FRACTALS, DIMENSION, AND NONSMOOTH ANALYSIS

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1. Fractional Dimension

"Fractal" = fractional dimension.

Intuition suggests dimension is an integer, e.g.,

- A line is 1-dimensional,
- a plane (or square) is 2-dimensional,
- a solid cube or ball is 3-dimensional, etc.

What does it mean to say something is 1.5-dimensional?

The example of a space-filling curve (e.g., BM, Peano): a curve so jaggedy that it "shades in" area.

What is meant by dimension? Has something to do with:
how measurements of a set change when that set is scaled.

"Scale" means to dilate by some factor, i.e., if we scale a subset $A \subseteq \mathbb{R}^n$ by 3, then any two points will now be three times further apart than they used to be:

$$3A = \{3x : x \in A\}$$

- Scale a line segment L by 2 and its measure doubles:
$$m(2L) = m(L) \times 2^1.$$
- Scale a square S by 2 and its measure increases by a factor of four:
$$m(2S) = m(S) \times 2^2.$$

- Scale a cube Q by 2 and its measure increases by a factor of eight:
$$m(2Q) = m(Q) \times 2^3.$$

So dimension corresponds to some exponent. If $A \subseteq \mathbb{R}^n$ had dimension 1.5, then its measure would increase by a factor of $2^{1.5}$ when scaled by 2.

How is this useful?

2. Motivation

Most objects in nature are irregular: the way trees branch, the way rivers fork, the way mud cracks, the structure of the venous system in the human body, the kidneys, the lungs, the brain.

Any organ in the human body which performs exchange functions tries to maximize surface area so that it can conduct more biological processes (improve metabolic efficiency) in a confined region,
hence highly perforated or branched structure,
hence fractal.

Example: lungs. Say you have two horses A and B, and A is a little larger. In fact, $A = 1.2B$.
Suppose you have measured the metabolic rate of the lungs of B to be 10 liters of air per minute.
The metabolic rate of A's lungs will be
$$10 \times 1.2^d,$$
where d is the effective dimension of the lungs of a horse.

Studying fractals allows for nonsmooth analysis.
Instead of approximating by something smoother, approximate by something rougher.

3. Discovery

I first became interested in the subject when I read James Gleick's "Chaos". Then, Dr. Lapidus suggested "Chaos and Fractals" by Peitgens, Jurgen, Saupe.

4. Getting more technical: measure theory

Earlier, I mentioned the "measure" of a line segment and the "measure" of a square.
Measure of a line segment: length.
Measure of a square: area.
Measure of a 1.5-dimensional set: ?

Note: there is not much useful information in the sentence "The area of this line segment is 0." or the sentence "The length of this square is infinite."

When you measure a set, you want to use the appropriate "measure" for the job.
So what is a measure?

In Analysis, you learn measure/Lebesgue theory.
How does one measure the following set:

$$\left(\frac{1}{2}, 1\right) \cup \left(\frac{1}{4}, \frac{1}{3}\right) \cup \ldots \left(\frac{1}{2n}, \frac{1}{2n-1}\right) \ldots ?$$

Length of $A = (a, b)$ is given by $\ell(A) = b - a$, and this extends to finite unions.
Lebesgue came up with a generalization that extends to infinite unions. It's called "Lebesgue measure", but it just means length like you're used to thinking about it.

For $A \subseteq \mathbb{R}$,

$$m(A) = \inf \left\{ \sum_{n=0}^{\infty} \ell(A_n) \, : \, A \subseteq \bigcup_{n=0} A_n \right\}.$$

The inf here is taken over all coverings of A by intervals $\{A_n\}$.

Actually, this is 1-dimensional Lebesgue measure. It can be extended to higher dimensions by covering the set with n-balls. For $A \subseteq \mathbb{R}^n$,

$$m(A) = \inf \left\{ \sum_{n=0}^{\infty} \mathrm{vol}_n(A_n) \, : \, A \subseteq \bigcup_{n=0} A_n \right\}.$$

$\mathrm{vol}_n(A_n)$ is n-dimensional volume of the ball A_n.

Lebesgue measure allows for a more robust theory of integration than Riemann's, but there are some things it can't really deal with. For example, the Koch curve.

What is the measure of this curve?
It has infinite length/Lebesgue measure, but
it has area/2-dim Lebesgue measure 0.

The Koch snowflake is a finite region with an infinitely long boundary!
The infinite length is because the curve is nowhere differentiable, i.e., every point on the curve is a corner/cusp.

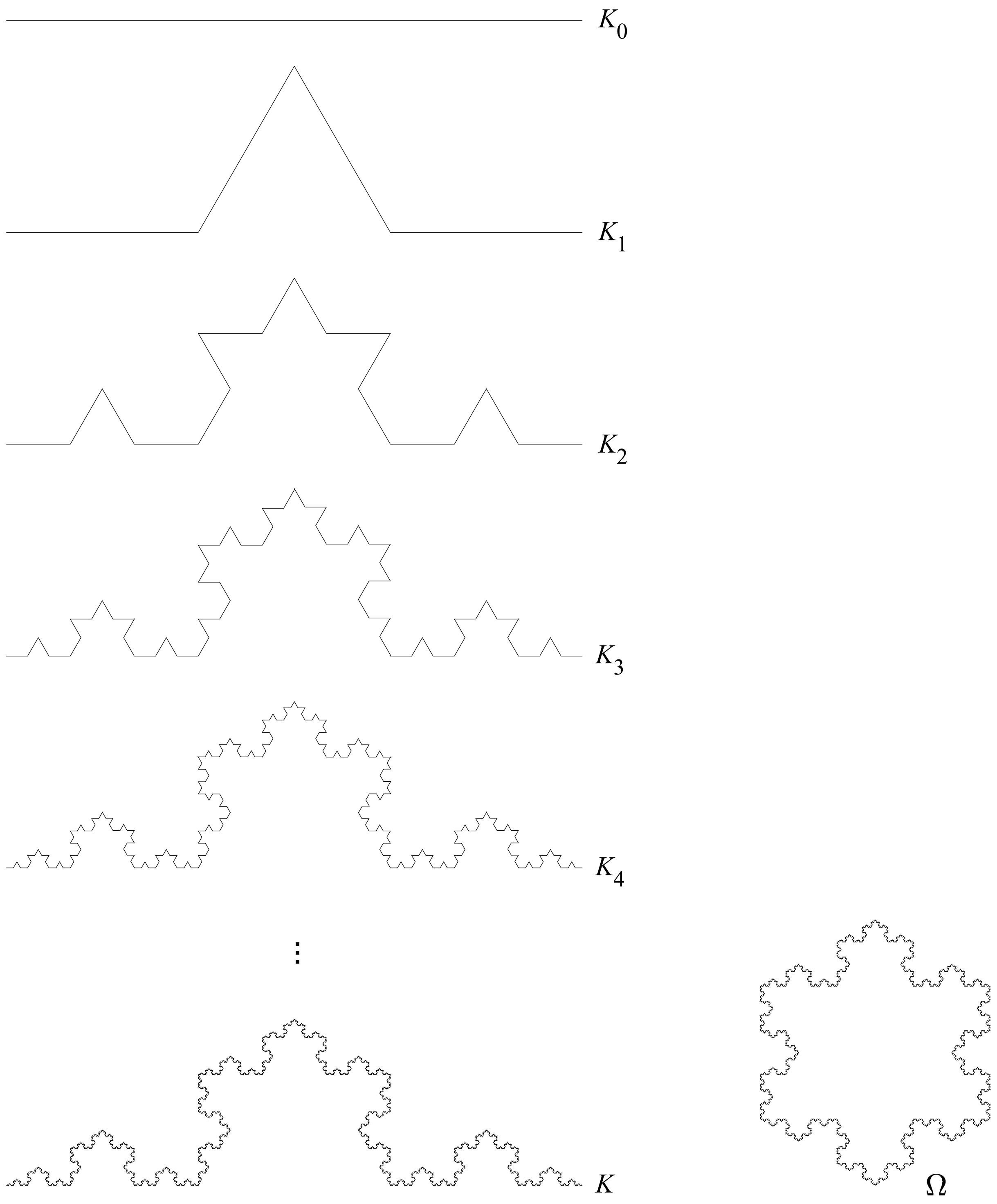


FIGURE 1. The Koch curve K (left) and the Koch snowflake Ω (right).

Hausdorff figured out a way to make sense of this: he realized that you can't meaningfully take the s-dimensional measure of an object that isn't of dimension s. You have to measure with respect to the appropriate dimension.

The s-dimensional Hausdorff measure is defined for any $s \in \mathbb{R}$. Suppose we want to measure a set $F \subseteq \mathbb{R}^n$. For $\delta > 0$,

$$H^s_\delta(F) = \inf \left\{ \sum_{n=0}^{\infty} |A_n|^s : A \subseteq \bigcup_{n=0} A_n, |A_n| < \delta \right\}$$

Three differences:

(1) the exponent s,
(2) the limit δ on the size of the covering sets,
(3) instead of $m(B)$, use the diameter

$$|B| = \sup\{|x - y| : x, y \in B\}.$$

Then

$$H^s(F) = \lim_{\delta \to 0} H^s_\delta(F).$$

Can be shown: for any F, there is a unique s at which $H^s(F)$ jumps from ∞ to 0. This s is the Hausdorff dimension of F. It is not hard to construct sets of any dimension $s \in \mathbb{R}$.

Example: Koch curve has Hausdorff dimension $D = \frac{\log 4}{\log 3}$.

Hausdorff measure turns out to be Lebesgue measure when s is an integer.

Hausdorff dimension is the most widely used notion of fractional dimension, but there are several others; there is no universally agreed upon "fractal dimension". I use Minkowski dimension most of the time because it has a couple of properties more suited to what I'm studying.

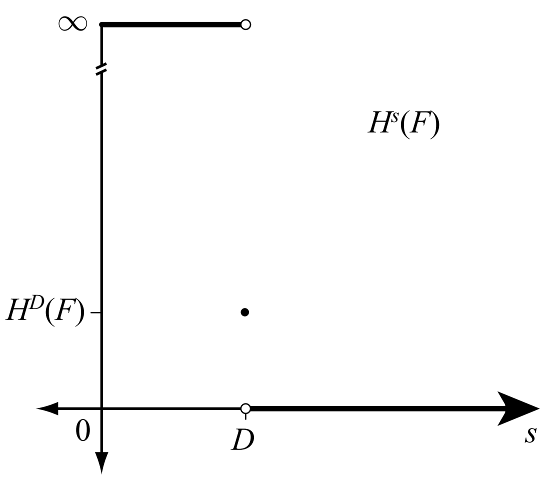


FIGURE 2. A graph of Hausdorff dimension $H^s(F)$ as a function of s. D is the Hausdorff dimension of the set F.

A *fractal string* is any bounded open subset of $\mathbb{R}$

$$\mathcal{L} := \{l_j\}_{j=1}^{\infty}, \quad \text{with } \sum_{j=1}^{\infty} l_j < \infty.$$

$$l_1 \geq l_2 \geq l_3 \geq \ldots,$$

or distinctly (with multiplicity):

$$l_1 > l_2 > l_3 > \ldots.$$

Idea/origin: comes from studying fractal subsets

$$\partial \mathcal{L} \subseteq \mathbb{R}.$$

FIGURE 3. The Cantor Set

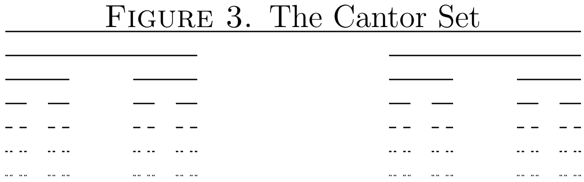
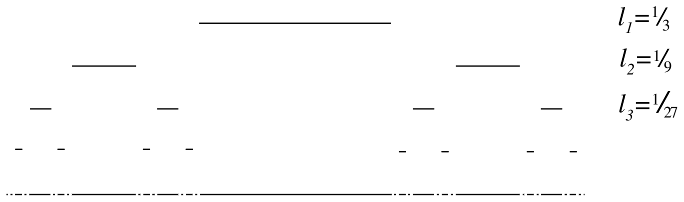


FIGURE 4. The Cantor String



The Cantor String example has lengths

$$\left\{3^{-(n+1)}\right\}$$

with multiplicities

$$w_{3^{-(n+1)}} = 2^n.$$

$$\mathcal{CS} = \left\{\frac{1}{3}, \frac{1}{9}, \frac{1}{9}, \frac{1}{27}, \frac{1}{27}, \frac{1}{27}, \frac{1}{27}, \ldots\right\}$$

The *geometric zeta function* of a string

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} l_j^s = \sum_{l} w_l l^s$$

encodes all this information.

Example:

$$\zeta_{\mathcal{CS}}(s) = \sum_{n=0}^{\infty} 2^n 3^{-(n+1)s} = \frac{3^{-s}}{1 - 2 \cdot 3^{-s}}.$$

Three key things about $\zeta_{\mathcal{L}}$:

(1) Relates to the dimension of $\partial\mathcal{L}$.

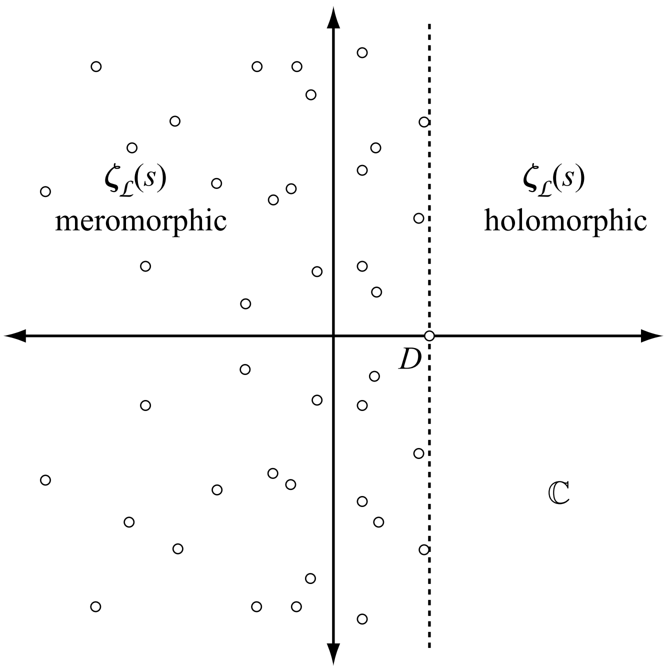


FIGURE 5. $D = \dim \partial\mathcal{L}$ is the abscissa of convergence of $\zeta_{\mathcal{L}}$.

$$D_{\partial\mathcal{L}} = \inf\{\sigma \geq 0 : \zeta_{\mathcal{L}}(\sigma) < \infty\}$$

Generalize and define the *complex dimensions*:

$$\mathcal{D} = \{\omega \in \mathbb{C} : \zeta_{\mathcal{L}} \text{ has a pole at } \omega\}$$

If the poles are periodic, then the underlying fractal is not measurable.

(2) Connects spectral and geometric properties.

Spectral properties: studying the spectrum, i.e, the distribution of the eigenvalues, of the Laplacian.

Consider the string to be vibrating, fixed at the endpoints. What are its fundamental modes?
A *frequency* of $\mathcal{L}$ is

$$f = \sqrt{\lambda}/\pi = \frac{k}{l_j}.$$

The *spectral zeta function* of $\mathcal{L}$ is

$$\zeta_\nu(s) = \sum_{j,k=1}^{\infty} (k \cdot l_j^{-1})^{-s} = \sum_f w_f f^{-s}$$

Also,

$$\zeta_\nu(s) = \zeta_{\mathcal{L}}(s)\zeta(s),$$

where ζ is the Riemann zeta function

$$\zeta(s) = \sum_{n=0}^{\infty} n^{-s}.$$

(3) Gives an explicit formula for $V(\varepsilon)$.

$$\begin{aligned} V(\varepsilon) &= \mathrm{vol}_n\{x \in \Omega : d(x, \partial\Omega) < \varepsilon\} \\ &= \sum_{\omega \in \mathcal{D}_{\mathcal{L}}} \mathrm{res}\left(\zeta_{\mathcal{L}}; \omega\right) \left(\frac{2^{1-\omega}}{\omega(1-\omega)}\right) \varepsilon^{1-\omega} + \mathcal{R}(\varepsilon). \end{aligned}$$

My goal: higher-dimensional analogues of these results.

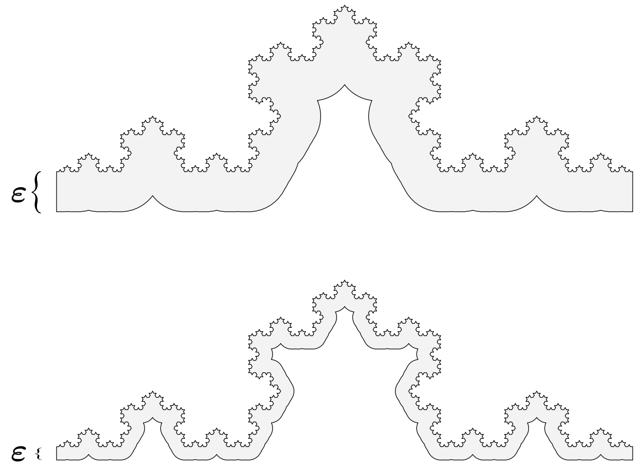


FIGURE 6. The ε-neighborhood of the Koch curve, for two different values of ε.

5. How this relates to different fields of mathematics

- Analysis: obviously, a bunch of measure theory is involved; lots of work with infinite series, distributions & integral transforms, geometric arguments.

- Complex analysis: much depends on the study of zeta functions, their poles and zeros. The explicit formulae are derived via a combination of complex contour integration and distributional arguments.

- Number theory: relations between the lengths of the strings are critical. Whether or not the l_j are rationally independent is critical.

- Analytical number theory: clearly, the study of zeta functions is crucial. Diophantine approximation and the search for solutions to Moran-type equations

$$r_1^s + \cdots + r_k^s = 1.$$

The context gives a geometric interpretation of the Riemann hypothesis.

- Dynamical systems: all of this has an interpretation in terms of dynamical systems (iteration of one or more functions) giving new proofs of prime and periodic orbit theorems, etc.

- Mathematical physics: spectral asymptotics are important when considering waves or diffusions on the fractal set.

  Fractals tend to have interesting lacunarity, connectivity, branching properties. Hence waves and diffusions can behave peculiarly.

- Quasicrystals, Penrose tilings: the set of complex dimensions shares features with these.

- Cohomology: a very deep analogy connecting self-similar geometries with finite (arithmetic or algebraic) geometries has been speculated.

  Relates varieties over finite fields, etale cohomology, to lattice-type strings. Possibly giving information about the structure of the space of zeta functions.

- Quantum mechanics and noncommutative geometry: building a ground for the development of noncommutative fractal geometry, connected to the role of noncommutative geometry in string theory.