Math 205C - Topology Homework 2 Erin Pearse

1. a) *Define a tangent vector on a manifold M.*

   A tangent vector on M at p is a mapping $X_p : C^\infty(p) \to \mathbb{R}$ satisfying

   i) $X_p(\alpha f + \beta g) = \alpha(X_p f) + \beta(X_p g)$, and

   ii) $X_p(f \cdot g) = (X_p f)\, g(p) + f(p)\,(X_p g)$,

   for all $\alpha, \beta \in \mathbb{R}$ and for all $f, g \in C^\infty(p)$, where $C^\infty(p)$ is the algebra of C^∞ functions whose domain of definition includes some open neighbourhood of p.

   b) *Define a vector field on M.*

   A C^r vector field on M is a function assigning to every point $p \in M$ a vector $X_p \in T_p(M)$ whose components in the frames of any local coordinates $\{(U_p, \varphi_p)\}$ are functions of class C^r on U_p.

   c) *Define the tangent space of M at p.*

   The tangent space $T_p(M)$ of M at p is the collection of all tangent vectors X_p (as defined above) with vector space operations defined by $(X_p + Y_p)\, f = X_p f + Y_p f$ and $(\alpha X_p)\, f = \alpha(X_p f)$.

   i) *What is* $\dim T_p(M)$*?*
   $\dim T_p(M) = m = \dim M$.

   ii) *Find a basis for* $T_p(M)$.
   $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$ is a basis for the tangent space at any point of $\mathbb{R}^n$, so we use the pullback of this basis (under the coordinate map) to define a basis for $T_p(M)$. Therefore, the collection of parametrized tangent vectors $\{\varphi_*^{-1} \circ \frac{\partial}{\partial x^1}, \ldots, \varphi_*^{-1} \circ \frac{\partial}{\partial x^m}\}$ forms a basis for $T_p(M)$, where φ_*^{-1} is the inverse of the isomorphism $\varphi_* : T_p(M) \to T_{\varphi(p)}(\mathbb{R}^n)$ of the tangent spaces, induced by the coordinate map φ.

   d) $\mathfrak{X}(M) = \{$*all vector fields on M*$\}$.

   i) *Does* $\mathfrak{X}(M)$ *form a vector space?*
   Yes. For vector fields $X, Y \in \mathfrak{X}(M)$ and $\forall a, b \in \mathbb{R}$, $Z = aX + bY \in \mathfrak{X}(M)$.

   ii) *What is* $\dim \mathfrak{X}(M)$*? (Provide proof)*
   $\dim \mathfrak{X}(M) = \infty$. Suppose that $\dim \mathfrak{X}(M) = n < \infty$. Then $\mathfrak{X}(M)$ has some basis $\{X_1, \ldots, X_n\}$, and any vector field on M can be written as a linear combination of these basis elements, i.e.,

   $$X = a_1 X_1 + \ldots + a_n X_n,\ \forall X \in \mathfrak{X}(M)$$

   Now choose one vector from $T_{p_i}(M)$ at each of n points $\{p_i\}_{i=1}^n$. Then we have

   $$X_{p_i} = a_1 X_1(p_i) + \ldots + a_n X_n(p_i) = a_1 X_{1p_i} + \ldots + a_n X_{np_i}$$

at each p_i. Hence, we have n equations and n unknowns, so we can solve for the a_i and get

$$X = \alpha_1 X_1 + \ldots + \alpha_n X_n \qquad (\star)$$

for some $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$. We have *completely* determined the vector field on M, by specifying it at these n points. Now choose some point $q \in M$ and determine X_q, based on $(\star)$. Let $Y_q \neq X_q$ be any other vector in $T_q(M)$. Consider the vectors $\{X_{p_1}, \ldots, X_{p_n}, Y_q\}$. Since we have specified only finitely many vectors, there must be some $X' \in \mathfrak{X}(M)$ which takes these values at each of the respective points $\{p_1, \ldots, p_n, q\}$. But clearly, this X' cannot be a linear combination of the "basis" $\{X_1, \ldots, X_n\}$. ■

2. a) *Let $X, Y \in \mathfrak{X}(M)$ and show that $[X, Y] = XY - YX \in \mathfrak{X}(M)$.*

For $f \in C^\infty$, define $Z \in \mathfrak{X}(M)$ by

$$Z_p f = (XY - YX)_p f = X_p(Yf) - Y_p(Xf).$$

This is a well-defined linear map $C^\infty(p) \to \mathbb{R}$ because $\mathfrak{X}(M)$ is a $C^\infty(M)$-module. To see that Z_p satisfies the Leibniz rule, note that

$$\begin{aligned}
Z_p fg &= (XY - YX)_p fg \\
&= X_p(Yfg) - Y_p(Xfg) \\
&= X_p(fYg + gYf) - Y_p(fXg + gXf) \\
&= (X_p f)(Yg)_p + f(p) X_p(Yg) + (X_p g)(Yf)_p \\
&\qquad + g(p) X_p(Yf) - (Y_p f)(Xg)_p - f(p) Y_p(Xg) \\
&\qquad - (Y_p g)(Xf)_p - g(p)(Y_p Xf) \\
&= f(p)(XY - YX)_p g + g(p)(XY - YX)_p f \\
&= f(p) Z_p g + g(p) Z_p f
\end{aligned}$$

b) *Is XY a vector field in general? If not, provide a counterexample.*

No, XY is not a vector field in general. Counterexample: define the vector fields $X = \frac{d}{dx}, Y = \frac{d}{dy}$ and check the product rule:

$$\begin{aligned}
XY(fg) &= \tfrac{d}{dx}\tfrac{d}{dy}(fg) \\
&= \tfrac{d}{dx}\left(g\tfrac{df}{dy} + f\tfrac{dg}{dy}\right) \\
&= \left(\tfrac{dg}{dx}\tfrac{df}{dy} + g\tfrac{d^2 f}{dxdy}\right) + \left(\tfrac{df}{dx}\tfrac{dg}{dy} + f\tfrac{d^2 g}{dxdy}\right) \\
(XYf)g + (XYg)f &= \tfrac{d^2 f}{dxdy}g + \tfrac{d^2 g}{dxdy}f
\end{aligned}$$

Since $XY(fg) - ((XYf)g + (XYg)f) = \frac{df}{dx}\frac{dg}{dy} + \frac{dg}{dx}\frac{df}{dy} \neq 0$,
we know that XY is not a vector field in this case.

□

3. a) *Define a Lie Algebra.*

A Lie Algebra $\mathfrak{L}$ is a vector space over $\mathbb{R}$ that is endowed with the additional structure of an operation $(X, Y) \to [X, Y] \in \mathfrak{L}$ satisfying

i) it is bilinear over $\mathbb{R}$:

$$[\alpha_1 X_1 + \alpha_2 X_2, Y] = \alpha_1 [X_1, Y] + \alpha_2 [X_2, Y]$$
$$[X, \alpha_1 Y_1 + \alpha_2 Y_2] = \alpha_1 [X, Y_1] + \alpha_2 [X, Y_2]$$

ii) it is skew commutative:

$$[X, Y] = -[Y, X]$$

iii) it satisfies the *Jacobi identity*:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

b) *Prove that $\mathfrak{X}(M)$ with the Lie bracket operation forms a Lie Algebra.*

Let $\alpha, \beta \in \mathbb{R}$ and suppose X_1, X_2, Y are vector fields. Then

$$\begin{aligned}
[\alpha X_1 + \beta X_2, Y] f &= (\alpha X_1 + \beta X_2) Y f - Y (\alpha X_1 + \beta X_2) f \\
&= \alpha X_1 Y f + \beta X_2 Y f - \alpha Y X_1 f - \beta Y X_2 f \\
&= \alpha X_1 Y f - \alpha Y X_1 f + \beta X_2 Y f - \beta Y X_2 f \\
&= \alpha (X_1 Y - Y X_1) f + \beta (X_2 Y - Y X_2) f \\
&= \alpha [X_1, Y] f + \beta [X_2, Y] f
\end{aligned}$$

shows that $[X, Y]$ is linear in the first variable. Then $[Y, X] = YX - XY = -XY + YX = -(XY - YX) = -[X, Y]$ shows that $[X, Y]$ is skew-commutative. Then

$$\begin{aligned}
[X, [Y, Z]] f &= X (Y (Zf)) - X (Z (Yf)) - Y (Z (Xf)) + Z (Y (Xf)) \\
[Y, [Z, X]] f &= Y (Z (Xf)) - Y (X (Zf)) - Z (X (Yf)) + X (Z (Yf)) \\
[Z, [X, Y]] f &= Z (X (Yf)) - Z (Y (Xf)) - X (Y (Zf)) + Y (X (Zf)) \\
&\implies [X, [Y, Z]] f + [Y, [Z, X]] f + [Z, [X, Y]] f = 0
\end{aligned}$$

□

c) *Give two examples of a Lie Algebra; one finite-dimensional and one infinite-dimensional.*

i) $(\mathbb{R}^3, \times)$ is a 3-dimensional Lie algebra.

ii) $\mathfrak{X}(M)$ with the commutator product $[X, Y]$ is a infinite-dimensional Lie algebra. (see 1(d) for proof of $\dim \mathfrak{X}(M) = \infty$). □

4. a) *$X, Y \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$.*
Prove the identity. $[fX, gY] = fg\,[X,Y] + f\,(Xg)\,Y - g\,(Yf)\,X$.

$$\begin{aligned}
[fX, gY] &= (fX)\,(gY) - (gY)\,(fX) \\
&= f\,(XgY + gXY) - g\,((Yf)\,X + fYX) \\
&= fXgY + fgXY - gYfX - gfYX \\
&= fg\,(XY - YX) + fXgY - gYfX \\
&= fg\,[X,Y] + f\,(Xg)\,Y - g\,(Yf)\,X
\end{aligned}$$

b) $X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$, $Y = z\frac{\partial}{\partial x} - y\frac{\partial}{\partial z}$, $Z = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$ *in* $\mathbb{R}^3$.

i) *Compute* $[X, Y]$.

$$[X,Y] = x\left(\frac{\partial^2 y}{\partial y \partial z} - \frac{\partial^2 z}{\partial x \partial y}\right) + y\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 x}{\partial y \partial z}\right) + z\left(\frac{\partial^2 x}{\partial x \partial y} - \frac{\partial^2 y}{\partial x^2}\right)$$

ii) *Compute* $[Y, Z]$.

$$[Y,Z] = \frac{\partial^2 y}{\partial z^2} - \frac{\partial^2 z}{\partial x \partial z} + \frac{\partial^2 y}{\partial y \partial z} - \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 y}{\partial x \partial z} - \frac{\partial^2 z}{\partial x^2}$$

iii) *Compute* $[X, Z]$.

$$[X,Z] = \frac{\partial^2 x}{\partial y \partial z} + \frac{\partial^2 x}{\partial y^2} + \frac{\partial^2 x}{\partial x \partial y} - \frac{\partial^2 y}{\partial x \partial z} - \frac{\partial^2 y}{\partial x \partial y} - \frac{\partial^2 y}{\partial x^2}$$

$\square$

5. a) *S^1 is the 1-dimensional sphere. Show that S^1 admits a nonvanishing vector field.*

Define S^1 by $S^1 = \{(\cos\theta, \sin\theta) \in \mathbb{R}^2 \vdots \theta \in \mathbb{R}\}$ and let X be a vector field defined on S^1 by $\frac{\partial}{\partial\theta}$. Then for any θ, $\frac{\partial}{\partial\theta} = (-\sin\theta, \cos\theta) \neq (0,0)$.
Equivalently, let $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. This is easily seen to be a nonvanishing vector field on S^1, as it only vanishes at $(0,0)$, and $(0,0) \notin S^1$. To see that it is tangent to S^1 everywhere, note that $(x,y)\cdot(-y,x) = -xy + xy = 0$.

b) *$T^2 = S^1 \times S^1$ is the torus. Show that T^2 also admits a nowhere vanishing vector field.*
Parametrize a torus as $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\Phi(u,v) = ((2+\cos u)\cos v, (2+\cos u)\sin v, \sin u)$$

Then, taking partial derivatives with respect to u and v, we get

$$\Phi_u(u,v) = (-\sin u\cos v, -\sin u\sin v, \cos u)$$
$$\Phi_v(u,v) = (-(2+\cos u)\sin v, (2+\cos u)\cos v, 0)$$

The vector field $X = (\Phi_u, \Phi_v)$ is thus everywhere tangential to T^2 by construction. To see that it is nowhere vanishing, compute

$$\begin{aligned}\|\Phi_u\| &= \sqrt{\sin^2 u\cos^2 v + \sin^2 u\sin^2 v + \cos^2 u} \\ &= \sqrt{\sin^2 u + \cos^2 u} \\ &= \sqrt{1} \\ &= 1\end{aligned}$$

and

$$\begin{aligned}\|\Phi_v\| &= \sqrt{4 + 4\cos u + \cos^2 u} \\ &= \sqrt{(2+\cos u)^2} \\ &= 2 + \cos u \\ &\geqslant 1\end{aligned}$$

Since each component always has magnitude at least 1, $\|X\| \neq 0$.

c) *How about the same question for S^2, the 2-dimensional sphere?*
It's darn tricky, this one is ...

6. a) *Define a cotangent vector field.*

A C^∞ cotangent vector field (or covector field) is a function σ which assigns to each point $p \in M$ a covector $\sigma_p \in T_p^*(M)$ in a smooth manner. I.e., for any coordinate neighbourhood (U, φ) with coordinate frames $\{E_1, \dots, E_n\}$, the functions $\sigma(E_i)$ are C^∞ on U.

b) *Let $F : M \to N$ be a C^∞ map between two differentiable manifolds M and N. Describe the induced map $F^* : \mathcal{T}^1(N) \to \mathcal{T}^1(M)$ where $\mathcal{T}^1(M)$ and $\mathcal{T}^1(N)$ are the collections of cofields on M and N respectively.*

F_* determines a linear map $F^* : T^*_{F(p)}(N) \to T^*_p(M)$, given by the formula

$$F^*\left(\sigma_{F(p)}\right)(X_p) = \sigma_{F(p)}\left(F_*\left(X_p\right)\right)$$

The pullback map F_* is of special significance, because it is always a well-defined homomorphism of algebras.

7. *Show that the restriction of $\sigma = x^1dx^2 = x^2dx^1 + x^3dx^4 - x^4dx^3$ of $\mathbb{R}^4$ to the sphere S^3 is never zero on S^3.*

Parametrize S^3 by solving for x_4 as $x_4 = \sqrt{1 - x_1^2 - x_2^2 - x_3^2}$ and defining

$$(x_1, x_2, x_3) \mapsto \left(x_1, x_2, x_3, \sqrt{1 - x_1^2 - x_2^2 - x_3^2}\right) = v.$$

Now taking the partial derivatives of v, we get

$$v_1 = \left(1, 0, 0, \frac{-x_1}{\sqrt{1-x_1^2-x_2^2-x_3^2}}\right) = \frac{\partial}{\partial x_1} - \frac{-x_1}{\sqrt{1-x_1^2-x_2^2-x_3^2}}\frac{\partial}{\partial x_4}$$
$$v_2 = \left(0, 1, 0, \frac{-x_2}{\sqrt{1-x_1^2-x_2^2-x_3^2}}\right) = \frac{\partial}{\partial x_2} - \frac{-x_2}{\sqrt{1-x_1^2-x_2^2-x_3^2}}\frac{\partial}{\partial x_4}$$
$$v_3 = \left(0, 0, 1, \frac{-x_3}{\sqrt{1-x_1^2-x_2^2-x_3^2}}\right) = \frac{\partial}{\partial x_3} - \frac{-x_3}{\sqrt{1-x_1^2-x_2^2-x_3^2}}\frac{\partial}{\partial x_4}$$

We evaluate σ on these vectors, using the rule $dx_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}$ to get

$$\sigma(v_1) = -x_2 - \frac{x_1x_3}{\sqrt{1-x_1^2-x_2^2-x_3^2}}$$
$$\sigma(v_2) = \quad x_1 - \frac{x_2x_3}{\sqrt{1-x_1^2-x_2^2-x_3^2}}$$
$$\sigma(v_3) = -x_4 - \frac{x_3^2}{\sqrt{1-x_1^2-x_2^2-x_3^2}}$$

Now to show that these are not all zero at the same time, we set them all equal to zero and derive a contradiction.

$$x_2 = -\frac{x_1x_3}{\sqrt{1-x_1^2-x_2^2-x_3^2}} \qquad \Longrightarrow \qquad x_2^2 = \frac{x_1^2x_3^2}{1-x_1^2-x_2^2-x_3^2}$$
$$x_1 = \quad \frac{x_2x_3}{\sqrt{1-x_1^2-x_2^2-x_3^2}} \qquad \Longrightarrow \qquad x_1^2 = \frac{x_2^2x_3^2}{1-x_1^2-x_2^2-x_3^2}$$
$$x_4 = -\frac{x_3^2}{\sqrt{1-x_1^2-x_2^2-x_3^2}} \qquad \Longrightarrow \qquad x_4^2 = \frac{x_3^4}{1-x_1^2-x_2^2-x_3^2}$$

$$\Longrightarrow 1 - x_1^2 - x_2^2 - x_3^2 = \frac{x_3^4}{1-x_1^2-x_2^2-x_3^2}$$
$$\Longrightarrow 1 - x_1^2 - x_2^2 - x_3^2 - x_2^2x_3^2 - x_1^2x_3^2 = x_3^4 + x_3^2 - x_3^2x_1^2 - x_2^2x_3^2 - x_3^4$$
$$\Longrightarrow 1 - x_1^2 - x_2^2 - x_3^2 = x_3^2$$
$$\Longrightarrow x_4^2 = x_3^2$$

■

8. *Let $F : M \to N$ be a C^∞ map between two differentiable manifolds.*

a) *Explain why $F_* : \mathfrak{X}(M) \to \mathfrak{X}(N)$ is in general not well-defined.*

case i) F may not be surjective.
In this case, $\exists q \in N$ such that $\nexists p \in M$ for which $F(p) = q$. Then if X is a vector field on M, $F_*(X)$ is not defined at $q \in N$.

case ii) F may not be injective.
In this case, suppose $p_1 \neq p_2$ are such that $F(p_1) = F(p_2) = q$. It could easily be the case that for some vector field X, $F_*(X_{p_1}) \neq F_*(X_{p_2})$. Since $F_*(X)$ cannot assign both X_{p_1} and X_{p_2} to q, $F_* : \mathfrak{X}(M) \to \mathfrak{X}(N)$ is not well-defined. □

b) *What about when F_* is a diffeomorphism?*
When F_* is a diffeomorphism, we know that F_* is both injective and surjective, and neither of the above cases can occur. F will have a well-defined inverse $G : N \to M$, and at each point p, we also have the isomorphism $F_* : T_p(M) \to T_{F(p)}(N)$ and its well-defined inverse G_*.
Now given a C^∞ vector field X on M, the vector $Y_q = F_*(X_{G(q)})$ is uniquely determined, for each $q \in N$. Finally, Y is of class C^∞ by direct application of [Boothby IV.1.5, p.110]. □

9. *Determine a subset of $\mathbb{R}^2$ on which $\sigma_1 = x_1 dx_1 + x_2 dx_2$ and $\sigma_2 = x_2 dx_1 + x_1 dx_2$ are linearly independent and find a frame field dual to σ_1, σ_2 over this set.*

   Suppose we have a linear combination $\alpha\sigma_1 + \beta\sigma_2 = 0$, so that

$$\begin{aligned} 0 &= \alpha\sigma_1 + \beta\sigma_2 \\ &= \alpha\left(x_1 dx_1 + x_2 dx_2\right) + \beta\left(x_2 dx_1 + x_1 dx_2\right) \\ &= \left(\alpha x_1 + \beta x_2\right) dx_1 + \left(\alpha x_2 + \beta x_1\right) dx_2. \end{aligned}$$

Since dx_1, dx_2 are linearly independent, we have a system of equations

$$\begin{aligned} \alpha x_1 + \beta x_2 &= 0 \\ \alpha x_2 + \beta x_1 &= 0 \end{aligned}$$

where we consider x_1, x_2 as coefficients, because we are solving for α, β. Consider that

$$\det \begin{vmatrix} x_1 & x_2 \\ x_2 & x_1 \end{vmatrix} = x_1^2 - x_2^2$$

So if $x_1^2 - x_2^2 \neq 0$, then $\alpha = \beta = 0$, so σ_1, σ_2 are linearly independent. Hence, σ_1, σ_2 are linearly independent on

$$U = \{(x_1, x_2) \in \mathbb{R}^2 \vdots\ |x_1| \neq |x_2|\}.$$

Now, to find a frame field dual to σ_1, σ_2 over U, we need to locate

$$\begin{aligned} \tau_1 &= a\tfrac{\partial}{\partial x_1} + b\tfrac{\partial}{\partial x_2} \\ \tau_2 &= c\tfrac{\partial}{\partial x_1} + d\tfrac{\partial}{\partial x_2} \end{aligned} \qquad \text{which satisfy} \qquad \begin{aligned} \sigma_1(\tau_1) &= 1 \qquad & \sigma_2(\tau_1) &= 0 \\ \sigma_1(\tau_2) &= 0 \qquad & \sigma_2(\tau_2) &= 1 \end{aligned}$$

This gives us the systems of equations

$$\begin{aligned} \sigma_1\left(\tau_1\right) &= x_1 a + x_2 b = 1 \\ \sigma_1\left(\tau_2\right) &= x_1 c + x_2 d = 0 \\ \sigma_2\left(\tau_1\right) &= x_2 a + x_1 b = 0 \\ \sigma_2\left(\tau_2\right) &= x_2 c + x_1 d = 1 \end{aligned}$$

from which we get

$$a = \frac{-x_1}{x_2^2 - x_1^2}, \quad b = \frac{x_2}{x_2^2 - x_1^2}, \quad c = \frac{-x_2}{x_1^2 - x_2^2}, \quad d = \frac{x_1}{x_1^2 - x_2^2}$$

by elementary algebra. Thus, our frame field is

$$\tau_1 = \frac{-x_1}{x_2^2 - x_1^2}\tfrac{\partial}{\partial x_1} + \frac{x_2}{x_2^2 - x_1^2}\tfrac{\partial}{\partial x_2}, \qquad \tau_2 = \frac{-x_2}{x_1^2 - x_2^2}\tfrac{\partial}{\partial x_1} + \frac{x_1}{x_1^2 - x_2^2}\tfrac{\partial}{\partial x_1}.$$

■