Self-similar tilings and their complex dimensions.

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July 6, 2006

The 21st Summer Conference on Topology and its Applications

References

[SST] "Canonical self-similar tilings by IFS", E. P. J. Pearse, preprint, Nov. 2005, 14 pages.
arXiv: math.MG/0606111

[CDS] "Tube formulas and complex dimensions of self-similar tilings", M. L. Lapidus and E. P. J. Pearse, preprint, Apr. 2006, 51 pages. arXiv: math.DS/0605527

[FGCD] "Fractal Geometry, Complex Dimensions and Zeta Functions: Geometry and Spectra of Fractal Strings", M. L. Lapidus and M. van Frankenhuijsen, Springer-Verlag, 2006.

[KTF] "A tube formula for the Koch snowflake curve, with applications to complex dimensions", M. L. Lapidus and E. P. J. Pearse, *J. London Math. Soc.* (in press).
arXiv: math-ph/0412029

Self-similarity

A *self-similar system* is $\Phi = \{\Phi_j\}_{j=1}^J$, where each Φ_j is a contractive similarity mapping, i.e.,

$$\Phi_j(x) = r_j A_j x + t_j, \; 0 < r_j < 1,$$

where $A_j \in O(d)$ is a rotation/reflection and $t_j \in \mathbb{R}^d$.

A set F is self-similar iff $F = \Phi(F) = \bigcup_{j=1}^J \Phi_j(F)$.
For any such Φ, $\exists! F \neq \varnothing$ and F is compact.

[SST] There is a tiling $\mathcal{T}$ of the convex hull $[F]$ which is canonically associated with Φ. Not a typical "fractal tiling":

- Tiles are not typically fractal.
- Only the region $[F]$ is tiled, not $\mathbb{R}^d$.
- Tiles may not be fractal, but $\mathcal{T}$ is.

$\mathcal{T}$ contains key geometric/dynamical information about Φ.

- $\mathcal{T}$ describes scaling/geometric oscillations.
- Curvature of the tiles relates to "curvature" of F.
- From $\mathcal{T}$, one can define $\zeta_{\mathcal{T}}$, a geometric zeta function associated with Φ.

$$\Phi_1(z) = \xi\overline{z}, \quad \Phi_2(z) = (1-\xi)(\overline{z}-1)+1, \quad \xi = \tfrac{1}{2} + \tfrac{\mathrm{i}}{2\sqrt{3}}.$$

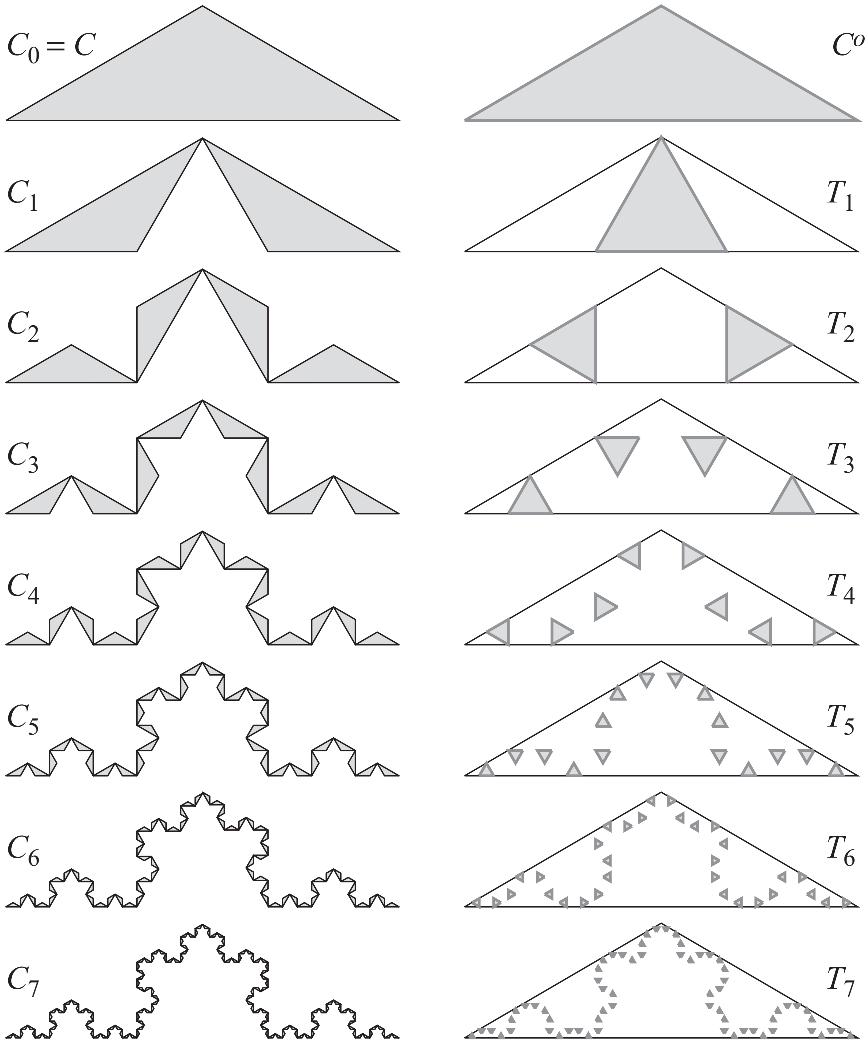


FIGURE 1. The Koch tiling, with unique generator $G = T_1$.

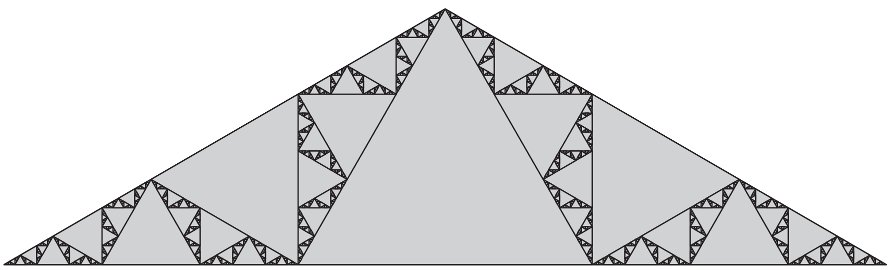


FIGURE 2. The tiling of $[K] \sim K$. The tiles exhaust the complement of the Koch curve within its convex hull.

Fractal strings [FGCD]

A *fractal string* $\mathcal{L} \subseteq \mathbb{R}$ is a bounded open subset

$$\mathcal{L} := \{\ell_n\}_{n=1}^{\infty}.$$

Translation invariance: $\ell_n \in \mathbb{R}$, and may assume

$$\ell_1 \geq \ell_2 \geq \ell_3 \geq \dots$$

Also assume $\ell_n > 0$, or else trivial.

Idea: $\partial\mathcal{L} = F$, where $F \subseteq \mathbb{R}$ is fractal.

A *self-similar fractal string* is when $\partial\mathcal{L} = F$ for some Φ with $d = 1$ and $A_j = \pm 1$.

The *geometric zeta function* $\zeta_{\mathcal{L}}$ is a Dirichlet generating function for the string.

$$\zeta_{\mathcal{L}}(s) := \sum_{n=1}^{\infty} \ell_n^s.$$

The *complex dimensions* of $\mathcal{L}$ are

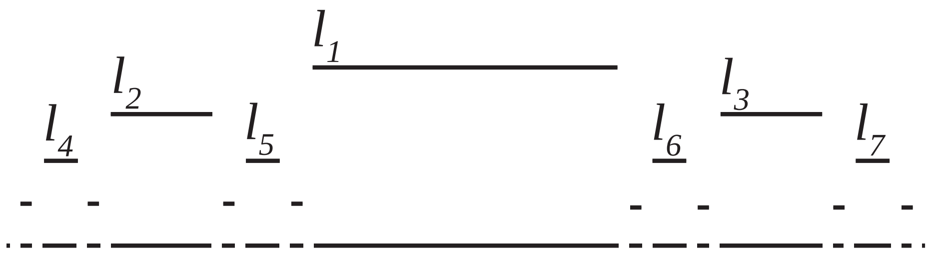
$$\mathcal{D}_{\mathcal{L}} := \{\text{poles of } \zeta_{\mathcal{L}}\}.$$

Important result: a *tube formula* for strings

$$\begin{aligned} V_A(\varepsilon) &= \mathrm{vol}_1\{x \in A \,\vdots\, dist(x, \partial A) < \varepsilon\}, \\ &= \sum_{\omega \in \mathcal{D}_{\mathcal{L}}} c_\omega \, \varepsilon^{1-\omega} + c_1 \varepsilon. \end{aligned}$$

c_ω is defined in terms of $\mathrm{res}\,(\zeta_{\mathcal{L}}; \omega)$.

Example: the Cantor String



$$\mathcal{CS} = \left\{\tfrac{1}{3}, \tfrac{1}{9}, \tfrac{1}{9}, \tfrac{1}{27}, \tfrac{1}{27}, \tfrac{1}{27}, \tfrac{1}{27}, \ldots\right\} = [0,1] \sim \mathcal{C}$$

$$\partial(\mathcal{CS}) = \text{the Cantor set } \mathcal{C}$$

$\mathcal{CS}$ has $l_k = \frac{1}{3^{k+1}}$ with $w_{l_k} = 2^k$.

$$\zeta_{\mathcal{CS}}(s) = \sum_{k=0}^{\infty} 2^k 3^{-(k+1)s} = \frac{3^{-s}}{1 - 2 \cdot 3^{-s}}.$$

$$V_{\mathcal{CS}}(\varepsilon) = \frac{1}{3\log 3} \sum_{\omega \in \mathcal{D}_{\mathcal{CS}}} \frac{6^{1-\omega}}{\omega(1-\omega)} \varepsilon^{1-\omega} - 2\varepsilon$$

$$= \sum_{\omega \in \mathcal{D}_{\mathcal{CS}} \cup \{1\}} c_\omega \varepsilon^{1-\omega}$$

$$\mathcal{D}_{\mathcal{CS}}(s) = \{D + \mathbb{i} n \mathbf{p} \,\vdots\, \mathbf{p} = \tfrac{2\pi}{\log 3}, n \in \mathbb{Z}\}.$$

$$D = \log_3 2 = \text{Minkowski dim of } \mathcal{C}.$$

As a self-similar string, $\mathcal{CS}$ is the tiling associated with the self-similar system $\{\Phi_1, \Phi_2\}$,

$$\Phi_1(x) = \tfrac{1}{3}x, \quad \Phi_2(x) = \tfrac{1}{3}x + \tfrac{2}{3}.$$

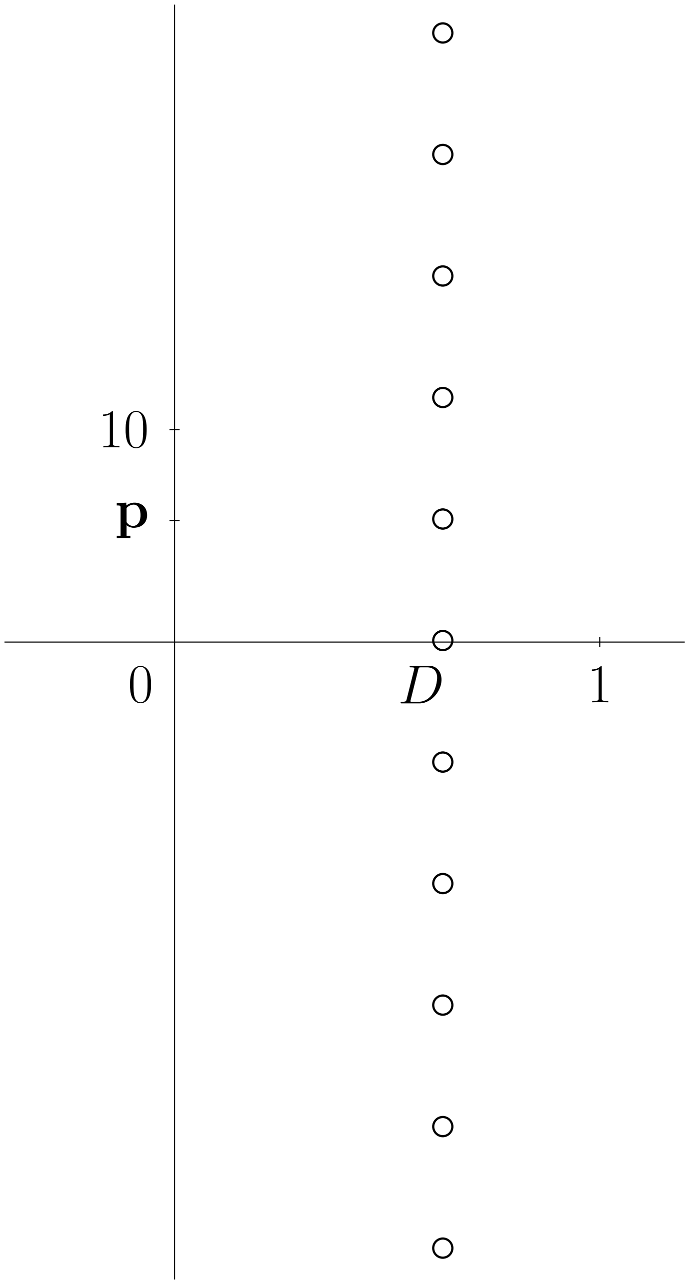


FIGURE 3. The complex dimensions of the Cantor string. $D = \log_3 2$ and $\mathbf{p} = 2\pi/\log 3$. (Plot from [FGCD])

Why call the poles of $\zeta_{\mathcal{L}}$ the "complex dimensions"?
First reason: relation to Minkowski/box dimension.

Theorem (Lapidus). Assuming $\ell_n > 0$ for all n,

$$D = \inf\{\sigma \geq 0 \colon \sum_{n=1}^{\infty} \ell_n^{\sigma} < \infty\}.$$

In fact, $\mathcal{D}_{\mathcal{L}} \cap \mathbb{R}^+ = \{D\}$.

Recall:
An *(inner) tube formula* for a set $A \subseteq \mathbb{R}$ is

$$V_A(\varepsilon) = \text{vol}_1\{x \in A \colon dist(x, \partial A) < \varepsilon\},$$

i.e., the volume of the inner ε-neighbourhood of A.

The *Minkowski dimension* of ∂A is then

$$D = \inf\{t \geq 0 \colon V_A(\varepsilon) = O(\varepsilon^{1-t}) \text{ as } \varepsilon \to 0^+\}.$$

A is said to be *Minkowski measurable* iff

$$\mathcal{M} = \lim_{\varepsilon \to 0^+} V(\varepsilon)\varepsilon^{-(1-D)}$$

exists, and has a value in $(0, \infty)$.

For $A \subseteq \mathbb{R}^d$, replace 1 by d in V_A, D, $\mathcal{M}$.

Why call the poles of $\zeta_{\mathcal{L}}$ the "complex dimensions"? Second reason: relation to classical geometry.

The Steiner formula for ε-nbd of $A \in \mathcal{K}^d$:

$$V_A(\varepsilon) = \sum_{i \in \{0,1,\ldots,d-1\}} c_i \, \varepsilon^{d-i}.$$

The classical (outer) tube formula is summed over the integral dimensions of A.

[FGCD] Tube formula for $\mathcal{L}$:

$$V_{\mathcal{L}}(\varepsilon) = \sum_{\omega \in \mathcal{D}_{\mathcal{L}} \cup \{1\}} c_\omega \, \varepsilon^{1-\omega}.$$

[CDS] Tube formula for $\mathcal{T}$:

$$V_{\mathcal{T}}(\varepsilon) = \sum_{\omega \in \mathcal{D}_{\mathfrak{s}} \cup \{0,1,\ldots,d\}} c_\omega \, \varepsilon^{d-\omega}.$$

The fractal tube formula is summed over the integral *and complex* dimensions of $\mathcal{L}$ or $\mathcal{T}$.

$$\Phi_1(z) = \xi\overline{z}, \quad \Phi_2(z) = (1-\xi)(\overline{z}-1)+1, \quad \xi = \tfrac{1}{2} + \tfrac{\mathbb{i}}{2\sqrt{3}}.$$

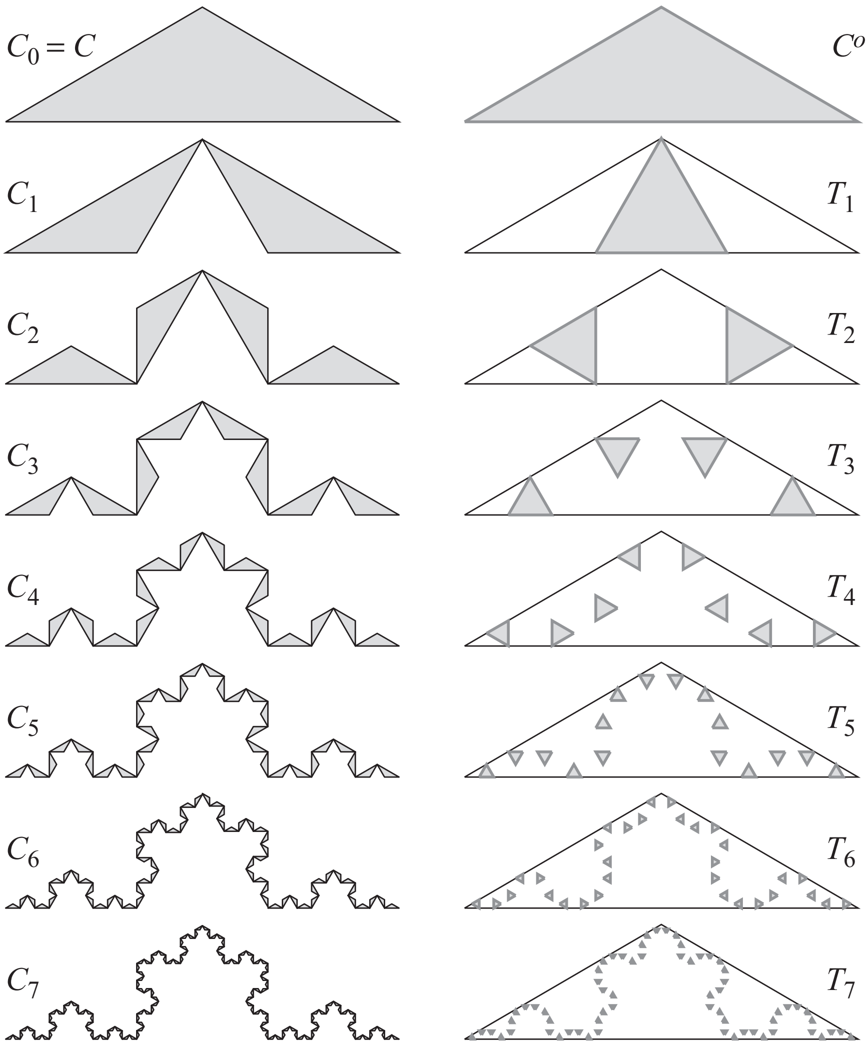


FIGURE 4. The Koch tiling, with unique generator $G = T_1$.

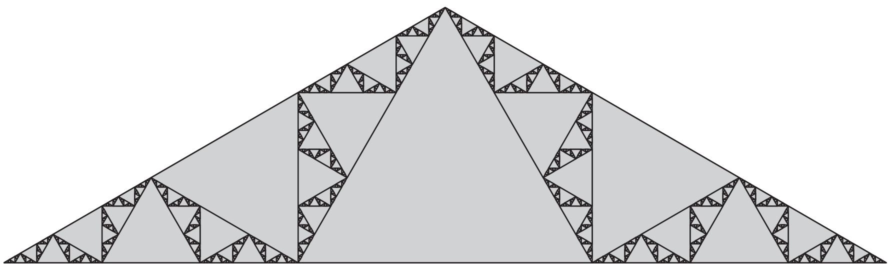


FIGURE 5. The tiles exhaust the complement of the Koch curve within its convex hull.

The self-similar tiling: extension to higher dimensions

From [SST]: Construct the *self-similar tiling* and produce a collection of tiles

$$\mathcal{T} = \{R_n\} = \{\Phi_w(G_q)\}.$$

$w \in \{1, 2, \ldots, J\}^k$ is a (finite) word, like $w = 3132$.

$$\Phi_{3132}(x) := \Phi_2 \circ \Phi_3 \circ \Phi_1 \circ \Phi_3(x), \; r_{3132} = r_1 r_2 r_3^2.$$

string $\mathcal{L} = \{\ell_n\}$	tiling $\mathcal{T} = \{R_n\}$
length ℓ_n	inradius ρ_n
$\zeta_{\mathcal{L}} = \sum \ell_n^s$	$\zeta_{\mathfrak{s}} = \sum r_w^s$
$\mathcal{D}_{\mathcal{L}} = \{$poles of $\zeta_{\mathcal{L}}\}$	$\mathcal{D}_{\mathfrak{s}} = \{$poles of $\zeta_{\mathfrak{s}}\}$

Idea: decompose the complement of the attractor F within its convex hull $[F]$.

1. Find the attractor F of Φ.

2. Take the (closed) convex hull $C = [F]$.

3. Define the generators G_q to be the connected components of $\text{relint}(C) \sim \Phi(C)$. (Note: $\Phi(C) \subseteq C$ [SST].)

4. The sets $\{\Phi_w(G_q)\}$ form a tiling of $C \sim F$.

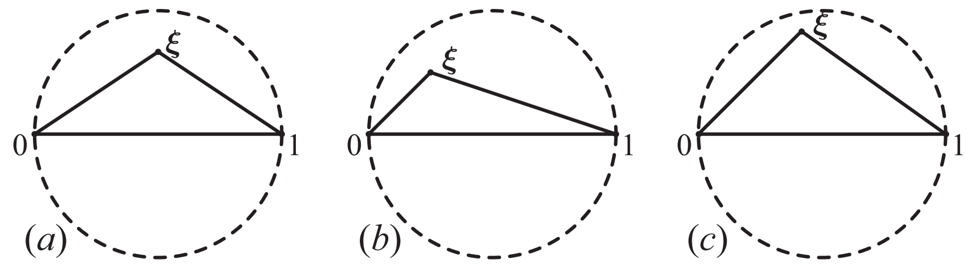


FIGURE 6. Parameters for nonstandard Koch tilings.

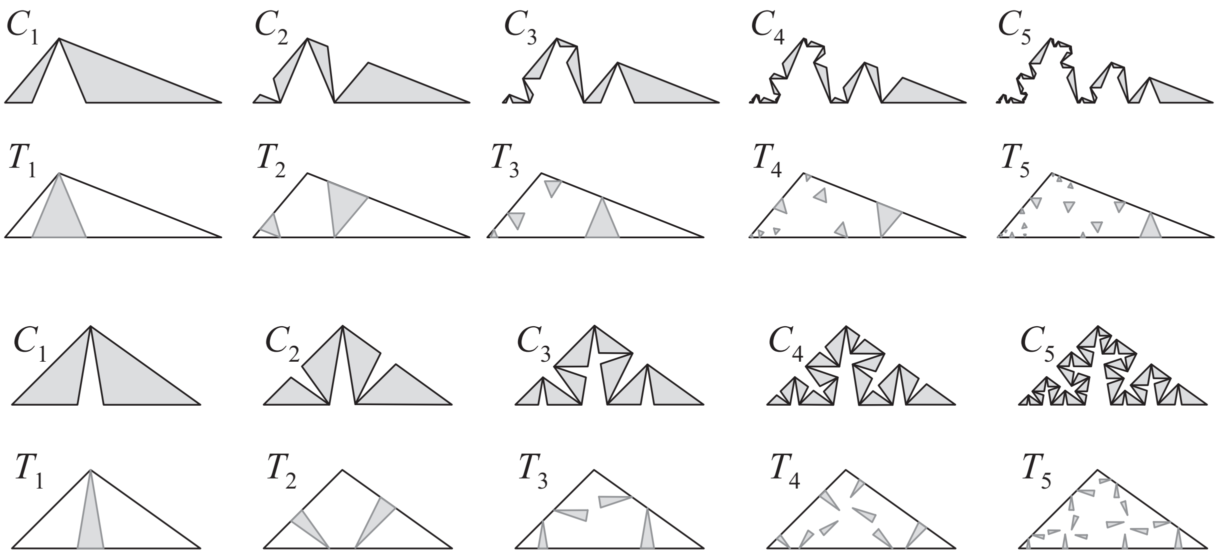


FIGURE 7. Nonstandard Koch tilings.

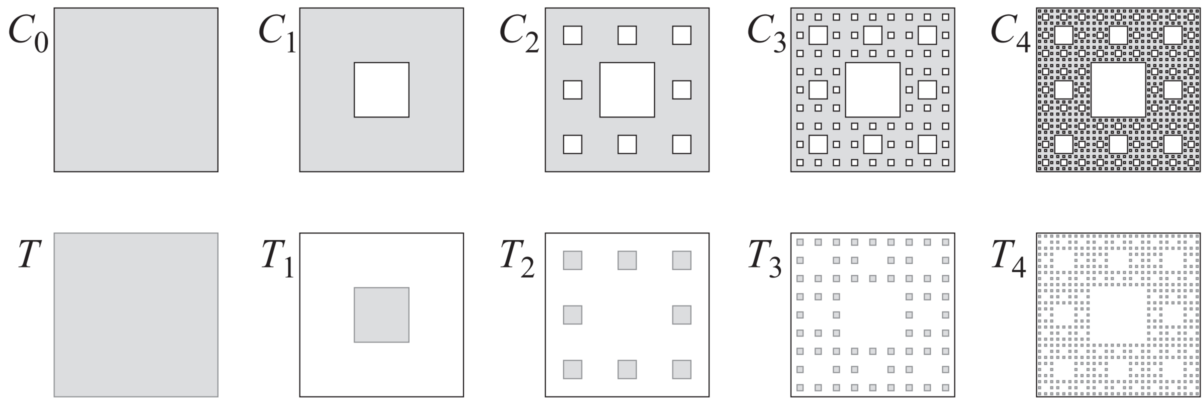


FIGURE 8. The Sierpinski Carpet tiling.

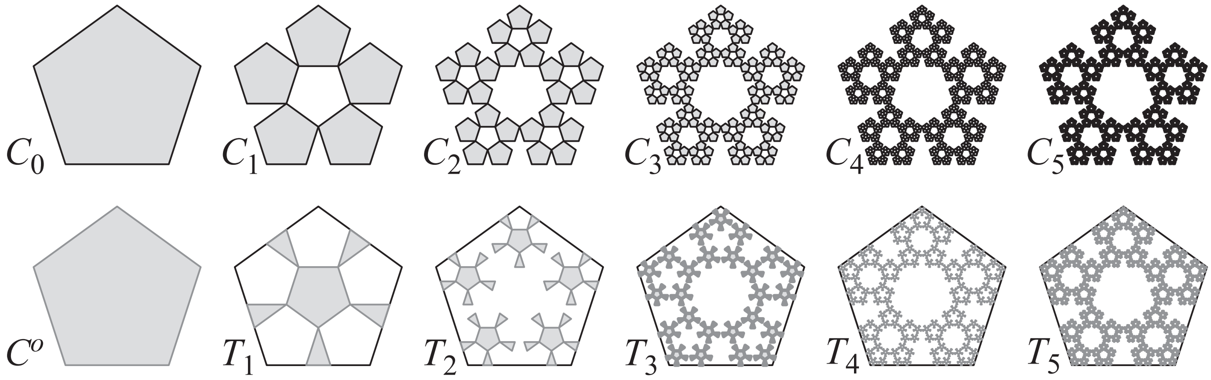


Figure 9. The Pentagasket tiling. This tiling has 6 generators; one pentagon G_1, and five congruent isoceles triangles $G_2, \ldots, G_6$. $T_1 = G_1 \cup G_2 \cup \cdots \cup G_6$.

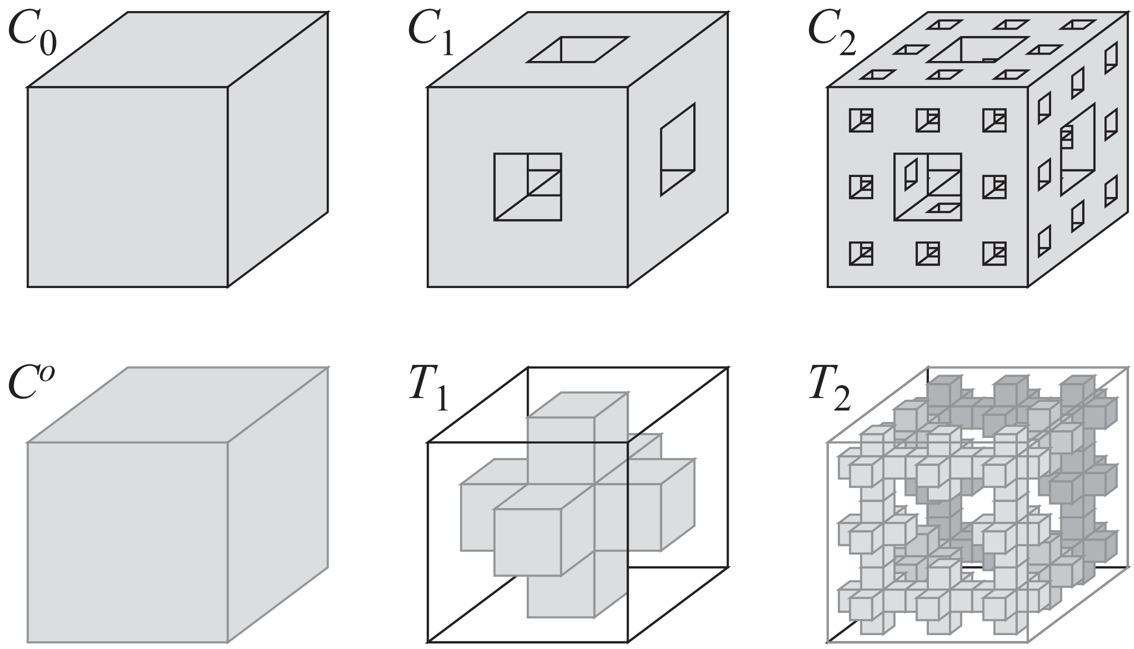


Figure 10. The Menger Sponge tiling. This tiling has the unique nonconvex generator $G = T_1$.

Tube formula of a self-similar tiling

Recall the form of $V_{\mathcal{T}}$ given earlier:

$$V_{\mathcal{T}}(\varepsilon) = \sum_{\omega \in \mathcal{D}_{\mathfrak{s}} \cup \{0,1,\ldots,d\}} c_\omega \, \varepsilon^{d-\omega}.$$

[CDS] The full form of $V_{\mathcal{T}}$:

$$V_{\mathcal{T}}(\varepsilon) = \sum_{\omega \in \mathcal{D}_{\mathcal{T}}} \operatorname{res}\left(\zeta_{\mathcal{T}}(\varepsilon, s); \omega\right).$$

$$\mathcal{D}_{\mathcal{T}} = \mathcal{D}_{\mathfrak{s}} \cup \{0, 1, \ldots, d\},$$

where $\mathcal{D}_{\mathfrak{s}} = \{\text{poles of } \zeta_{\mathfrak{s}}\}$.

Once the residues are evaluated, these are the same.

For sets with no fine structure (trivially self-similar),

$$\zeta_{\mathfrak{s}} \text{ holomorphic } \implies \mathcal{D}_{\mathfrak{s}} = \varnothing$$

and we recover the "inner Steiner formula".

In the formula for $V_{\mathcal{T}}$, $\zeta_{\mathcal{T}}$ is a meromorphic distribution-valued function; a generating function for the geometry of $\mathcal{T}$.

Ingredients of the Geometric Zeta Function

The *tiling zeta function* is the matrix product

$$\zeta_{\mathcal{T}}(\varepsilon, s) = \vec{g}(s) \cdot \boldsymbol{\kappa}(\varepsilon) \cdot \boldsymbol{\mathcal{E}}(\varepsilon, s).$$

The *generator inradii* form a Q-vector:

$$\vec{g}(s) := [g_1^s, \ldots, g_Q^s]\, \zeta_{\mathfrak{s}}(s).$$

The *curvature matrix* is the $Q \times (d+1)$ matrix

$$\boldsymbol{\kappa}(\varepsilon) := [\kappa_{qi}(\varepsilon)]$$

($\kappa_{qi}(\varepsilon)$ is the "i^{th} curvature" of the q^{th} generator;
i.e., $\kappa_{qi}(\varepsilon)$ is the coefficient of ε^{d-i} in γ_q.)

The *boundary terms* compose a $(d+1)$-vector

$$\boldsymbol{\mathcal{E}}(\varepsilon, s) := \left[\tfrac{1}{s}, \tfrac{1}{s-1}, \ldots, \tfrac{1}{s-d}\right] \varepsilon^{d-s}.$$

Then $c_\omega\, \varepsilon^{d-\omega}$ is a residue of the matrix product:

$$c_\omega\, \varepsilon^{d-\omega} = \operatorname{res}\left(\zeta_{\mathcal{T}}(\varepsilon, s); \omega\right).$$

Idea behind the tube formula

Obtain a tube formula for $\mathcal{T}$ using the idea

$$V_{\mathcal{T}}(\varepsilon) = \langle \eta_{\mathfrak{g}}, \gamma_G \rangle = \int_0^\infty \gamma_G(x, \varepsilon) \, d\eta_{\mathfrak{g}}(x).$$

$\gamma_G(x, \varepsilon)$ is the volume of the ε-neighbourhood of a tile with inradius $1/x$; i.e., the contribution of an tile at scale r_w^{-1}.

$\eta_{\mathfrak{g}}(x)$ is a measure giving the density of geometric states: it is supported at reciprocal tile inradii. Each generator has its own associated $\eta_{\mathfrak{g}}$:

$$\eta_{\mathfrak{g}}(x) = [\eta_{\mathfrak{s}}(x/g_1), \ldots, \eta_{\mathfrak{s}}(x/g_Q)]$$

$\eta_{\mathfrak{s}}(x)$ is the scaling measure: it is supported at the reciprocal scales.

Obtain a tube formula

$$V_{\mathcal{T}}(\varepsilon) = \sum_{\omega \in \mathcal{D}_{\mathcal{T}}} c_\omega \, \varepsilon^{d-\omega}$$

for $\mathcal{D}_{\mathcal{T}} = \mathcal{D}_{\mathfrak{s}} \cup \{0, 1, \ldots, d\}$.

Contributions to $V_{\mathcal{T}}(\varepsilon) = \int_0^\infty \gamma_G(x,\varepsilon)\, d\eta_{\mathfrak{g}}(x)$

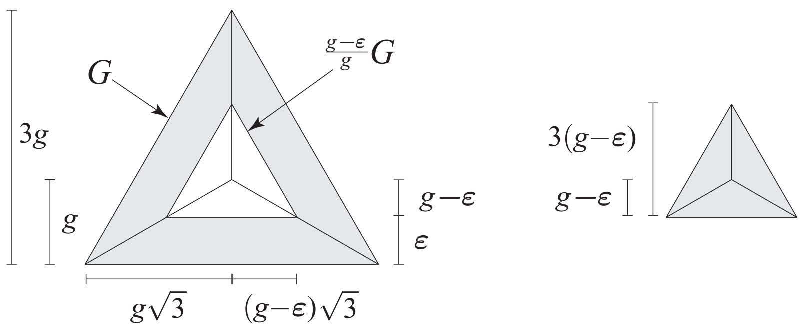


FIGURE 11. The volume $V_G(\varepsilon) = \gamma_G(1/g, \varepsilon)$ of the generator of the Koch tiling.

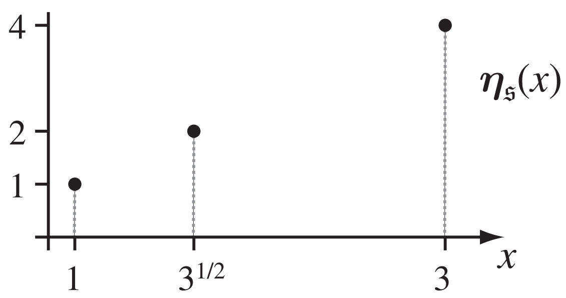


FIGURE 12. The Koch scaling measure.

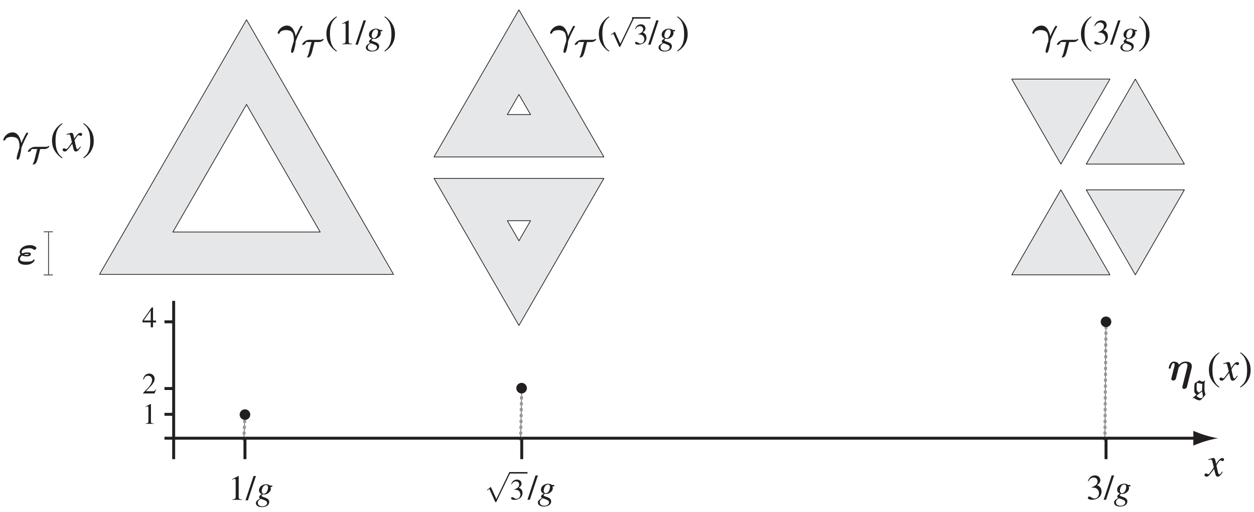


FIGURE 13. The Koch geometric measure and contributions to the integral.

Q: ***What's different in*** $\mathbb{R}^d$***?***
A: The generators.

$$\Phi_j(x) = r_j A_j x + t_j,\ 0 < r_j < 1.$$

New: $A_j \in SO(d)$ is a rotation/reflection.

- Since generators may have different shapes, must distinguish between tile of different type, but with the same inradius.
- Different generators may have different inner tube formulas. (Intervals all have 2ε.)

Must distinguish scales from sizes: $\eta_{\mathfrak{s}}$ vs. $\eta_{\mathfrak{g}}$.

$$\eta_{\mathfrak{s}} = \sum_{w \in \mathcal{W}} \delta_{r_w^{-1}} \qquad \eta_{\mathfrak{g}} = [\eta_{\mathfrak{s}}(x/g_q)]_{q=1}^{Q}$$

$$\eta_{\mathfrak{g}q} = \sum_{n=1}^{\infty} \delta_{\rho_n(G_q)^{-1}} = \sum_{w \in \mathcal{W}} \delta_{(g_q r_w)^{-1}}$$

Must calculate each congruency class of generators separately. Cannot integrate against the same measure for each.

Q: ***What's the same in*** $\mathbb{R}^d$***?***

A: The scaling ratios.

Properties of $r_1, \ldots, r_J$ are still of key importance.

$\zeta_{\mathfrak{s}}$ is formally identical to ζ_η for η with $g = 1$.

$$\zeta_{\mathfrak{s}}(s) = \sum_{w \in \mathcal{W}} r_w^s = \frac{1}{1 - \sum_{j=1}^J r_j^s}.$$

$\mathcal{D}_{\mathfrak{s}}$ depends entirely on $r_1, \ldots, r_J$.

Lattice/nonlattice dichotomy still holds.

The structure theorem for $\mathcal{D}_{\mathfrak{s}}$ is the same.

The distributional explicit formulas which applied to η in the 1-dimensional case apply to each $\eta_{\mathfrak{g}q}$ in the d-dimensional case.

Convex Geometry

The *Steiner Formula* for convex bodies $A \in \mathcal{K}^d$.

$$V_A(\varepsilon) = \sum_{i=0}^{d-1} \binom{d}{i} W_{d-i}(A) \varepsilon^{d-i}$$
$$= \sum_{i=0}^{d-1} \mu_{d-i}(B^{d-i}) \mu_i(A) \varepsilon^{d-i}.$$

- W_i are the Minkowski functionals.
- B^i is the i-dimensional unit ball.
- μ_i are invariant/intrinsic measures.
  Homogeneous: $\mu_i(rA) = r^i \mu_i(A)$,

Basic idea:

$$V_A(\varepsilon) = \sum_{\omega \in \{0,1,\ldots,d-1\}} c_\omega \, \varepsilon^\omega$$

where c_ω is related to the curvature of A.

Curvature measures in convex geometry

$$V_A(\varepsilon) = \sum_{i=0}^{d-1} \mu_i(A)\mu_{d-i}(B^{d-i})\varepsilon^{d-i}.$$

For convex bodies $A \in \mathcal{K}^d$,

$$\kappa_i(A) = d \cdot \mu_i(A)\mu_{d-i}(B^{d-i}) = \binom{d}{i} C_i(A).$$

Here, the C_i are curvature measures and the κ_i are curvatures (as def'd in convex/integral geometry).

The κ_i are homogeneous and translation invariant because the μ_i are.

$C_i(A)$ is the *total curvature of* A; a special case of the generalized curvature measure

$$C_i(A) := C_i(A, \mathbb{R}^d) = \Theta_i(A, \mathbb{R}^d \times S^{d-1}).$$

Θ_i is defined on $U(\mathbb{K}^d) \times \mathcal{B}(\Sigma)$, where $U(\mathbb{K}^d)$ is the ring of polyconvex sets of dimension $\leq d$.