Math 205C - Topology Homework 3 Erin Pearse

1. V is an n-dimensional vector space over $\mathbb{R}$.

   a) Define a tensor of type (r,s).

   First, we define a multilinear map to be a function $\sigma : V_1 \times \ldots \times V_r \to \mathbb{R}$ which is linear in each V_i:

   $$\sigma(v_1, \ldots, av_i + bu_i, \ldots, v_r) = a\sigma(v_1, \ldots, v_i, \ldots, v_r) + b\sigma(v_1, \ldots, u_i, \ldots, v_r)$$

   Then for $r, s \in \mathbb{N}$, σ is a tensor of type (r,s) iff it is a multilinear map

   $$\sigma : \underbrace{V \times \ldots \times V}_{r \text{ times}} \times \underbrace{V^* \times \ldots \times V^*}_{s \text{ times}} \to \mathbb{R}$$

   r is called the *covariant order*, and s the *contravariant order*, of σ.

   b) Explain how $\mathcal{T}^r_s(V)$ forms a vector space of dimension n^{r+s}.

   c) i) For $\mathcal{T}^\star(V) = \bigoplus_{k=0}^{\infty} \mathcal{T}^k(V)$, explain how $(\mathcal{T}^\star(V), +, \otimes)$ forms an associative algebra.

   See [Booth] p.207, Thm 6.2

   ii) Is the tensor product $\otimes$ commutative?

   No, it is not commutative. Let $V = \mathbb{R}^2$, and let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be vectors in V. Define $\varphi : V \times V \to \mathbb{R}$ and $\psi : V \times V \to \mathbb{R}$ by

   $$\begin{aligned} \varphi(u,v) &= \langle u, v \rangle = u_1 v_1 + u_2 v_2 \\ \psi(u,v) &= \det(u,v) = u_1 v_2 - u_2 v_1 \end{aligned}$$

   For an example, we use four randomly selected vectors of $\mathbb{R}^2$:

   $$s = (0,1), t = (1,1), u = (1,3), v = (1,0)$$

   and compute

   $$\begin{aligned} \varphi \otimes \psi(s,t,u,v) &= \varphi \otimes \psi((0,1),(1,1),(1,3),(1,0)) \\ &= \varphi((0,1),(1,1))\,\psi((1,3),(1,0)) \\ &= (0+1)(0-3) \\ &= -3 \end{aligned}$$

   and

   $$\begin{aligned} \psi \otimes \varphi(s,t,u,v) &= \psi \otimes \varphi((0,1),(1,1),(1,3),(1,0)) \\ &= \psi((0,1),(1,1))\,\varphi((1,3),(1,0)) \\ &= (0-1)(1+0) \\ &= -1 \end{aligned}$$

2. Explain the tensor fields $\mathcal{T}^r_s(M), \mathcal{T}^r(M), \mathcal{T}_s(M)$ over a manifold M. How is a tensor field written in local coordinates?

   See [Booth] p.209

3. a) Define the alternating and symmetric tensors.

   Let V be a vector space, and let $\varphi \in \mathcal{T}^r(V)$ be some tensor. We say φ is *symmetric* iff $\forall i, j = 1, 2, \ldots, r$, we have

   $$\varphi(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_r) = \varphi(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_r)$$

   We say φ is *alternating* iff $\forall i, j = 1, 2, \ldots, r$, we have

   $$\varphi(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_r) = -\varphi(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_r)$$

   b) Define the alternating and symmetrizing operators.

   The action of the alternating operator $\mathcal{A} : T^r(V) \to T^r(V)$ on a tensor φ is defined pointwise as

   $$(\mathcal{A}\varphi)(v_1, \ldots, v_r) = \tfrac{1}{r!} \sum\nolimits_{\sigma \in \mathfrak{S}_r} (\operatorname{sgn} \sigma) \varphi\left(v_{\sigma(1)}, \ldots, v_{\sigma(r)}\right)$$

   The action of the symmetrizing operator $\mathcal{S} : T^r(V) \to T^r(V)$ on a tensor φ is defined pointwise as

   $$(\mathcal{S}\varphi)(v_1, \ldots, v_r) = \tfrac{1}{r!} \sum\nolimits_{\sigma \in \mathfrak{S}_r} \varphi\left(v_{\sigma(1)}, \ldots, v_{\sigma(r)}\right)$$

   $\mathfrak{S}_r$ is the symmetric group of all permutations on r letters.

   c) Define the wedge product: $\bigwedge : \bigwedge^k(V) \times \bigwedge^l(V) \to \bigwedge^{k+l}(V)$.

   $$(\varphi, \psi) \xrightarrow{\wedge} (\varphi \wedge \psi) \overset{def}{=} \tfrac{(r+s)!}{r!s!} \mathcal{A}(\varphi \otimes \psi)$$

   d) Give an example $\omega \in \bigwedge^k, \eta \in \bigwedge^l$ such that $\omega \otimes \eta \notin \bigwedge^{k+l}$.

   Consider the tensors $dx, dy \in \bigwedge^1$, considered as alternating tensors on V, and observe how they act on the vectors $u = (u_1, u_2)$ and $v = (v_1, v_2)$:

   $$dx \otimes dy\left((u_1, u_2), (v_1, v_2)\right) = dx(u_1, u_2) \cdot dy(v_1, v_2) = u_1 v_2$$

   but

   $$dx \otimes dy\left((v_1, v_2), (u_1, u_2)\right) = dx(v_1, v_2) \cdot dy(u_1, u_2) = v_1 u_2.$$

   Since we don't necessarily have that $u_1 v_2 = v_1 u_2$, $dx \otimes dy \notin \bigwedge^2$.

4. a) If $\dim_{\mathbb{R}} = n$, then what is the dimension of $\bigwedge^k(V)$? What is its basis?

If the vector space V has basis $\{v_1, \ldots, v_n\}$, then the basis of $\bigwedge^k(V)$ is

$$\mathcal{B} = \{v_{i_1} \wedge \ldots \wedge v_{i_k} \vdots\, i_1 < \ldots < i_k\}.$$

Every element of $\bigwedge^k(V)$ can be written in the form $\sum_{i=1}^n a_i v_i$, so using the rule $x \wedge y = -y \wedge x$, it is clear that $v_1 \wedge \ldots \wedge v_n$ generates $\bigwedge^k(V)$. Since the v_i are linearly independent, $\mathcal{B}$ forms a basis. The final thing to note is that the rule $x \wedge y = -y \wedge x$ forces any basis element to have distinct factors, and no more than k. This is because if $v_i = v_j$, then $v_1 \wedge \ldots \wedge v_i \wedge v_j \wedge v_k = 0$. See the next problem for more details.

Based on this, the cardinality of the basis can be calculated as simple counting argument. Since any basis element looks like $v_{i_1} \wedge \ldots \wedge v_{i_k}$, there are k components to choose and n possibilities to choose from. Order does not matter, because $x \wedge y = -y \wedge x$, so as generators of a vector space, $x \wedge y$ and $y \wedge x$ are essentially equivalent. Hence, the number of elements in any basis, and thus the dimension of $\bigwedge^k(V)$ is $\binom{n}{k}$.

b) Prove that $\bigwedge^k(V) = \{0\}$ if $k > n$.

Since the basis of $\bigwedge^k(V)$ is $\mathcal{B} = \{v_{i_1} \wedge \ldots \wedge v_{i_k} \vdots\, i_1 < \ldots < i_k\}$ and $k > n$, every basis element looks like:

$$v_{i_1} \wedge \ldots \wedge v_{i_n} \wedge v_{i_{n+1}} \ldots \wedge v_{i_k}$$

Consider the possibilities for $v_{i_{n+1}}$. $v_{i_{n+1}}$ must be one of the $\{v_i\}_{i=1}^n$ that form a basis of V. Suppose $v_{i_{n+1}} = v_j$. But v_j is already one of the initial $v_{i_1} \wedge \ldots \wedge v_{i_n}$, so we are in the case where v_j occurs twice, which implies immediately that the entire basis element $v_{i_1} \wedge \ldots \wedge v_{i_n} \wedge v_{i_{n+1}} \ldots \wedge v_{i_k}$ is 0. To see this, first note that we can reorder the basis element so that the two v_js are adjacent, without changing the value of the element. Then by swapping positions of the two v_j, we get

$$v_{i_1} \wedge \ldots \wedge v_j \wedge v_j \ldots \wedge v_{i_k} = -v_{i_1} \wedge \ldots \wedge v_j \wedge v_j \ldots \wedge v_{i_k}$$

which shows that both sides are 0. Since every basis element is 0, the entire basis is 0, and thus $\bigwedge^k(V) = \{0\}$.

c) For $\bigwedge^\star(V) = \bigotimes_{k=0}^\infty \bigwedge^k(V)$, what is $\dim \bigwedge^\star(V)$? Explain how $(\bigwedge(V), +, \wedge)$ forms an associative algebra. Is it commutative?

We compute the dimension of using a technique from algebra:

$$[\textstyle\bigwedge^\star(V) : \mathbb{R}] = [\bigwedge^\star(V) : C^\infty(V)] \cdot [C^\infty(V) : \mathbb{R}] = n \cdot \infty = \infty$$

d) Is $(\bigwedge(V), +, \wedge)$ a subalgebra of $(\mathcal{T}^\star(V), +, \otimes)$?

Yes. Clearly $(\bigwedge(V), +, \wedge) \subset (\mathcal{T}^\star(V), +, \otimes)$, and we have shown above that $\bigwedge(V)$ actually does form an algebra under these operations. The key fact that distinguishes this case from 3(d) is that for tensors φ and ψ, $\varphi \otimes \psi$ may not be alternating, but the wedge product is defined in terms of the alternating operator, so that $\varphi \wedge \psi$ is alternating by construction. This is why $\bigwedge(V)$ is closed under $\wedge$.

5. a) Explain a k-form $\omega \in \bigwedge^k(M)$ on a manifold.

b) Explain the exterior/Grassman algebra on a manifold.

c) Let $F : M \to N$ be a C^∞ map between two manifolds M, N. Describe the pullback map $F^* : \bigwedge(N) \to \bigwedge(M)$.

d) Prove that F^* is an algebra homomorphism.

$$F^*(\omega + \eta) = (\omega + \eta) \circ F = \omega \circ F + \eta \circ F = F^*\omega + F^*\eta$$

Now we need to show $F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$, so we begin by showing $F^*(\omega \otimes \eta) = F^*\omega \otimes F^*\eta$ as follows:

$$\begin{aligned}
F^*(\omega \otimes \eta)(u,v) &= ((\omega \otimes \eta) \circ F)(u,v) && \text{def of } F^* \\
&= (\omega \otimes \eta)(F(u), F(v)) \\
&= \omega \circ F(u) \cdot \eta \circ F(v) && \text{def of } \omega \otimes \eta \\
&= F^*\omega(u) \cdot F^*\eta(v) && \text{def of } F^* \text{ again} \\
&= (F^*\omega \otimes F^*\eta)(u,v) && \text{def of } \omega \otimes \eta \text{ again}
\end{aligned}$$

Now we proceed with $\wedge$:

$$\begin{aligned}
&F^*(\omega \wedge \eta)(u,v) \\
&\quad = (\omega \wedge \eta)(F(u), F(v)) && \text{def of } F^* \\
&\quad = \tfrac{(r+s)!}{r!s!}\mathcal{A}(\omega \otimes \eta)(F(u), F(v)) && \text{def of } \wedge \\
&\quad = \tfrac{(r+s)!}{r!s!r!}\sum_{\sigma \in \mathfrak{S}_{k+j}} (\operatorname{sgn}\sigma)(\omega \otimes \eta)(F(u), F(v)) && \text{def of } \mathcal{A} \\
&\quad = \tfrac{(r+s)!}{r!s!r!}\sum_{\sigma \in \mathfrak{S}_{k+j}} (\operatorname{sgn}\sigma)(F^*\omega \otimes F^*\eta)(u,v) && \text{by above} \\
&\quad = \tfrac{(r+s)!}{r!s!}\mathcal{A}(F^*\omega \otimes F^*\eta)(u,v) && \text{def of } \mathcal{A} \text{ again} \\
&\quad = (F^*\omega \wedge F^*\eta)(u,v) && \text{def of } \wedge \text{ again}
\end{aligned}$$

Thus, $F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$. $\square$

6. a) Define the exterior differentiation operator $d : \bigwedge^k(M) \to \bigwedge^{k+1}(M)$.

d is the unique $\mathbb{R}$-linear map $d : \bigwedge^\star(M) \to \bigwedge^\star(M)$ satisfying

i) for $f \in C^\infty(M) = \bigwedge^0(M)$, we have $d(f) = df$, the differential of f.

ii) $d(\theta \wedge \sigma) = d\theta \wedge \sigma + (-1)^r \theta \wedge d\sigma$, $\forall \theta \in \bigwedge^r, \forall \sigma \in \bigwedge^k$

iii) $d^2 = 0$

b) Prove that $d^2 = 0$.

For $\omega \in \bigwedge^p(M)$, we can write $\omega = \sum \omega_{i_1 \cdots i_p} dx_{i_1} \wedge \ldots \wedge dx_{i_p}$ in local coordinates. Since d is $\mathbb{R}$-linear, it will suffice to consider the case $d\left(d\left(\omega_{i_1 \cdots i_p} dx_{i_1} \wedge \ldots \wedge dx_{i_p}\right)\right)$. Then

$$\begin{aligned}
d^2\omega_i &= d\left(d\left(\omega_{i_1 \cdots i_p} dx_{i_1} \wedge \ldots \wedge dx_{i_p}\right)\right) \\
&= d\left[\sum_{k=1}^n \left(\tfrac{\partial \omega_{i_1 \cdots i_p}}{\partial x_k} dx_k\right) \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_p}\right] \\
&= d\left[\sum_{k=1}^n \tfrac{\partial \omega_{i_1 \cdots i_p}}{\partial x_k} dx_k \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_p}\right] \\
&= \sum_{k=1}^n \sum_{j=1}^n \left[\tfrac{\partial^2 \omega_{i_1 \cdots i_p}}{\partial x_j \partial x_k} dx_j \wedge dx_k \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_p}\right] \\
&= \sum_{1 \leqslant k < j \leqslant n} \left[\tfrac{\partial^2 \omega_{i_1 \cdots i_p}}{\partial x_j \partial x_k} - \tfrac{\partial^2 \omega_{i_1 \cdots i_p}}{\partial x_k \partial x_j} dx_j \wedge dx_k \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_p}\right].
\end{aligned}$$

but then $\frac{\partial^2 \omega_{i_1 \cdots i_p}}{\partial x_j \partial x_k} = \frac{\partial^2 \omega_{i_1 \cdots i_p}}{\partial x_k \partial x_j}$, by the equality of mixed partial derivatives. So each term of the sum, and hence the entire sum, is 0.

c) Define the closed and exact differential forms.

$\omega \in \bigwedge^p(M)$ is a closed form iff $d\omega = 0$, i.e., $\omega \in \operatorname{Ker} d$.
$\omega \in \bigwedge^p(M)$ is an exact form iff $\exists \eta \in \bigwedge^{p-1}(M)$ such that $d\eta = \omega$, i.e., $\omega \in \operatorname{Im} d$.

$$\textstyle\bigwedge^{p-1}(M) \xrightarrow{\ d\ } \bigwedge^p(M) \xrightarrow{\ d\ } \bigwedge^{p+1}(M)$$

7. a) Let $\omega = \sum_{i=1}^n \omega_i dx_i$ be a 1-form in $\mathbb{R}^n$. Show ω is closed $\iff$ $\frac{\partial \omega_i}{\partial x_j} = \frac{\partial \omega_j}{\partial x_i}$, $\forall i, j$.

$\boxed{\Rightarrow}$ Assume ω is closed so that $d\omega = 0$. Then

$$\begin{aligned}
0 &= d\omega \\
&= \sum_{k=1}^n (d\omega_k) \wedge dx_i \\
&= \sum_{k=1}^n \left(\sum_{i=1}^n \tfrac{\partial \omega_i}{\partial x_k} dx_k\right) \wedge dx_i \\
&= \sum_{1 \leqslant k < i \leqslant n} \left[\left(\tfrac{\partial \omega_i}{\partial x_k} - \tfrac{\partial \omega_k}{\partial x_i}\right) dx_k \wedge dx_i\right]
\end{aligned}$$

But the basis elements $\{dx_k \wedge dx_i\}$ are linearly independent, so this implies

$$\tfrac{\partial \omega_i}{\partial x_k} - \tfrac{\partial \omega_k}{\partial x_i} = 0 \ \ \forall i,k \quad \Longrightarrow \quad \tfrac{\partial \omega_i}{\partial x_k} = \tfrac{\partial \omega_k}{\partial x_i} \ \ \forall i,k.$$

$\boxed{\Leftarrow}$ $\frac{\partial \omega_i}{\partial x_k} = \frac{\partial \omega_k}{\partial x_i} \quad \Rightarrow \quad \frac{\partial \omega_i}{\partial x_k} - \frac{\partial \omega_k}{\partial x_i} = 0$, which immediately gives

$$d\omega = \sum_{1 \leqslant k < i \leqslant n} \left(\frac{\partial \omega_i}{\partial x_k} - \frac{\partial \omega_k}{\partial x_i} \right) dx_k \wedge dx_i = 0.$$

$\square$

b) Find necessary and sufficient conditions for a 2-form $\omega = Cdx \wedge dy + Ady \wedge dz + Bdz \wedge dx$ to be closed in $\mathbb{R}^3$, where A, B, C are functions in $\mathbb{R}^3$.

We put $\omega_{12} = C, \omega_{23} = A, \omega_{13} = -B$, and consider $\omega = \sum_{1 \leqslant i < j \leqslant 3} \omega_{ij} dx_i \wedge dx_j$. Now

$$\begin{aligned} d\omega = &(\tfrac{\partial C}{\partial x} dx) \wedge dx \wedge dy + (\tfrac{\partial C}{\partial y} dy) \wedge dx \wedge dy + (\tfrac{\partial C}{\partial z} dz) \wedge dx \wedge dy + d(dx \wedge dy) \\ &+ (\tfrac{\partial A}{\partial x} dx) \wedge dy \wedge dz + (\tfrac{\partial A}{\partial y} dy) \wedge dy \wedge dz + (\tfrac{\partial A}{\partial z} dz) \wedge dy \wedge dz + d(dy \wedge dz) \\ &+ (\tfrac{\partial B}{\partial x} dx) \wedge dz \wedge dx + (\tfrac{\partial B}{\partial y} dy) \wedge dz \wedge dx + (\tfrac{\partial B}{\partial z} dz) \wedge dz \wedge dx + d(dz \wedge dx). \end{aligned}$$

Now by the basic properties of the alternating product,

$$dx \wedge dy = -dy \wedge dx \quad \Longrightarrow \quad dx \wedge dx = -dx \wedge dx \quad \Longrightarrow \quad dx \wedge dx = 0$$

Using the rule that $dx \wedge dx = 0$, we see that only the third, fourth, and eighth terms of the above sum are nonzero, i.e.,

$$\begin{aligned} d\omega &= \left(\tfrac{\partial C}{\partial z} dz\right) \wedge dx \wedge dy + \left(\tfrac{\partial A}{\partial x} dx\right) \wedge dy \wedge dz + \left(\tfrac{\partial B}{\partial y} dy\right) \wedge dz \wedge dx \\ &= \tfrac{\partial A}{\partial x} dx \wedge dy \wedge dz + \tfrac{\partial B}{\partial y} dx \wedge dy \wedge dz + \tfrac{\partial C}{\partial z} dx \wedge dy \wedge dz \\ &= \left(\tfrac{\partial A}{\partial x} + \tfrac{\partial B}{\partial y} + \tfrac{\partial C}{\partial z} \right) dx \wedge dy \wedge dz \end{aligned}$$

Thus we obtain $d\omega = 0 \iff \frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} = 0.$ $\square$

c) Show that a 2-form $\omega = \sum_{i<j} a_{ij} dx_i \wedge dx_j$ in $\mathbb{R}^n$ is closed $\iff \frac{\partial a_{ij}}{\partial x_k} - \frac{\partial a_{jk}}{\partial x_j} + \frac{\partial a_{jk}}{\partial x_i} = 0 \ \forall i,j,k$

$$d\omega = \sum\nolimits_{1 \leqslant i < j \leqslant n} \sum\nolimits_{k=1}^{n} \tfrac{\partial \omega_{ij}}{\partial x_k} dx_k \wedge dx_i \wedge dx_j$$

8. Consider the 1-form $\omega = \frac{xdy - ydx}{x^2+y^2}$ in $\mathbb{R}^2 \backslash \{(0,0)\}$. Is ω closed? Is ω exact?

$$\omega = \sum\nolimits_{i=1}^{2} \omega_i dx_i = \tfrac{-y}{x^2+y^2} dx + \tfrac{x}{x^2+y^2} dy$$

So applying d gives

$$\begin{aligned} d\omega &= \sum\nolimits_{i=1}^{n} \sum\nolimits_{k=1}^{n} \tfrac{\partial \omega_i}{\partial x_k} dx_i \\ &= \sum\nolimits_{1 \leqslant k < i \leqslant n} \left(\tfrac{\partial \omega_i}{\partial x_k} - \tfrac{\partial \omega_k}{\partial x_i} \right) dx_k \wedge dx_i \end{aligned}$$

Which shows that ω will be closed precisely when

$$\tfrac{\partial}{\partial y} \left(\tfrac{-y}{x^2+y^2} \right) = \tfrac{\partial}{\partial x} \left(\tfrac{x}{x^2+y^2} \right).$$

Now

$$\tfrac{\partial}{\partial y} \left(\tfrac{-y}{x^2+y^2} \right) = \frac{-\left(x^2+y^2\right) + y\,(2y)}{\left(x^2+y^2\right)^2} = \frac{y^2 - x^2}{\left(x^2+y^2\right)^2}$$

and

$$\tfrac{\partial}{\partial x} \left(\tfrac{x}{x^2+y^2} \right) = \frac{\left(x^2+y^2\right) - x\,(2x)}{\left(x^2+y^2\right)^2} = \frac{y^2 - x^2}{\left(x^2+y^2\right)^2}$$

shows that ω is closed.

9. a) In $\mathbb{R}^3$, determine which of the following forms are closed and which are exact[1]:

i) $\varphi = yzdx + xzdy + xydz$

Consider $\varphi = \omega_1 dx_1 + \omega_2 dx_2 + \omega_3 dx_3$. Then φ is closed because

$$\begin{aligned} d\varphi &= \sum\nolimits_{1 \leqslant k < i \leqslant 3} \left[\tfrac{\partial \omega_i}{\partial x_k} - \tfrac{\partial \omega_k}{\partial x_i} \right] dx_k \wedge dx_i \\ &= \left(\tfrac{\partial \omega_2}{\partial x_1} - \tfrac{\partial \omega_1}{\partial x_2} \right) dx_1 \wedge dx_2 \\ &\qquad + \left(\tfrac{\partial \omega_3}{\partial x_1} - \tfrac{\partial \omega_1}{\partial x_3} \right) dx_1 \wedge dx_3 + \left(\tfrac{\partial \omega_3}{\partial x_2} - \tfrac{\partial \omega_2}{\partial x_3} \right) dx_2 \wedge dx_3 \\ &= \left(\tfrac{\partial (xz)}{\partial x} - \tfrac{\partial (yz)}{\partial y} \right) dx \wedge dy \\ &\qquad + \left(\tfrac{\partial (xy)}{\partial x} - \tfrac{\partial (yz)}{\partial z} \right) dx \wedge dz + \left(\tfrac{\partial (xy)}{\partial y} - \tfrac{\partial (xz)}{\partial z} \right) dy \wedge dz \\ &= (z - z)\, dx \wedge dy + (y - y)\, dx \wedge dz + (x - x)\, dy \wedge dz \\ &= 0 \end{aligned}$$

If we define $\psi \in \bigwedge^0(\mathbb{R}^3)$ by $\psi = xyz$, then φ is exact because

$$\begin{aligned} d\psi &= \tfrac{\partial}{\partial x}(xyz)\, dx + \tfrac{\partial}{\partial y}(xyz)\, dy + \tfrac{\partial}{\partial z}(xyz)\, dz \\ &= yzdx + xzdy + xydz \\ &= \varphi \end{aligned}$$

■

[1] Note that φ exact $\implies \varphi$ closed, so some of these calculations are unnecessary. ☺

ii) $\varphi = xdx + x^2y^2dy + xzdz$

φ is not closed because

$$\begin{aligned} d\varphi &= \left(\frac{\partial\left(x^2y^2\right)}{\partial x} - \frac{\partial\left(x^2y^2\right)}{\partial y}\right) dx \wedge dy \\ &\qquad + \left(\frac{\partial(xz)}{\partial x} - \frac{\partial(x)}{\partial z}\right) dx \wedge dz + \left(\frac{\partial(xz)}{\partial y} - \frac{\partial\left(x^2y^2\right)}{\partial z}\right) dy \wedge dz \\ &= \left(2xy^2 - 0\right) dx \wedge dy + (z-0)\, dx \wedge dz + (0-0)\, dy \wedge dz \\ &= 2xy^2 dx \wedge dy + z dx \wedge dz \\ &\neq 0 \end{aligned}$$

Since not closed $\implies$ not exact, we know that φ cannot be exact. ■

iii) $\varphi = 2xy^2 dx \wedge dy + z dy \wedge dz$

φ is closed because

$$\begin{aligned} d\varphi &= \sum_{1 \leqslant i_2 < i_2 \leqslant 3} \left[\left(\sum_{k=1}^{3} \frac{\partial \omega_{i_1 i_2}}{\partial x_k} dx_k\right) \wedge dx_{i_1} \wedge dx_{i_2}\right] \\ &= \left(\frac{\partial\left(2xy^2\right)}{\partial x} dx\right) \wedge dx \wedge dy \\ &\qquad + \left(\frac{\partial\left(2xy^2\right)}{\partial y} dy\right) \wedge dx \wedge dy + \left(\frac{\partial\left(2xy^2\right)}{\partial z} dz\right) \wedge dx \wedge dy \\ &\qquad + \left(\frac{\partial z}{\partial x} dx\right) \wedge dy \wedge dz + \left(\frac{\partial z}{\partial y} dy\right) \wedge dy \wedge dz + \left(\frac{\partial z}{\partial z} dz\right) \wedge dy \wedge dz \\ &= 2y^2 dx \wedge dx \wedge dy + 4xy dy \wedge dx \wedge dy + 0 dz \wedge dx \wedge dy \\ &\qquad + 0 dx \wedge dy \wedge dz + 0 dy \wedge dy \wedge dz + 1 dz \wedge dy \wedge dz \\ &= 2y^2 dx \wedge dx \wedge dy + 4xy dy \wedge dx \wedge dy + 1 dz \wedge dy \wedge dz \\ &= 0 \end{aligned}$$

If we define $\psi \in \bigwedge^1(\mathbb{R}^3)$ by $\psi = cdx + x^2y^2dy + zydz$, where $c \in \mathbb{R}$, then φ is exact because

$$\begin{aligned} d\psi &= \left(\frac{\partial c}{\partial x} dx\right) \wedge dx + \left(\frac{\partial c}{\partial y} dy\right) \wedge dx + \left(\frac{\partial c}{\partial z} dz\right) \wedge dx \\ &\qquad + \left(\frac{\partial\left(x^2y^2\right)}{\partial x} dx\right) \wedge dy + \left(\frac{\partial\left(x^2y^2\right)}{\partial y} dy\right) \wedge dy + \left(\frac{\partial\left(x^2y^2\right)}{\partial z} dz\right) \wedge dy \\ &\qquad + \left(\frac{\partial(zy)}{\partial x} dx\right) \wedge dz + \left(\frac{\partial(zy)}{\partial y} dy\right) \wedge dz + \left(\frac{\partial(zy)}{\partial z} dz\right) \wedge dz \\ &= 0 dx \wedge dx + 0 dy \wedge dx + 0 dz \wedge dx \\ &\qquad + 2xy^2 dx \wedge dy + 2x^2 y dy \wedge dy + 0 dz \wedge dy \\ &\qquad + 0 dx \wedge dz + z dy \wedge dz + y dz \wedge dz \\ &= 2xy^2 dx \wedge dy + z dy \wedge dz \\ &= \varphi \end{aligned}$$

■

10. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ taking $(x,y,z) \xrightarrow{f} (s,t)$ be defined by
$$f(x,y,z) = (xy, yz+1).$$

a) Let $\varphi = stds + dt$ be a 1-form in $\mathbb{R}^2$. Compute $f^*(\varphi)$.

$f^*(\varphi) = \varphi \circ f$, so $s = xy$ and $t = yz+1$, so we find the other components of φ in terms of x, y, z as

$$\begin{aligned} ds &= \tfrac{\partial(xy)}{\partial x}dx + \tfrac{\partial(xy)}{\partial y}dy + \tfrac{\partial(xy)}{\partial z}dz = ydx + xdy \\ dt &= \tfrac{\partial(yz+1)}{\partial x}dx + \tfrac{\partial(yz+1)}{\partial y}dy + \tfrac{\partial(yz+1)}{\partial z}dz = zdy + ydz \end{aligned}$$

Then

$$\begin{aligned} f^*(\varphi) &= \overbrace{(xy)}^{s}\overbrace{(yz+1)}^{t}\overbrace{(ydx+xdy)}^{ds} + \overbrace{(zdy+ydz)}^{dt} \\ &= x^2y^2zdy + xy^3zdx + x^2ydy + xy^2dx + ydz + zdy \\ &= \left(xy^3z + xy^2\right)dx + \left(x^2y^2z + z + x^2y\right)dy + ydz \end{aligned}$$

□

b) Let $\varphi = st(ds \wedge dt)$ be a 1-form in $\mathbb{R}^2$. Compute $f^*(\varphi)$.

Using ds, dt as calculated above, we obtain

$$\begin{aligned} f^*(\varphi) &= (xy)(yz+1)((ydx+xdy)\wedge(zdy+ydz)) \\ &= \left(xy^2z + xy\right)\left(yzdx\wedge dy + y^2dx\wedge dz + xzdy\wedge dy + xydy\wedge dz\right) \\ &= \left(xy^3z^2 + xy^2z\right)dx\wedge dy + \left(xy^4z + xy^3\right)dx\wedge dz + \left(x^2y^3z + x^2y^2\right) \end{aligned}$$

□

11. Let ω be a 1-form. For vector fields X, Y, prove the formula
$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]).$$

$X, Y \in \mathfrak{X}(M)$ and $\omega \in \bigwedge^1(M)$, so let $\omega = fdg$ where $f, g \in C^\infty$. It will suffice to prove that the formula is true locally, i.e., in a coordinate neighbourhood of each point. In each such neighbourhood, with coordinates $x_1, \ldots, x_n$, we have $\omega = \sum_{i=1}^n a_i dx_i$, by the definition of ω as a 1-form. Now the left side of the formula becomes

$$\begin{aligned} d\omega(X,Y) &= df \wedge dg(X,Y) \\ &= df(X)dg(Y) - dg(X)df(Y) \\ &= (Xf)(Yg) - (Xg)(Yf) \end{aligned}$$

and the right side of the formula becomes

$$\begin{aligned} X\omega(Y) - Y\omega(X) - \omega([X,Y]) &= X(fdg(Y)) - Y(fdg(X)) - fdg([X,Y]) \\ &= X(f(Yg)) - Y(f(Xg)) - f(XYg - YXg) \\ &= (Xf)(Yg) - (Xg)(Yf) \end{aligned}$$

■

12. a) Define a Riemannian metric.

A Riemannian metric on a differentiable manifold M is not actually a metric at all. Instead, it is the (rather misleading) name given to any positive definite covariant symmetric tensor of type (2,0). More formulaically,

$$ds^2 = \sum_{1 \leqslant i < j \leqslant n} g_{ij} dx_i \otimes dx_j, \quad \text{where } g_{ij} = g_{ji}, \forall i, j$$

b) Describe two ways to construct a Riemannian metric on a manifold M.

i) Using the Whitney Imbedding Theorem, we can imbed M into some $\mathbb{R}^n$. Let dS_E^2 denote the standard Euclidean metric of $\mathbb{R}^n$. If we restrict dS_E^2 to M, we get a Riemannian metric on M.

ii) First, we choose an open cover $\{A_\alpha\}$ of M and use Boothby V.4.1 to produce a regular covering $\{U_i, V_i, \varphi_i\}$ and Boothby V.4.4 to produce a C^∞ partition of unity $\{f_i\}$ subordinate to this cover. In a given coordinate neighbourhood U_i, we define a "local" Riemannian metric by

$$\Phi_i = \varphi_i^*(\psi),$$

where $\psi = dx_1^2 + dx_2^2 + \ldots + dx_n^2$ is the Euclidean metric. Finally, define

$$ds^2 = \Phi = \sum_i f_i \Phi_i$$

to obtain a globally-defined Riemannian metric.

c) How does one make a metric out of the Riemannian metric?

Let Φ be a a Riemannian metric defined on M. By simply denoting $\langle x, y \rangle = \Phi(x, y)$, the Riemannian metric gives an inner product to the tangent space $T_p(M), \forall p \in M$. Let $\gamma(t)$ be a C^1 curve on M, and define $p_0 = \gamma(0)$ and $p_1 = \gamma(1)$. Define the length L of this curve, from p_0 to p_1, by

$$L = \int_0^1 \left(\Phi\left(\tfrac{d\gamma}{dt}, \tfrac{d\gamma}{dt}\right)\right)^{1/2} dt = \int_0^1 \sqrt{\left\langle \tfrac{d\gamma}{dt}, \tfrac{d\gamma}{dt} \right\rangle}\, dt = \int_0^1 \|\gamma'(t)\|\, dt$$

We now obtain a metric $d(x, y)$ by defining the distance from p_0 to p_1 as

$$d(p_0, p_1) = \inf_{\gamma \in C^1} \left\{ \int_0^1 \|\gamma'(t)\|\, dt \right\}$$

13. a) Define a volume element.

b) Compute the volume (with the induces metric of $\mathbb{R}^2$) in terms of the coordinates given by

i) Stereographic projection.

ii) Spherical coordinates (with $\rho = 1$).