= -\langle T_n, \varphi' \rangle \\ &\xrightarrow{n \to \infty} -\langle T, \varphi' \rangle \\ &= \langle T', \varphi \rangle \end{aligned}$$

□

Theorem 7. $\mathcal{D}'$ is complete.

Theorem 8. $\mathcal{D}$ is dense in $\mathcal{D}'$.
(Given $T \in \mathcal{D}'$, $\exists\{\varphi_k\} \subseteq \mathcal{D}$ such that $\varphi_k \to T$ as distributions.)

Proposition 9. If f has classical derivative f' and f' is integrable on Ω, then

$$T_f' = T_{f'}\,.$$

Theorem 10. $\{f_k\} \subseteq L^1_{loc}$, $f_k \xrightarrow{ae} f$, and $|f_k| \leq g \in L^1_{loc}$. Then $f_k \xrightarrow{\mathcal{D}'} f$.
(i.e., $T_{f_k} \xrightarrow{\mathcal{D}'} T_f$)

Definition 11. A sequence of functions $\{f_k\}$ such that $f_k \xrightarrow{\mathcal{D}'} \delta$ is a *delta-convergent sequence* , or δ*-sequence.*

Theorem 12. Let $f \geq 0$ be integrable on $\mathbb{R}^n$ with $\int f = 1$. Define

$$f_\lambda(x) = \lambda^{-n} f\left(\tfrac{x}{\lambda}\right) = \lambda^{-n} f\left(\tfrac{x_1}{\lambda}, \ldots, \tfrac{x_n}{\lambda}\right)$$

for $\lambda > 0$. Then $f_\lambda \xrightarrow{\mathcal{D}'} \delta$ as $\lambda \to 0$.

Example.

$$\int_{\mathbb{R}} \frac{dx}{1+x^2} = \pi \qquad \Longrightarrow \qquad \text{define } f(x) = \frac{1}{\pi(1+x^2)}.$$

Then obtain the delta-sequence

$$f_\lambda = \tfrac{1}{\lambda} \cdot \tfrac{1}{\pi(1+(x/\lambda)^2)} = \tfrac{\lambda}{\pi(x^2+\lambda^2)}.$$

Example.

$$\int_{\mathbb{R}} e^{-x^2} = \pi^{1/2} \qquad \Longrightarrow \qquad \text{define } f(x) = \tfrac{e^{-x^2}}{\sqrt{\pi}}.$$

Then obtain the δ-sequence $f_\lambda = \frac{e^{-x^2/\lambda}}{\sqrt{\pi\lambda}}$. Further,

$$\begin{aligned} \int_{\mathbb{R}^n} e^{-|x|^2} dx &= \int_{\mathbb{R}^n} \prod_{k=1}^{n} e^{-x_k^2} dx_k \\ &= \prod_{k=1}^{n} \int_{\mathbb{R}^n} e^{-x_k^2} dx_k \\ &= \pi^{n/2} \end{aligned}$$

gives the n-dimensional δ-sequence

$$f_\lambda(x) = \frac{e^{-|x|^2/\lambda}}{(\pi\lambda)^{n/2}}. \qquad (1)$$

IV.1. **Extension from test functions to distributions.**

Approximation and adjoint identities

Approximation: For $T \in \mathcal{D}'$, and an operator S defined on $\mathcal{D}$, find arbitrary $\varphi_n \to T$ and define $ST = \lim S\varphi_n$.

Example. (Translation)
Define $S = \tau_h$ on $\mathcal{D}$ by $\tau_h(\varphi) = \langle \tau_h, \varphi \rangle = \varphi(x - h)$.
To define $\tau_h T$: find an arb sequence $\{\varphi_n\} \subseteq \mathcal{D}$, with $\varphi_n \xrightarrow{\mathcal{D}'} T$.
Then $\langle T, \varphi \rangle = \int T(x)\varphi(x)\,dx = \lim \int \varphi_n(x)\varphi(x)\,dx$, so

$$\begin{aligned}
\langle \tau_h T, \varphi \rangle &= \lim \int \tau_h \varphi_n(x)\varphi(x)\,dx \\
&= \lim \int \varphi_n(x - h)\varphi(x)\,dx \\
&= \lim \int \varphi_n(x)\varphi(x + h)\,dx \\
&= \lim \int \varphi_n(x)\tau_{-h}\varphi(x)\,dx \\
&= \langle T, \tau_{-h}\varphi \rangle
\end{aligned}$$

Example. (Differentiation)
Define $S = \frac{d}{dx}$ on $\mathcal{D}(\mathbb{R})$.
Let I denote the identity, use $\tau_h \xrightarrow{h\to 0} I$.
Then $\frac{d}{dx} = \lim_{h\to 0} \frac{1}{h}(\tau_h - I)$, so

$$\begin{aligned}
\langle \tfrac{d}{dx}T, \varphi\rangle &= \lim_{h\to 0} \tfrac{1}{h}\left(\langle \tau_h T, \varphi\rangle - \langle T, \varphi\rangle\right)\\
&= \lim_{h\to 0} \tfrac{1}{h}\left(\langle T, \tau_{-h}\varphi\rangle - \langle T, \varphi\rangle\right)\\
&= \lim_{h\to 0} \langle T, \tfrac{1}{h}\left(\tau_{-h}\varphi - \varphi\right)\rangle\\
&= \langle T, \lim_{h\to 0} \tfrac{1}{h}\left(\tau_{-h} - I\right)\varphi\rangle\\
&= \langle T, -\tfrac{d}{dx}\varphi\rangle
\end{aligned}$$

Adjoint Identities: Let $T \in \mathcal{D}'$ and let R be an operator such that $R\varphi \in \mathcal{D}$. Define S as the operator which satisfies

$$\langle ST, \varphi\rangle = \langle T, R\varphi\rangle, i.e.$$
$$\int ST(x)\varphi(x)\,dx = \int T(x)R\varphi(x)\,dx.$$

Definition 13. S is the *adjoint* of R.
If $S = R$, we say R is *self-adjoint*.
Example: The Laplacian Δ. $\langle \Delta T, \varphi\rangle = \langle T, \Delta\varphi\rangle$.

Example.
(Translation) Define $S = \tau_h$ by $\langle S\psi, \varphi\rangle := \langle \psi, \tau_{-h}\varphi\rangle$.

$$\int \tau_h\psi(x)\varphi(x)\,dx = \int \psi(x)\tau_{-h}\varphi(x)\,dx$$

Example.
(Differentiation) Define $S = \frac{d}{dx}$ by $\langle S\psi, \varphi\rangle := -\langle \psi, \frac{d}{dx}\varphi\rangle$.

$$\int \left(\tfrac{d}{dx}\psi(x)\right)\varphi(x)\,dx = -\int \psi(x)\left(\tfrac{d}{dx}\varphi(x)\right)\,dx$$

Example.
(Multiplication) Define S by $\langle S\psi, \varphi\rangle := \langle \psi, f\cdot\varphi\rangle$.

$$\int (f(x)\psi(x))\,\varphi(x)\,dx = \int \psi(x)\,(f(x)\varphi(x))\,dx$$

Note: this only works when $f\cdot\varphi \in \mathcal{D}$! So require $f \in C^\infty$.

Example of trouble: let $f(x) = \operatorname{sgn}(x)$ so f is discontinuous at 0. Then $f\cdot\delta$ cannot be defined:

$$\langle f\cdot\delta, \varphi\rangle = \langle \delta, f\cdot\varphi\rangle = f(0)\varphi(0)$$

but $f(0)$ is undefined. Thus, the product of two arbitrary distributions is undefined.

IV.2. **Multiplication.** There is no way of defining the product of two distributions as a natural extension of the product of two functions. ☹

Definition 14. For $T \in \mathcal{D}'$, $f \in C^\infty$, can define their product fT as the linear functional

$$\langle fT, \varphi \rangle = \langle T, f\varphi \rangle, \quad \forall \varphi \in \mathcal{D}.$$

When T is regular,

$$\begin{aligned} \langle fT_g, \varphi \rangle &= \langle T_g, f\varphi \rangle \\ &= \int gf\varphi \\ &= \langle fg, \varphi \rangle \end{aligned}$$

since fg is also in L^1_{loc}. Thus $fT_g = T_{fg}$ in this case.

Example. (Leibniz Rule)

$$\begin{aligned} \langle D_k(fT), \varphi \rangle &= -\langle fT, D_k\varphi \rangle \\ &= -\langle T, fD_k\varphi \rangle \\ &= -\langle T, D_k(f\varphi) - (D_kf)\varphi \rangle \\ &= -\langle T, D_k(f\varphi) \rangle + \langle T, (D_kf)\varphi \rangle \\ &= \langle fD_kT, \varphi \rangle + \langle (D_kf)T, \varphi \rangle \end{aligned}$$

Thus,

$$D_k(fT) = fD_kT + (D_kf)T.$$

V. Things from last time

A *distribution* is a continuous linear functional on $C_c^\infty(\Omega)$

$$\langle T, \varphi \rangle = T(\varphi) := \int T(x)\varphi(x)\, dx.$$

$$\langle T', \varphi \rangle = -\langle T, \varphi' \rangle, \qquad \langle \partial^\alpha T, \varphi \rangle = (-1)^{|\alpha|}\langle T, \partial^\alpha \varphi \rangle.$$

$$\langle \delta, \varphi \rangle = \varphi(0)$$

$$\langle \delta', \varphi \rangle = -\varphi'(0).$$

For $x_+ = \max\{0, x\}$ and the Heaviside function $H(x)$,

$$x_+'' = H' = \delta.$$

The translation operator

$$\langle \tau_h, \varphi \rangle = \tau_h(\varphi) = \varphi(x - h)$$

$$\langle \tau_h T, \varphi \rangle = \langle T, \tau_{-h}\varphi \rangle.$$

The multiplication-by-a-smooth-function operator

$$\langle f \cdot T, \varphi \rangle = \langle T, f \cdot \varphi \rangle.$$

Adjoint Identities: Let $T \in \mathcal{D}'$ and let R be an operator such that $R\varphi \in \mathcal{D}$. Define S as the operator which satisfies

$$\langle ST, \varphi \rangle = \langle T, R\varphi \rangle.$$

VI. Convolutions

Convolution is a smoothing process: convolve anything with a C^m function and the result is C^m, even if $m = \infty$.

Strategy: T may be nasty, but $T * \varphi$ is lovely (C^∞), so work with it instead.

Even better, find $\{\varphi_k\}$ such that $T * \varphi_k \to T$ and use it to extend the usual rules of calculus & DEs.

Need $\varphi(x-y)$ to discuss convolutions, so consider functions of 2 variables for a moment:

$$\varphi(x, y) \in \mathcal{D}(\Omega_1 \times \Omega_2).$$

Let T_i be a distribution on Ω_i ($T_i \in \mathcal{D}(\Omega_i)$).
For fixed $y \in \Omega_2$, the function $\varphi(\cdot, y)$ is in $\mathcal{D}(\Omega_1)$, and T_1 maps $\varphi(\cdot, y)$ to the number

$$\langle T_1, \varphi(\cdot, y)\rangle = T_1(\varphi)(y).$$

Theorem 15. For φ, T_i as above, $T_1(\varphi) \in \mathcal{D}(\Omega_2)$ and $\partial_y^\beta T_1(\varphi) = T_1\left(\partial_y^\beta \varphi\right)$.

So $T_i : \varphi \mapsto T_i(\varphi)$ preserves the smoothness of $\varphi \in \mathcal{D}$.

Corollary 16. If $\varphi \in C^\infty(\Omega_1 \times \Omega_2)$ has compact support as a function of x and y separately, then $T_1(\varphi) \in C^\infty(\Omega_2)$ for every $T_1 \in \mathcal{D}'(\Omega_1)$.

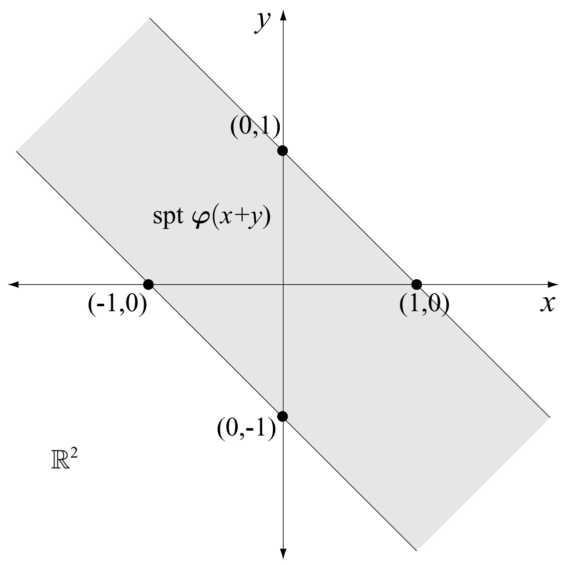

FIGURE 4. For $\varphi \in \mathcal{D}(\mathbb{R})$ with support $\text{spt}(\varphi) = [-1, 1]$, the function $\varphi(x + y)$ is defined on $\mathbb{R}^2$ and does not have compact support.

Definition 17. *Convolution* of a C_c^∞ function φ with an L^1_{loc} function f is

$$(\varphi * f)(x) := \int \varphi(x - y) f(y)\, dy = \int \varphi(y) f(x - y)\, dy.$$

Now extend convolution to distributions:

$$(T_1 * T_2)(\varphi) = \langle T_1 * T_2, \varphi \rangle := T_1(T_2(\varphi(x+y))), \quad \varphi \in \mathcal{D}(\mathbb{R}^n).$$

Suppose T_i are regular, and defined by $f_i \in L^1_{loc}(\mathbb{R}^n)$, where at least one of the f_i has compact support. Then

$$\begin{aligned}(T_1 * T_2)(\varphi) &= \langle T_1 * T_2, \varphi\rangle \\ &= \big\langle T_1, \langle T_2, \varphi(x+y)\rangle\big\rangle \\ &= \int f_1(x) \int f_2(y)\varphi(x+y)\,dy\,dx \\ &= \int\int f_1(x-y)f_2(y)\varphi(x)\,dy\,dx \quad x \mapsto x-y \\ &= \langle f_1 * f_2, \varphi\rangle\end{aligned}$$

with $f_1 * f_2$ as above.
Note: $f_1 * f_2 = f_2 * f_1 \quad \Longrightarrow \quad T_1 * T_2 = T_2 * T_1$.

VI.1. **Properties of the convolution.**

Let $T \in \mathcal{D}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Then

$$\begin{aligned}\langle \delta * T, \varphi\rangle &= \langle T * \delta, \varphi\rangle \\ &= \big\langle T, \langle \delta, \varphi(x+y)\rangle\big\rangle \\ &= \langle T, \varphi(y)\rangle\end{aligned}$$

shows $\delta * T = T * \delta = T$.
Additionally,

$$\begin{aligned}\langle (D^\alpha\delta) * T, \varphi\rangle &= \big\langle T, \langle (D^\alpha\delta), \varphi(x+y)\rangle\big\rangle \\ &= \langle T, (-1)^{|\alpha|} D^\alpha\varphi(y)\rangle \\ &= \langle D^\alpha T, \varphi\rangle\end{aligned}$$

The Magic Property:

$$(D^\alpha \delta) * T = D^\alpha T = D^\alpha(\delta * T) = \delta * D^\alpha T.$$

Further properties of $*$:[2]

1. $\operatorname{spt}(T_1 * T_2) \subseteq \operatorname{spt} T_1 + \operatorname{spt} T_2$.

2. $T_1 * (T_2 * T_3) = (T_1 * T_2) * T_3 = T_1 * T_2 * T_3$.

3. $D^\alpha(T_1 * T_2) = (D^\alpha \delta) * T_1 * T_2$
$$= (D^\alpha T_1) * T_2 = T_1 * (D^\alpha T_2).$$

4. $\tau_h(T_1 * T_2) = \delta_h * T_1 * T_2$
$$= (\tau_h T_1) * T_2 = T_1 * (\tau_h T_2).$$

To see the last, note that $\tau_h \delta = \delta_h$ by

$$\begin{aligned} \langle \tau_h \delta, \varphi \rangle &= \langle \delta, \tau_{-h} \varphi \rangle \\ &= \langle \delta, \varphi(x+h) \rangle \\ &= \varphi(h) \\ &= \langle \delta_h, \varphi \rangle, \end{aligned}$$

[2] Assuming one of the T_i has compact support.

so $\delta_h * T = \tau_h T$ by

$$\begin{aligned}\langle \delta_h * T, \varphi \rangle &= \Big\langle T, \langle \delta_h, \varphi(x+y) \rangle \Big\rangle \\ &= \langle T, \varphi(x+h) \rangle \\ &= \langle T, \tau_{-h}\varphi \rangle \\ &= \langle \tau_h T, \varphi \rangle.\end{aligned}$$

Conclusion:
$(\mathcal{D}', +, *)$ is a commutative algebra with unit δ.

Example. Let $S * H = \delta$. Then

$$\delta' = (S * H)' = S * H' = S * \delta = S,$$

shows $H^{-1} = \delta'$. Similarly, $(\delta')^{-1} = H$.

Definition 18. For f on $\mathbb{R}^n$, its *reflection* in 0 is $\tilde{f}(x) = f(-x)$. For distributions, we extend the defn by

$$\langle \tilde{T}, \varphi \rangle = \langle T, \tilde{\varphi} \rangle.$$

Theorem 19. The convolution $(T * \psi)(x) = T(\tau_x \tilde{\psi})$ is in $C^\infty(\mathbb{R}^n)$.

Proof. For any $\varphi \in \mathcal{D}(\mathbb{R})$, we have

$$\langle T * \psi, \varphi \rangle = \Big\langle T, \langle \psi(y), \varphi(x+y) \rangle \Big\rangle .$$

Thus

$$\begin{aligned}
\langle \psi(y), \varphi(x+y)\rangle &= \int \psi(y)\varphi(x+y)\,dy \\
&= \int \psi(\xi - x)\varphi(\xi)\,d\xi \\
&= \langle \psi(\xi - x), \varphi(\xi)\rangle \\
&= \langle \tilde{\psi}(x-\xi), \varphi(\xi)\rangle \\
&= \langle \tau_\xi \tilde{\psi}(\xi), \varphi(\xi)\rangle
\end{aligned}$$

shows

$$\begin{aligned}
\langle T * \psi, \varphi\rangle &= \Big\langle T, \langle \tau_\xi \tilde{\psi}(x), \varphi(\xi)\Big\rangle \\
&= \langle T(\tau_\xi \tilde{\psi}), \varphi(\xi)\rangle \\
&= \langle T(\tau_x \tilde{\psi}), \varphi\rangle.
\end{aligned}$$

Finally, note that

$$(T * \psi)(x) = T(\tau_x \tilde{\psi}) = T(\psi(x-y))$$

is smooth by Cor. 16. □

Corollary 20. $T(\varphi) = (T * \tilde{\varphi})(0)$.

VI.2. **Applications of the convolution.**

The C^∞ function

$$\alpha(x) = \begin{cases} \exp\left(-\frac{1}{1-|x|^2}\right), & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$$

has support in the closed unit ball $\overline{B} = \overline{B}(0,1)$.

$$\beta(x) = \alpha(x)\left[\int \alpha(y)\,dy\right]^{-1}$$

is another C^∞ function with support in the closed unit ball $\overline{B}$ which satisfies $\int \beta = 1$. Then take the δ-sequence

$$\beta_\lambda(x) = \frac{1}{\lambda^n}\beta\left(\frac{x}{\lambda}\right).$$

Theorem 21. The C^∞ function $T*\beta_\lambda$ converges strongly[3] to T as $\lambda \longrightarrow 0$.

Definition 22. The convolution of f (or T) with β_λ is called a *regularization* of f (or T).

$$T * \beta_{1/k} = T * \gamma_k$$

is a *regularizing sequence* for the distribution $T \in \mathcal{D}'$.

[3] $\langle T * \beta_\lambda, \varphi\rangle \xrightarrow{unif} \langle T, \varphi\rangle$ on every bounded subset of $\mathcal{D}$.

Proposition 23. Suppose $T' = 0$. Then $T \underset{\text{ae}}{=} c \in \mathbb{R}$.

Proof. Let γ_k be a regularizing sequence for δ. Then

$$(T * \gamma_k)' = T' * \gamma_k = 0$$

for every k, so $T * \gamma_k = c_k$.
$c_k = T * \gamma_k \xrightarrow{\mathcal{D}'} T$, but still must show $\{c_k\}$ converges in $\mathbb{C}$.
Pick $\varphi \in \mathcal{D}$ with $\int \varphi = 1$. Then $c_k = \langle c_k, \varphi \rangle$ converges in $\mathbb{C}$; hence its limit $c = \lim c_k$ coincides with T. $\square$

Note: in general, $f_k \xrightarrow{\mathcal{D}'} f$ does not imply $f_k \xrightarrow{pw} f$ (it doesn't even imply f is a function!). $f_k =$ constant is a special case.

Proposition 24. Suppose $T'' = 0$. Then T is linear a.e.

Proof. For $\varphi \in \mathcal{D}$, $T * \varphi \in C^\infty$, and

$$(T * \varphi)'' = T'' * \varphi = 0.$$

Thus $T * \varphi$ is a linear function of the form $(T * \varphi)(x) = ax + b$.

Let $h(x) = ax + b$. Then

$$\begin{aligned}(h * \beta)(x) &= \int h(x-y)\beta(y)\,dy \\ &= \int [a(x-y)+b]\beta(y)\,dy \\ &= ax + b\end{aligned}$$

since $\int y\beta(y)\,dy = 0$. Thus $h * \beta = h$, so

$$(T * \beta) * \gamma_k = (T * \gamma_k) * \beta = T * \gamma_k.$$

As $k \to \infty$, this gives $T * \beta \underset{\text{ae}}{=} T$. $\square$

Note: this generalizes immediately to

$$T^{(n)} = 0 \quad \Longrightarrow \quad T \underset{\text{ae}}{=} P(x) = a_0 + a_1 x + \cdots + a_m x^m,$$

for $m < n$.

Definition 25. For $T \in \mathcal{D}'$, a distribution S satisfying $D_k S(\varphi) = T(\varphi)$ for all $\varphi \in \mathcal{D}$ is called a *primitive* (*antiderivative*) of T.

Theorem 26. Any distribution in $\mathcal{D}'(\mathbb{R})$ has a primitive which is unique up to an additive constant.

Another notion of regularization.

Definition 27. If f has a pole at x_0, the *Cauchy principal value* of the divergent integral $\int f(x)\,dx$ is

$$pv \int f(x)\,dx = \lim_{\varepsilon \to 0} \int_{|x-x_0|\geq\varepsilon} f(x)\,dx.$$

Definition 28. Obtaining the distributional derivative of $f \in L^1_{loc}$ from a divergent integral by taking the principal value is called *regularizing* the integral. If $f \in L^1_{loc}$ but $D^\alpha f \notin L^1_{loc}$, then $D^\alpha T_f$ is a *regularization* of $T_{D^\alpha f}$.

Example. The function

$$f(x) = \begin{cases} 1/x, & x > 0, \\ 0, & x \leq 0 \end{cases}$$

does not define a distribution on $\mathbb{R}$ However, $f|_{\mathbb{R}^+} \in L^1_{loc}$ and so defines a regular distribution.

As classical derivatives,

$$\tfrac{d}{dx} \log|x| = \tfrac{1}{x}, \quad x \neq 0$$

so what is the relation of the distributional derivative of $\log|x|$ to $1/x$?

$$
\begin{aligned}
\langle \tfrac{d}{dx}\log|x|, \varphi\rangle &= -\langle \log|x|, \varphi'\rangle \\
&= -\int_{\mathbb{R}} \log|x|\varphi'(x)\,dx \qquad (\log|x| \in L^1_{loc}) \\
&= -\lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon} \log|x|\varphi'(x)\,dx \\
&= -\lim_{\varepsilon\to 0}\left[[\log|x|\varphi(x)]_{\varepsilon}^{-\varepsilon} - \int_{|x|\geq\varepsilon}\frac{\varphi(x)}{x}\,dx\right] \\
&= \lim_{\varepsilon\to 0}\left[2\varepsilon\log\varepsilon\tfrac{\varphi(\varepsilon)-\varphi(-\varepsilon)}{2\varepsilon} + \int_{|x|\geq\varepsilon}\frac{\varphi(x)}{x}\,dx\right] \\
&= \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}\frac{\varphi(x)}{x}\,dx \\
&= pv\int\frac{\varphi(x)}{x}\,dx
\end{aligned}
$$

since φ is differentiable at $x = 0$. Hence, we say that

$$(\log|x|)' = pv\,1/x.$$

is the distributional derivative of $\log|x|$, and is not a function.

VII. Solving Differential Equations with Distributions

Consider a DE

$$Lu = f, \tag{2}$$

where L is some linear[4] differential operator of order m.

Definition 29. A *classical solution* to (2) is an m-times differentiable u defined on Ω which satisfies the equation in the sense of equality between functions. If $u \in C^m(\Omega)$ (so Lu is continuous) then u is a *strong solution* .

Definition 30. A *weak solution* to (2) is $u \in \mathcal{D}'(\Omega)$ which satisfies the equation in the sense of distributions:

$$\langle Lu, \varphi \rangle = \langle f, \varphi \rangle, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Note: every strong soln is a weak soln, but converse is false.

[4] Restr to linear is nec because we cannot define mult in $\mathcal{D}'$ as a natural extn of mult of functions.

Example. For $\Omega = \mathbb{R}$, consider

$$xu' = 0.$$

Strong soln: $u = c_1$.
Weak soln: consider $u = c_2 H$.

$$\begin{aligned} u' &= c_2\delta \\ \langle xu', \varphi\rangle &= c_2\langle x\delta, \varphi\rangle \\ &= c_2\langle \delta, x\varphi\rangle \\ &= 0 \qquad \forall \varphi \in \mathcal{D}. \end{aligned}$$

So $u = c_2 H$ is also a weak soln, and

$$u = c_1 + c_2 H.$$

is the (weak) general solution (but not a strong solution).

What else is odd about this example?

Example. Solve $u'' = \delta$ on $\mathbb{R}$.
We found the solution

$$x_+ = xH(x).$$

Any other solution will satisfy the homogeneous equation

$$D_x^2\left[u - xH(x)\right] = 0,$$

so will have the form

$$\zeta(x) = xH(x) + ax + b.$$

Boundary conditions will determine the constants, e.g.,

$$\zeta(0) = 0, \zeta(1) = 1 \quad \Longrightarrow \quad \zeta(x) = xH(x).$$

$\hookrightarrow$ We use ζ to denote the general solution of $u'' = \delta$.

Now use this to solve more general equations.

Example. For $\Omega = (0,1)$, solve

$$u'' = f. \tag{3}$$

Consider $f = 0$ outside $(0,1)$ so $f \in L^1(\mathbb{R})$. Then

$$\begin{aligned}(f * \zeta)'' &= f * \zeta'' \\ &= f * \delta \\ &= f.\end{aligned}$$

So one solution to (3) is

$$\begin{aligned}u(x) &= (f * \zeta)(x) \\ &= \int_0^1 (x-\xi)H(x-\xi)f(\xi)\,d\xi \\ &= \int_0^x (x-\xi)f(\xi)\,d\xi \qquad\qquad 0 \le x \le 1,\end{aligned}$$

and the general solution is thus

$$u(x) = \int_0^x (x-\xi)f(\xi)\,d\xi + ax + b.$$

If we have initial conditions

$$u'(0) = a, \quad u(0) = b,$$

this becomes

$$u(x) = \int_0^x (x-\xi) f(\xi)\, d\xi + u'(0)x + u(0).$$

Alternatively, boundary conditions at $x = 0$ and $x = 1$ can be expressed

$$\begin{aligned} u(0) &= b \\ u(1) &= \int_0^1 (1-\xi) f(\xi)\, d\xi + a + b. \end{aligned}$$

Definition 31. ζ is the *fundamental solution* of the operator D^2.
$E \in \mathcal{D}'$ is the fundamental solution of the differential operator

$$L = \sum_{|\alpha| \le m} c_\alpha(x) D^\alpha$$

iff $LE = \delta$. Reason:

$$\begin{aligned} L(f * E) &= f * LE \\ &= f * \delta \\ &= f \end{aligned}$$

shows $f * E$ is a solution to $Lu = f$.

Example. On $\mathbb{R}^2$, is $\log|x|$ a weak solution to

$$\Delta u = 0?$$

For $x \neq 0$,

$$\begin{aligned}\Delta \log|x| &= D_1\left(\frac{1}{|x|}D_1|x|\right) + D_2\left(\frac{1}{|x|}D_2|x|\right) \qquad (4)\\ &= D_1\left(\frac{x_1}{|x|^2}\right) + D_2\left(\frac{x_2}{|x|^2}\right)\\ &= 0,\end{aligned}$$

so one might think so ...

$\log|x| \in L^1_{loc}(\mathbb{R}^2_0)$, so compute the (dist) Laplacian $\Delta \log|x|$.

$$\begin{aligned}\langle \Delta \log|x|, \varphi\rangle &= \langle \log|x|, \Delta\varphi\rangle\\ &= \int_{\mathbb{R}^2} \log|x| \Delta\varphi(x)\, dx\\ &= \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon} \log|x|\Delta\varphi(x)\, dx\end{aligned}$$

Now choose Ω to contain $\operatorname{spt}\varphi$ and $\overline{B}(0,\varepsilon)$, for $\varepsilon > 0$. Use Green's formula[5] on the open set

$$\Omega_\varepsilon = \Omega\setminus\overline{B}(0,\varepsilon) = \{x \in \Omega : |x| > \varepsilon\}$$

[5] $\int_\Omega (u\Delta v - v\Delta u) = \int_{\partial\Omega}(uD_\nu v - vD_\nu u)$.

to get

$$\int_{\Omega_\varepsilon} \log|x|\Delta\varphi(x)\,dx = \int_{\Omega_\varepsilon} \varphi(x)\Delta\log|x|\,dx$$
$$+\int_{\partial\Omega_\varepsilon} \left[\log|x| D_\nu\varphi(x) - \varphi(x) D_\nu \log|x|\right]\,d\sigma$$

where D_ν is outward normal on $\partial\Omega_\varepsilon$.

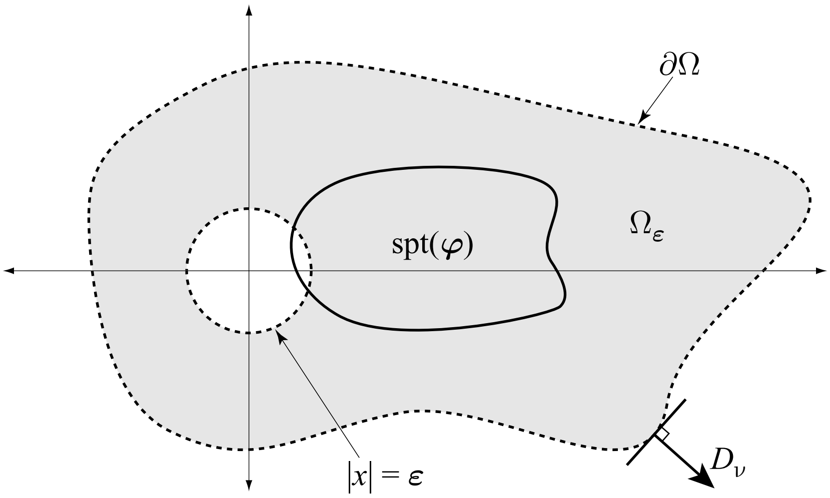

FIGURE 5. The domains Ω and Ω_ε.

But φ and $D_\nu\,\varphi$ vanish on (and outside) $\partial\Omega$, so

$$\int_{|x|\geq\varepsilon} \log|x|\Delta\varphi(x)\,dx = \int_{|x|\geq\varepsilon} \varphi(x)\Delta\log|x|\,dx$$
$$+\int_{|x|=\varepsilon} \left[\log|x| D_\nu\varphi(x) - \varphi(x) D_\nu \log|x|\right]\,d\sigma$$

By (4), the first integral on the right drops out. With $|x| = \left(x_1^2 + x_2^2\right)^{1/2} = r$, we have $D_\nu = -D_r$ on the circle $|x| = \varepsilon$; and so

$$\int_{|x|\geq\varepsilon} \log|x|\Delta\varphi(x)\,dx$$

$$= \int_{|x|=\varepsilon} \left[-\log\varepsilon D_r\varphi(x) + \tfrac{\varphi(x)}{\varepsilon}\right]\,d\sigma$$

Since $\varphi \in \mathcal{D}(\mathbb{R}^2)$, $|D_r\varphi| \leq M$ on $\mathbb{R}^2$. Thus

$$\left|\int_{|x|=\varepsilon} \log\varepsilon D_r\varphi(x)\,d\sigma\right| \leq |\log\varepsilon| \cdot M \cdot 2\pi\varepsilon$$

$$\xrightarrow{\varepsilon\to 0} \quad 0.$$

The other integral is

$$\tfrac{1}{\varepsilon}\int_{|x|=\varepsilon} \varphi(x)\,d\sigma$$

$$=\tfrac{1}{\varepsilon}\int_{|x|=\varepsilon} (\varphi(x) - \varphi(0))\,\,d\sigma + \tfrac{1}{\varepsilon}\int_{|x|=\varepsilon} \varphi(0)\,d\sigma$$

$$\xrightarrow{\varepsilon\to 0} \quad 0 + 2\pi\varphi(0),$$

since φ is continuous at 0. Conclusion:

$$\langle\Delta\log|x|, \varphi\rangle = 2\pi\varphi(0), \quad \forall\varphi \in \mathcal{D},$$

hence

$$\Delta\log|x| = 2\pi\delta.$$

Thus, $\frac{1}{2\pi}\log|x|$ is a fundamental solution of Δ in $\mathbb{R}^2$.

VIII. Things from last time

Convolution

$$(\varphi * f)(x) := \int \varphi(x-y) f(y)\, dy = \int \varphi(y) f(x-y)\, dy.$$

$$(T_1 {*} T_2)(\varphi) = \langle T_1 {*} T_2, \varphi \rangle := T_1(T_2(\varphi(x+y)), \quad \varphi \in \mathcal{D}(\mathbb{R}^n).$$

Key properties:

$$\delta * T = T * \delta = T.$$

$$T_1 * T_2 = T_2 * T_1.$$

$$T_1 * (T_2 * T_3) = (T_1 * T_2) * T_3.$$

$$D^\alpha (T_1 * T_2) = (D^\alpha T_1) * T_2 = T_1 * (D^\alpha T_2).$$

$T \in \mathcal{D}'(\mathbb{R}) \implies \exists S \in \mathcal{D}'$ s.t. $D_k S(\varphi) = T(\varphi), \forall \varphi$

$$pv \int f(x)\, dx = \lim_{\varepsilon \to 0} \int_{|x-x_0| \geq \varepsilon} f(x)\, dx.$$

Definition 32. $E \in \mathcal{D}'$ is the *fundamental solution* of the differential operator L iff $LE = \delta$. Reason:

$$L(f * E) = f * LE = f * \delta = f$$

shows $f * E$ is a solution to $Lu = f$.

$\frac{1}{2\pi} \log |x|$ is a fundamental solution of Δ in $\mathbb{R}^2$.

Example. On $\mathbb{R}^3$, is $|x|^{-1}$ a weak solution to $\Delta u = 0$?

$$\begin{aligned}
\Delta|x|^{-1} &= (D_1^2 + D_2^2 + D_3^2)(x_1^2 + x_2^2 + x_3^2)^{-1/2} \\
&= \sum_{j=1}^{3} D_j \frac{-x_j}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} \\
&= \sum_{j=1}^{3} \left(\frac{3x_j^2}{(x_1^2 + x_2^2 + x_3^2)^{5/2}} - \frac{1}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} \right) \qquad (5) \\
&= 3\sum_{j=1}^{3} \frac{x_j^2}{(x_1^2 + x_2^2 + x_3^2)^{5/2}} - \sum_{j=1}^{3} \frac{1}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} \\
&= 0, \text{ for } x \neq 0
\end{aligned}$$

so one might think so ...

Since $|x|^{-1} \in L^1_{loc}(\mathbb{R}^3)$,

$$\begin{aligned}
\left\langle \Delta|x|^{-1}, \varphi \right\rangle &= \left\langle |x|^{-1}, \Delta\varphi \right\rangle \\
&= \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} |x|^{-1} \Delta\varphi(x)\, dx \\
&= \lim_{\varepsilon \to 0} \left[\int_{|x| \geq \varepsilon} \Delta|x|^{-1} \varphi(x)\, dx \right. \\
&\qquad \left. + \int_{|x| = \varepsilon} \Big(|x|^{-1} D_\nu \varphi(x) - D_\nu |x|^{-1} \varphi(x) \Big)\, d\sigma \right]
\end{aligned}$$

by Green's formula. Again, the first integral vanishes by (5). With $D_\nu = -D_r$,

$$\int_{|x|\geq\varepsilon} |x|^{-1}\Delta\varphi(x)\,dx$$
$$= -\frac{1}{\varepsilon}\int_{|x|=\varepsilon} D_r\varphi(x)\,d\sigma - \frac{1}{\varepsilon^2}\int_{|x|=\varepsilon}\varphi(x)\,d\sigma.$$

Since $|D_r\varphi| \leq M$ on $\mathbb{R}^3$,

$$\left|\frac{1}{\varepsilon}\int_{|x|=\varepsilon} D_r\varphi(x)\,d\sigma\right| \leq \frac{M}{\varepsilon}\int_{|x|=\varepsilon} d\sigma = 4\pi\varepsilon M \xrightarrow{\varepsilon\to 0} 0.$$

Finally,

$$\frac{1}{\varepsilon^2}\int_{|x|=\varepsilon}\varphi(x)\,d\sigma$$
$$= \frac{1}{\varepsilon^2}\left(\int_{|x|=\varepsilon}\Big(\varphi(x)-\varphi(0)\Big)\,d\sigma + \frac{1}{\varepsilon^2}\int_{|x|=\varepsilon}\varphi(0)\,d\sigma\right)$$
$$= \frac{1}{\varepsilon^2}\int_{|x|=\varepsilon}\Big(\varphi(x)-\varphi(0)\Big)\,d\sigma + 4\pi\varphi(0)$$
$$\xrightarrow{\varepsilon\to 0} 0 + 4\pi\varphi(0)$$

Thus $\langle\Delta|x|^{-1},\varphi\rangle = -4\pi\varphi(0)$ shows $\Delta\frac{1}{|x|} = -4\pi\delta$, and hence $-\frac{1}{4\pi|x|}$ is a fundamental solution of Δ on $\mathbb{R}^3$.

By the preceding results, the Poisson equation

$$\Delta u = f$$

has a solution given by

$$u = f * \left(-\frac{1}{4\pi|x|}\right)$$

when the conv is well-defined. Recall, this is because for such a u,

$$\begin{aligned}\Delta u &= \Delta\left(f * \left(-\frac{1}{4\pi|x|}\right)\right)\\ &= f * \Delta\left(-\frac{1}{4\pi|x|}\right)\\ &= f * \delta\\ &= f\end{aligned}$$

The solution may be interpreted physically as the potential generated by f, e.g., gravitational potential due to a mass density distribution f.

When $f \in L^1_K$,

$$u(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{f(\xi)}{|x-\xi|}d\xi.$$

Example. Temperature distribution on a slender, infinite conducting bar is described by

$$\begin{cases} u_t = u_{xx} \\ u(x,0) = \varphi(x), \end{cases}$$

where $\varphi(x)$ is the initial heat distribution at $t = 0$.

To get the general solution $u = f * E$, we must find the fundamental solution E satisfying

$$(D_t - D_x^2)E(x,t) = 0 \tag{6}$$

on the upper half plane $\mathbb{R} \times \mathbb{R}_+$, and

$$E(x,0) = \delta_{(x,0)} \tag{7}$$

on the boundary $t = 0, x \in \mathbb{R}$.

Such an E is given by Fourier theory:

$$E(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}.$$

This satisfies (6) by direct computation.
In order to satisfy (7), it suffices to show $E(x,t)$ is a δ-sequence as $t \to 0^+$, but we did this in (1).

Example. The motion of an infinite vibrating string solves

$$D_t^2 u = D_x^2 u, \quad \text{where } x \in \mathbb{R}, t > 0.$$

Suppose the string is released with initial shape u_0 and initial velocity u_1.

Let

$$\begin{aligned} E_0 &= \tfrac{1}{2}\left[H(x+t) - H(x-t)\right] \\ E_1 &= D_t E_0 = \tfrac{1}{2}\left[\delta(x+t) + \delta(x-t)\right]. \end{aligned}$$

Then

$$\begin{aligned} (D_t^2 - D_x^2)E_0 &= 0 \\ (D_t^2 - D_x^2)E_1 &= 0 \end{aligned}$$

clearly hold in the upper half plane. When $t = 0$,

$$E_0 = 0, E_1 = \delta, D_t E_1 = 0.$$

Consequently,

$$u = u_0 * E_1 + u_1 * E_0$$

satisfies the initial conditions:

$$\begin{aligned} u(x,0) &= u_0 * \delta + u_1 * 0 = u_0 \\ u_t(x,0) &= u_0 * D_t E_1 + u_1 * D_t E_0 = u_0 * 0 + u_1 * \delta = u_1 \end{aligned}$$

When $u_0 \in C^2$ and $u_1 \in C^1 \cap L^1$,

$$u = \tfrac{1}{2}\left[u_0(x-t) + u_1(x+t)\right] + \tfrac{1}{2}\int u_1(\xi)\left[H(x+t-\xi) - H(x-t-\xi)\right] d\xi$$

$$= \tfrac{1}{2}\left[u_0(x-t) + u_1(x+t)\right] + \tfrac{1}{2}\int_{x-t}^{x+t} u_1(\xi)\, d\xi$$

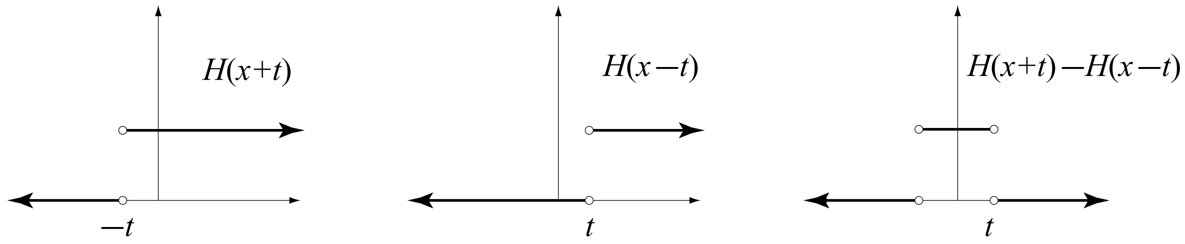

FIGURE 6. Construction of $H(x+t) - H(x-t)$, which is convolved against u_1.

The more general wave equation

$$D_t^2 u = c^2 D_x^2 u$$

is solved from this one via change of coordinates:

$$u = \tfrac{1}{2}\left[u_0(x-ct) + u_1(x+ct)\right] + \tfrac{1}{2c}\int_{x-ct}^{x+ct} u_1(\xi)\, d\xi.$$

If the string is released from rest ($u_1 = 0$), the solution is the average of two travelling waves $u_0(x-ct)$, $u_1(x-ct)$, both having the same shape u_0 but travelling in opposite directions with velocity $\pm c$.

IX. The Descent Method

The "Descent Method" was so coined in [La-vF] and is reminiscent of the method used by Schwartz to establish the convergence of the Fourier series associated to a periodic distribution.

General idea of the Descent Method:

(1) Begin with a formula you would like to manipulate, but cannot due to some issue of convergence.
(2) Prove that the pointwise manipulations hold under some sufficiently restrictive conditions.
(3) Integrate multiple times (say q times), until these conditions are met, so everything is sufficiently smooth and converges nicely.
(4) Perform the desired manipulations.
(5) Differentiate distributionally (q times) until you obtain the formula you need.

Resulting identity will hold distributionally, but may not make sense pointwise.

We will be using periodic distributions:
$\mathcal{D}'(\mathbb{R}^n)_{per} := \{T \in \mathcal{D}'(\mathbb{R}^n) \,\vdots\, \tau_\kappa T = T, \kappa \in \mathbb{Z}^n\} = \mathcal{D}'(\mathbb{T}^n)$, where $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Properties true for $\mathbb{R}^n$ will remain true on $\mathbb{T}^n$ when of a local nature; convolution product has all the familiar properties.

Theorem 33. If $T \in \mathcal{D}'$ has compact support, then there is a continuous function f and a multi-index $\alpha \in \mathbb{N}^n$ such that

$$\langle T, \varphi \rangle = \langle D^\alpha f, \varphi \rangle, \quad \forall \varphi \in \mathcal{D}.$$

Theorem 34. (Dirichlet) If a periodic function f has a point x_0 for which both

$$f(x_0-) = \lim_{x \to x_0^-} f(x) \qquad \text{and} \qquad f(x_0+) = \lim_{x \to x_0^+} f(x)$$

exist and are finite, then its Fourier series converges at x_0 to the value

$$\tfrac{1}{2}[f(x_0-) + f(x_0+)].$$

In particular, if f is continuous on $[a, b]$, then the Fourier series converges pointwise to the value of the function on $[a, b]$.

Theorem 35. The Fourier series associated to a periodic distribution converges if and only if the Fourier coefficients are of slow growth[6]

Combining these results, we know that we will be able to integrate any periodic distribution until it becomes a continuous fn, whence we can integrate it until it is C^m. This is the key that allows the descent method.

Example. Suppose you have the Fourier series of two periodic distributions:

$$\sum_{\alpha\in\mathbb{Z}} P_\alpha, \qquad \sum_{\beta\in\mathbb{Z}} Q_\beta.$$

What is the product

$$\left(\sum_{\alpha\in\mathbb{Z}} P_\alpha\right)\left(\sum_{\beta\in\mathbb{Z}} Q_\beta\right)?$$

The product will contain a coefficients $R_{\alpha,\beta}$ for each point $(\alpha,\beta)\in\mathbb{Z}^2$.

We'd like to say

$$\left(\sum_{\alpha\in\mathbb{Z}} P_\alpha\right)\left(\sum_{\beta\in\mathbb{Z}} Q_\beta\right) = \sum_{N\in\mathbb{N}}\left(\sum_{|\alpha|+|\beta|=N} R_{\alpha,\beta}\right).$$

6 "Slow growth" means that they do not grow faster than polynomially, i.e., $D^\alpha\varphi(x)| \leq C_\alpha(1+|x|)^{N(\alpha)}, \forall\alpha$.

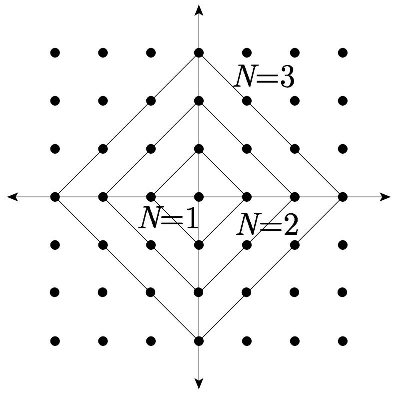

FIGURE 7. Writing the Cauchy product of doubly infinite series.

If the series do not converge absolutely, this rearrangement is not justified: need the Descent Method!

(1) Start with

$$\left(\sum_{\alpha\in\mathbb{Z}} P_\alpha\right)\left(\sum_{\beta\in\mathbb{Z}} Q_\beta\right) = \sum_{(\alpha,\beta)\in\mathbb{Z}^2} R_{\alpha,\beta}.$$

(2) We know that such rearrangements are valid when the series is absolutely convergent.
(3) Integrate the series term-by-term, q times. For large enough q, the series will converge absolutely, and even normally.
(4) Rearrange the terms of the integrated series so that they are indexed/ordered in the concentric form mentioned above.
(5) Differentiate the reordered series, term-by-term, q times.

We have just shown that as distributions,

$$\left(\sum_{\alpha\in\mathbb{Z}} P_\alpha\right)\left(\sum_{\beta\in\mathbb{Z}} Q_\beta\right) = \sum_{N\in\mathbb{N}}\left(\sum_{|\alpha|+|\beta|=N} R_{\alpha,\beta}\right),$$

even though the sum on the right may not converge pointwise.

Example. Suppose for $x \in [0, y]$ you have an expression of the form

$$\sum_{m=0}^{\infty}\left(a_m \sum_{n\in\mathbb{N}}(-1)^{f_1(n,m)} b_{n,m} x^n - c_m \sum_{n\in\mathbb{N}}(-1)^{f_2(n,m)} d_{n,m} x^n\right)$$

and you would like to factor out the powers of x.

Pointwise, such a manipulation would have no justification unless we had some strong convergence conditions, positivity of the terms, etc. However, we interpret the series as a distribution and apply the descent method. Then we get the series

$$\sum_{n\in\mathbb{N}}\left[\sum_{m=0}^{\infty}\left(a_m(-1)^{f_1(n,m)} b_{n,m} - c_m(-1)^{f_2(n,m)} d_{n,m}\right)\right] x^n$$

which is equal, as a distribution, to the original expression, and shows the coefficients of the x^n much more clearly.

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