Curvature and Convexity I

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1. Introduction

For smooth surfaces (at least C^2, anyway), understood by generalizing

$$\tfrac{d^2}{dt^2} = \left[\tfrac{d}{dt}\right]'$$

to higher dimensions. Three perspectives on curvature:

1. How does a curve/surface "bend away" from a tangent line/plane?
2. How does the measure of a curve/surface change when it is distorted in a normal direction?
3. Consider a small disk around a point, and its image under normal distortion by some distance ε. How does the volume of the region between these two surfaces change, as a function of ε?

Today: look at (1) & (2); no convexity 'til next week!
This part follows [Morg] closely; some sections are direct quotes.

2. Curvature

2.1. **1 dimension.**

Let $\boldsymbol{x} : \mathbb{R} \to \mathbb{R}^2$ be a smooth curve with velocity $\boldsymbol{v} = \dot{\boldsymbol{x}}$.
The curvature of $\boldsymbol{x}(t)$ is the change in the unit tangent vector $T = \frac{\boldsymbol{v}}{|\boldsymbol{v}|}$.

The curvature vector $\boldsymbol{\kappa}$ points in the direction in which a unit tangent T is turning.

$$\boldsymbol{\kappa} = \frac{dT}{ds} = \frac{dT/dt}{ds/dt} = \frac{1}{|\boldsymbol{v}|}\dot{T}.$$

The scalar curvature is the rate of turning

$$\kappa = |\boldsymbol{\kappa}| = |d\boldsymbol{n}/ds|.$$

When parametrized by arc length, curvature is

$$\boldsymbol{\kappa} = \frac{d^2\boldsymbol{x}}{ds^2}.$$

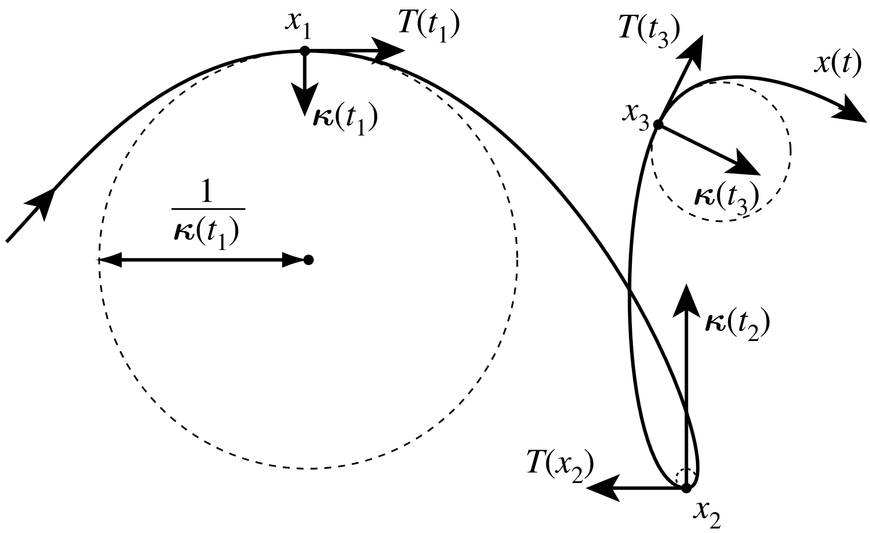

FIGURE 1. Tangents and curvatures. The *radius of curvature* at t_1 is $1/\boldsymbol{\kappa}(t_1)$.

If the curve is the graph of a function $f : \mathbb{R} \to \mathbb{R}^{n-1}$ tangent to the x-axis at the origin 0, then

$$\boldsymbol{\kappa}(0) = f''(0) \in \mathbb{R}^{n-1}.$$

Without tangency hypothesis, scalar curvature is

$$\kappa = \frac{|f''|\sqrt{1 + |f'|^2 \sin\theta}}{(1 + |f'|^2)^{3/2}},$$

where θ is the angle between f' and f''. In $\mathbb{R}^2$, $f : \mathbb{R} \to \mathbb{R}$ and $\theta = 0$, so

$$\kappa = \frac{|f''|}{(1 + |f'|^2)^{3/2}}.$$

"Curvature tells how the length of a curve changes as the curve is deformed. If an infinitesimal piece of a planar curve ds is pushed a distance du in the direction of κ, the length changes by a factor of $1-\kappa du$. Indeed, the original arc lies to second order on a circle of radius $1/\kappa$, and the new one on a circle of radius $1/\kappa - du = (1/\kappa)(1-\kappa du)$." [Morg]

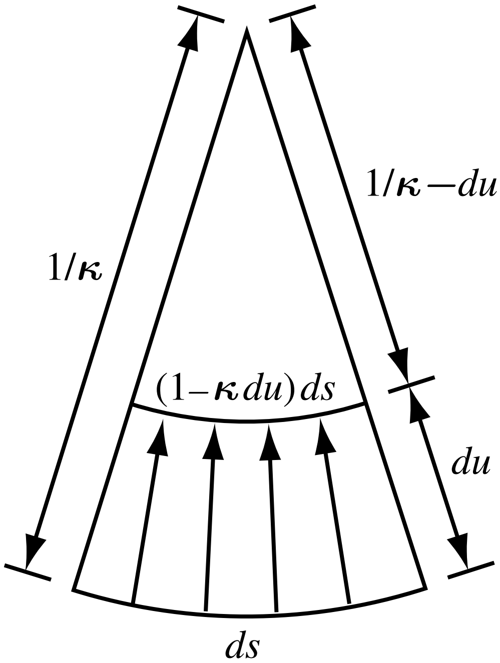

FIGURE 2. Change in an infinitesimal curve.

In Fig. 2, κ is pointing upward (opposite to $1/\kappa$).

2.2. **2 dimensions.**
A surface can curve different amounts in different directions, possibly even with different signs, e.g., a saddle. The *principal curvatures* are

$$\kappa_1 = \text{most upward}, \qquad \kappa_2 = \text{most downward},$$

Note: κ_1, κ_2 always occur orthogonally.

$$\begin{aligned} \text{mean curvature:} \quad & H = \kappa_1 + \kappa_2 \\ \text{Gauss curvature:} \quad & G = \kappa_1 \kappa_2 \end{aligned}$$

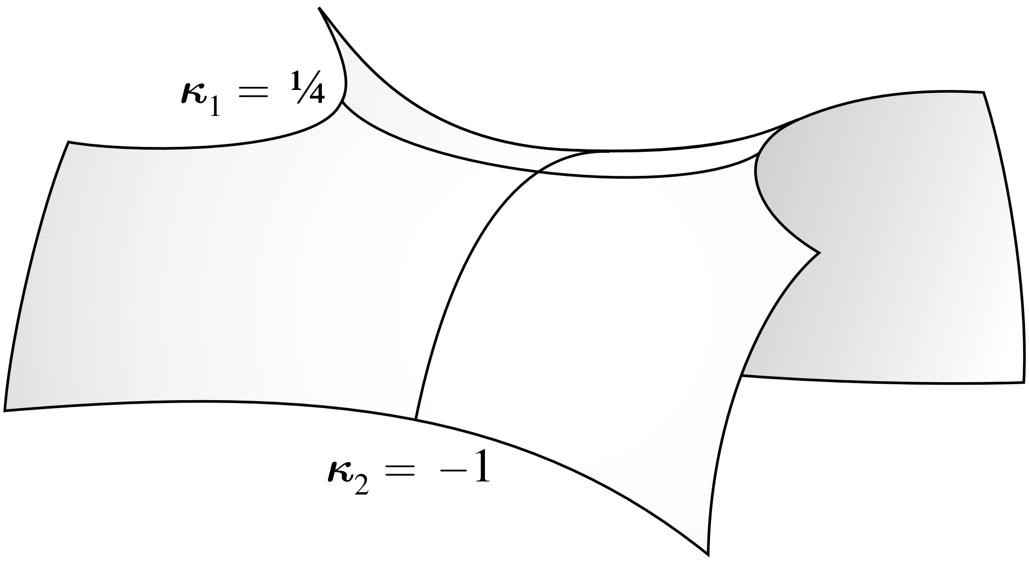

FIGURE 3. At the origin, this saddle has principal curvatures $\kappa_1 = \frac{1}{4}, \kappa = -1$, mean curvature $H = -\frac{3}{4}$, and Gauss curvature $G = -\frac{1}{4}$.

Note: Gauss curvature is negative iff κ_1, κ_2 have different signs.

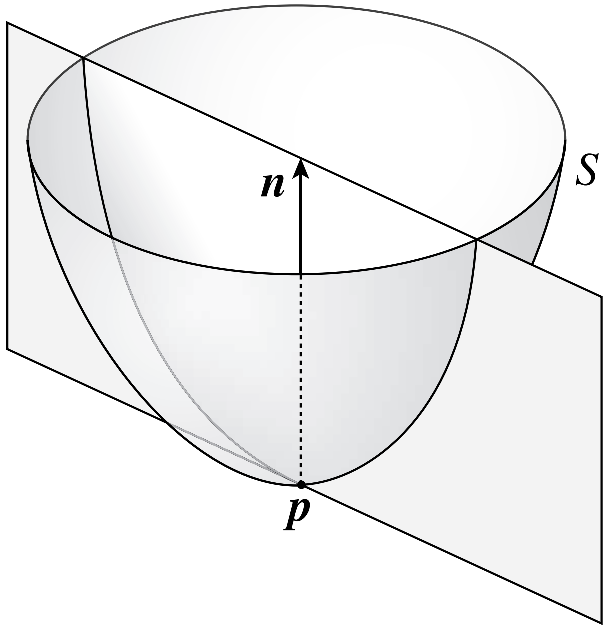

FIGURE 4. The curvature of a surface S at a point $\boldsymbol{p}$ is measured by the curvature of its slices by planes.

The curvature at $(\boldsymbol{p}, \boldsymbol{v})$ is given by the *second fundamental form*

$$\mathrm{II}(\boldsymbol{p}, \boldsymbol{v}) = (D^2 f)_{\boldsymbol{p}}(\boldsymbol{v}, \boldsymbol{v}) := \boldsymbol{v}^\top \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(\boldsymbol{p}) & \frac{\partial^2 f}{\partial x \partial y}(\boldsymbol{p}) \\ \frac{\partial^2 f}{\partial x \partial y}(\boldsymbol{p}) & \frac{\partial^2 f}{\partial y^2}(\boldsymbol{p}) \end{bmatrix} \boldsymbol{v}.$$

So if $\boldsymbol{v} = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$, for example, then $\kappa = \frac{\partial^2 f}{\partial x^2}$.

II is symmetric. When diagonalized (e.g., by choosing good coordinates),

$$\mathrm{II} = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix},$$

so that mean curvature is $\operatorname{trace} \mathrm{II}$.

Theorem (Euler)**.** Curvature in direction $\boldsymbol{v} = (\cos\theta, \sin\theta)$ is

$$\kappa = \mathrm{II}(\boldsymbol{p}, \boldsymbol{v}) = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta,$$

a weighted average of the principal curvatures.

Analogy:

(a) Rate of change of a function is directional derivative and depends on direction.
(b) Rate of change of area of a surface is *first variation* and depends on a vector field $\boldsymbol{V}$.

Theorem. The first variation S with respect to a compactly supported vector field $\boldsymbol{V}$ on S is

$$\delta^1(S) = \tfrac{d}{dt}\,\text{area}\ (S + t\boldsymbol{V})\big|_{t=0} = -\int_S \boldsymbol{V} \cdot H\boldsymbol{n}.$$

Sketch of proof. Since formula is linear in $\boldsymbol{V}$, consider normal and tangential variations separately. For tangential variations, the surface slides along itself, and $\delta^1(S) = 0$. Let $V\boldsymbol{n}$ be a small normal variation. Infinitesimally, to first order we have

$$\begin{aligned}(1 - V\kappa_1)dx \cdot (1 - V\kappa_2)dy &\approx (1 - VH)dx\,dy \\ &= (1 - \boldsymbol{V} \cdot H\boldsymbol{n})dx\,dy.\end{aligned}$$ □

Consequence of Theorem: an area-minimizing surface must have vanishing mean curvature.

Definition. S is a *minimal surface* iff

$$\partial S = \partial T \quad \Longrightarrow \quad area(S) \leq area(T).$$

2.3. **Higher dimensions.**

No more boldfaced vector notation (too many vectors!)

2.3.1. *2-dimensional surfaces in* $\mathbb{R}^n$.
The tangent plane T_pS to S at p is the x_1x_2-plane and the orthogonal complement $T_pS^\perp$ is the $x_3 \ldots x_n$-plane, and S is locally the graph of a function

$$f : T_pS \to T_pS^\perp.$$

The *second fundamental tensor*

$$\mathrm{II} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix},$$

where now the entries are vectors $a_{ij} \in T_pS^\perp$.

The *mean curvature vector* is

$$H = \operatorname{trace} \mathrm{II} = a_{11} + a_{22} \in T_pS^\perp.$$

The *Gauss curvature* is the scalar

$$G = \det \mathrm{II} = a_{11} \cdot a_{22} - a_{12} \cdot a_{12}.$$

And again, it is a theorem that

$$\delta^1(S) = -\int_S \boldsymbol{V} \cdot H.$$

Note: for $\mathbb{R}^3$, second fundamental tensor is second fundamental form times $\boldsymbol{n}$.

2.3.2. *m-dimensional surfaces in $\mathbb{R}^n$.*
S is still locally the graph of a function

$$f : T_pS \to T_pS^\perp,$$

but now T_pS is the $x_1 \dots x_m$-plane.

The *second fundamental tensor* is a symmetric $m \times m$ matrix

$$\mathrm{II} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_m} & \cdots & \frac{\partial^2 f}{\partial x_m^2} \end{bmatrix},$$

with entries in $T_pS^\perp$.

For hypersurfaces ($n = m + 1$), second fundamental tensor is second fundamental form (a scalar matrix) times $\boldsymbol{n}$ and for some coordinates

$$\mathrm{II} = \begin{bmatrix} \kappa_1 & & \\ & \ddots & \\ & & \kappa_m \end{bmatrix}.$$

Then

$$H = \operatorname{trace} \mathrm{II} = \kappa_1 + \cdots + \kappa_m.$$

And again, it follows that

$$\delta^1(S) = -\int_S \boldsymbol{V} \cdot H$$

by considering the infinitesimal displacement

$$(1 - |\boldsymbol{V}|\kappa_1)\, dx_1 \dots (1 - |\boldsymbol{V}|\kappa_m)\, dx_m \approx (1 - \boldsymbol{V} \cdot H)\, dx_1 \dots dx_m$$

2.4. **Conclusions.** Consider the case of an immersion

$$f : M^n \to N^{n+1}.$$

- Then $f(M) \subseteq N$ is a hypersurface.
- II is a real symmetric scalar matrix (times $\boldsymbol{n}$)
- II has an orthonormal basis of eigenvectors with real eigenvalues

$$\kappa_1, \dots, \kappa_m.$$

- The symmetric functions of $\kappa_1, \dots, \kappa_m$ are the invariants of the immersion f.

In other words,

- κ_i gives 1-dimensional curvature information
- $\kappa_1 + \cdots + \kappa_m$ gives 2-dimensional curvature information
- $\sum_{m_j < m_k} \kappa_{m_j} \kappa_{m_k}$ gives 3-dimensional info
- $G = \kappa_1 \dots \kappa_m$ gives n-dimensional info.

Note: Gauss' Theorema Egregium shows that G is intrinsic.

Moral:
Curvatures are the coefficients in some polynomial that expresses change in volume under small deviations.
Let $\varepsilon \boldsymbol{n}$ be a normal variation of small magnitude ε. Then

$$(1 - \varepsilon\kappa_1)dx \cdot (1 - \varepsilon\kappa_2)dy = 1 - (\kappa_1 + \kappa_2)\varepsilon + \kappa_1\kappa_2\varepsilon^2 \, dx \, dy$$

2.5. **Application to tubes.**

From [Gray].
Suppose $\gamma : [a, b] \to \mathbb{R}^2$ is smooth plane curve.
Define $J(x, y) := (-y, x)$ (in $\mathbb{C}$, this is $z \mapsto iz$).
Then the curvature is the function $\kappa(u)$ such that

$$\gamma''(u) = \kappa(u) J\gamma'(u).$$

Put $T := \gamma'$ and $N := J\gamma'$, so that

$$\begin{aligned} T' &= \kappa N \\ N' &= -\kappa T. \end{aligned}$$

Now for

$$X_t(u) = \gamma(u) + tN(u),$$

the curve $t \mapsto X_t(u)$ traces out a parallel curve to γ at distance $|t|$, for small t.

$$X_t'(u) = T(u) + tN'(u) = (1 - \kappa(u)t)T(u)$$

$$Length(X_t) = \int_a^b \|X_t'(u)\| \, du = \int_a^b (1 - \kappa(u)t) \, du$$

Then

$$\begin{aligned} V_\gamma(\varepsilon) &= \int_{-\varepsilon}^{\varepsilon} Length(X_t) \, dt \\ &= \int_a^b \int_{-\varepsilon}^{\varepsilon} (1 - \kappa(u)t) \, dt \, du \\ &= \int_a^b 2\varepsilon du \\ &= 2\varepsilon \, Length(\gamma) \end{aligned}$$

This is valid for small ε (no self-intersections).

Now suppose $\gamma : [a,b] \to \mathbb{R}^3$ is smooth curve in $\mathbb{R}^3$, and

$$\begin{aligned} T' &= \kappa N, \\ N' &= -\kappa T + \tau B, \\ B' &= -\tau N, \end{aligned}$$

where τ is the torsion of γ. The tube at distance t is parametrized by

$$X^t(u,v) = \gamma(u) + t\cos v N(u) + t \sin v B(u)$$

with partial derivatives

$$\begin{aligned} X^t_u(u,v) &= (1 - t\kappa(u)\cos v)T(u) - t\tau(u)\sin v N(u) \\ &\qquad\qquad\qquad + t\tau(u)\cos v B(u), \\ X^t_v(u,v) &= -t\sin v N(u) + t\cos v B(u), \end{aligned}$$

Then for small $t \geq 0$,

$$\begin{aligned} X^t_u(u,v) \times X^t_v(u,v) &= -t\sin v(1 - t\kappa(u)\cos(v))B(u) \\ &\qquad\qquad - t\cos v(1 - t\kappa(u)\cos v)N(u), \\ \|X^t_u(u,v) \times X^t_v(u,v)\| &= t(1 - \kappa(u) t\cos v). \end{aligned}$$

$$\int_0^{2\pi} \|X^t_u(u,v) \times X^t_v(u,v)\| = 2\pi t.$$

$$\mathrm{vol}_2(P_t) = \int_a^b \int_0^{2\pi} \|X^t_u \times X^t_v\| \, dv \, du = \int_a^b 2\pi t \, du = 2\pi t Length(\gamma)$$

So for small ε,

$$V_\gamma(\varepsilon) = \int_0^\varepsilon \mathrm{vol}_2(P_t) \, dt = \pi\varepsilon^2 Length(\gamma).$$

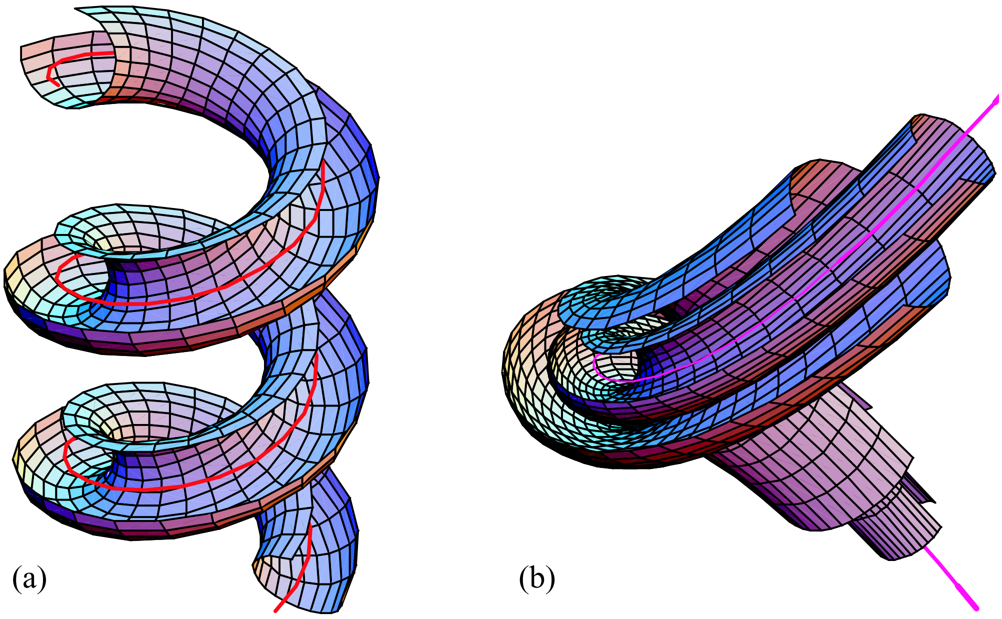

FIGURE 5. (a) A tubular surface about the helix $t \mapsto (\cos t, \sin t, t/4)$. (b) A tube about a twisted cubic. [Gray]

REFERENCES

[Gray] A. Gray *Tubes* Birkhäuser, Boston, 2000.

[Morg] F. Morgan, *Riemannian Geometry: a beginner's guide, 2nd ed.,* A. K. Peters, Natick, Massachusetts, 1998.