. January 2005

Part I.

1. (a) Let $U \subseteq K$ be contained in a Hausdorff space X, with U open and K compact, and assume that $f : X \to Y$ is a continuous map into another Hausdorff space Y. Prove that the closure of $f(U)$ is a compact subset of $f(K)$.

   Solution. Since K is compact in X, the continuity of f implies $f(K)$ will be compact in Y. Y is Hausdorff, so $f(K)$ is closed, i.e., $\overline{f(K)} = f(K)$.
   Since $U \subseteq K$ implies $f(U) \subseteq f(K)$, which implies $\overline{f(U)} \subseteq \overline{f(K)} = f(K)$, we see that $\overline{f(U)}$ is a closed subset of a compact set and is thus compact. □

   (b) Let f, X, Y be as in the preceding problem and assume in addition that f is an onto, open mapping and X is locally compact. Prove that Y is also locally compact.

   Solution. Using the 'local' definition of locally compact, we choose a neighbourhood U of y such that $y \in U \subseteq Y$ and find a neighbourhood V of y such that $\bar{V}$ is compact and $\bar{V} \subseteq U$.
   f continuous implies $f^{-1}(U)$ is a neighbourhood of $f^{-1}(y)$, so for $x \in f^{-1}(y)$ we can find a neighbourhood W of x with compact closure $\bar{W} \subseteq f^{-1}(U)$ by the local compactness of X. Since f is open, $f(W)$ will be a neighbourhood of y. And since f is continuous, $f(\bar{W})$ will be compact in $f(f^{-1}(U)) = U$. This equality follows by the surjectivity of f. Also, $W \subseteq \bar{W}$ implies that $f(W) \subseteq f(\bar{W}) \subseteq U$.
   Now $f(\bar{W})$ is a compact subspace of the Hausdorff space Y, so it is closed, i.e., $f(\bar{W}) = \overline{f\left(\bar{W}\right)}$. Then

   $$\begin{aligned} W \subseteq \bar{W} &\implies f(W) \subseteq f(\bar{W}) = \overline{f\left(\bar{W}\right)} \\ &\implies \overline{f(W)} \subseteq \overline{f\left(\bar{W}\right)} \\ &\implies \overline{f(W)} \text{ is compact.} \end{aligned}$$

   So put $V = f(W)$ and see that Y is locally compact. □

2. Let S be the set of all connected subsets of the cartesian plane. Show that the cardinality of S is strictly greater than the cardinality of the plane. (*Hint*: consider all subsets of the form
   $$(0,1)^2 \cup \{B \times \{1\}\}$$
   where B is a subset of $(0,1)$. Under what conditions is such a subset connected? Why?

   Solution. Following the hint, first note that $(0,1)^2 \cup \{B \times \{1\}\}$ is connected for any arbitrary subset $B \subseteq (0,1)$. To see this, pick $b \in B$ and observe that any open neighbourhood $O = U \times V$ ("cartesian plane" indicates the usual topology on $\mathbb{R}^2$) must intersect $(0,1)^2$, hence is in the same component as $(0,1)^2$. See Thm. 25.1.

Since there is a continuous bijection $[0,1] \to [0,1]^2$ by the Peano map, we have

$$\begin{aligned}
|S| &\geq \left|(0,1)^2 \cup \{B \times \{1\}\}\right| \\
&= |\{B \subseteq (0,1)\}| \\
&= \left|2^{(0,1)}\right| \\
&= \left|2^{[0,1]}\right| && \text{2 points don't do much} \\
&= \left|2^{[0,1]^2}\right| && \text{by the Peano map} \\
&> \left|[0,1]^2\right| && \text{by some famous theorem} \\
&= \left|\mathbb{R}^2\right|.
\end{aligned}$$

□

3. The logical implications of the statements below have the form $A \implies B \implies D$ and $A \implies C \implies D$ if they are suitably labeled as A, B, C, D. Give a labeling of the statements for which this is true.
   (1) The topological space X is compact and metrizable.
   (2) The topological space X is first countable.
   (3) The topological space X is metrizable.
   (4) The topological space X is second countable.

Solution by Chui (Zhi) Yao. We label the statements:
(A) The topological space X is compact and metrizable.
(D) The topological space X is first countable.
(B) The topological space X is metrizable.
(C) The topological space X is second countable.

It is clear that $(A) \implies (B)$. To see that $(B) \implies (D)$, fix $x \in X$ and consider the collection of metric balls $\{B_n(x)\}_{n=1}^{\infty}$, where $B_n(x) = B(x, 1/n)$.

To see $(A) \implies (C)$, let $\mathcal{A}_n$ be a finite covering of X by $1/n$-balls (any covering by $1/n$-balls has a finite subcover, by compactness). Then $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$ is a basis for X. To see that it really is a basis, use Lemma 13.2: pick an open set $U \subseteq X$. Then we can find a metric ball $B(x, \varepsilon)$ with $x \in B(x, \varepsilon) \subseteq U$, for $0 < \varepsilon < \infty$. Then choose n sufficiently large that $1/n < \varepsilon/2$ and find an element of $\mathcal{A}_n$ containing x. We have satisfied the requirements of the Lemma. It is clear that $(C) \implies (D)$; if $\mathcal{B}$ is a countable basis, then just take the subcollection consisting of sets of $\mathcal{B}$ which contain x, as a basis at x. □

Part II.

4. Let F_n be the free group of rank n. Show that for all $n \geq 2$, F_2 contains a subgroup isomorphic to F_n.

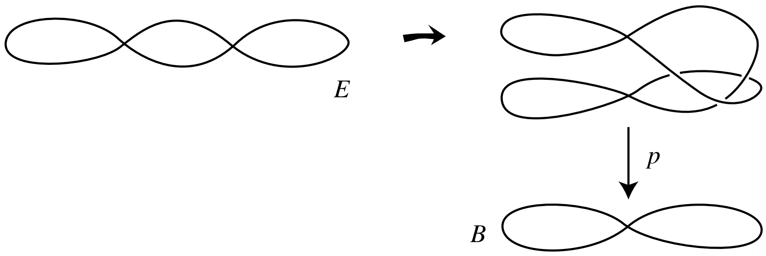

Figure 1. A covering space for a wedge of 2 circles.

Solution. The space E depicted in Figure 1 is a covering space for the wedge of two circles B. Note that E and a wedge of 3 circles are both deformation retracts of the triply punctured plane, hence they have the same fundamental groups. In a similar manner, it is possible to show that a wedge E_n of n circles is a covering space for a wedge of 2 circles, for any $n \geq 2$.

By Thm. 54.6, the covering map $p : E_n \to B$ gives a monomorphism $p_* : \pi_1(E_n, e_0) \to \pi_1(B, b_0)$. This gives an injective homomorphism

$$p_* : F_n = \pi_1(E_n, e_0) \to F_2 = \pi_1(B, b_0),$$

so $p_*(\pi_1(E_n, e_0))$ is a subgroup of $\pi_1(B, b_0) = F_2$ which is isomorphic to F_n. □

Note, this does not produce an explicit homomorphism, as is requested in the similar problem June 2004 #6, so here is an alternative approach:

Solution. Let the generators of F_n be $a_1, a_2, \ldots, a_n$, so $F_n = F(a_1, a_2, \ldots, a_n)$; and let the generators of F_2 be x, y so that $F_2 = F(x, y)$. We construct a homomorphism $\varphi : F_n \to F_2$ by defining it on the generators:

$$\begin{aligned} a_1 &\mapsto yxy^{-1} \\ a_2 &\mapsto y^2x^2y^{-2} \\ &\vdots \\ a_n &\mapsto y^nx^ny^{-n}. \end{aligned}$$

Then use $\varphi(ab) = \varphi(a)\varphi(b)$ to extend φ to all of F_n. Now $H = \mathrm{Im}(\varphi)$ is a subgroup of F_2 and $\varphi : F_n \to Im(\varphi)$ is an epimorphism. It only remains to show φ is injective.

Consider that an element in the image of φ has the form

$$\varphi(\alpha) = y^{\delta_1}x^{\varepsilon_1}y^{\delta_2}x^{\varepsilon_2}y^{\delta_3}x^{\varepsilon_3}\ldots y^{\delta_m}x^{\varepsilon_m}y^{\delta_{m+1}}.$$

We can recover α via the following decoding algorithm:

(i) $\delta_1 = 1, 2, \ldots, n$ indicates that the first letter of α is $a_1, a_2, \ldots, a_n$, respectively.

(ii) ε_1/δ_1 is the exponent of the first letter of α.

(iii) $\delta_2' = \delta_1 + \delta_2$ indicates whether the second letter of α is $a_1, a_2, \ldots, a_n$, as in (i).

(iv) ε_2/δ_2' is the exponent of the second letter of α.

$\vdots$

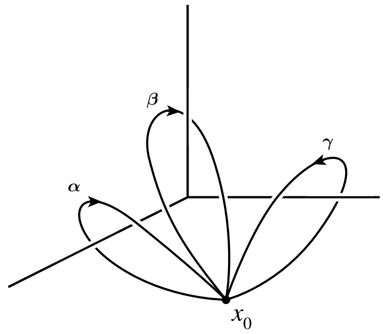

FIGURE 2. Loops around the axes in $\mathbb{R}^3$.

This algorithm clearly stops after m steps. Since this procedure for constructing the inverse exists, φ is injective. This procedure works because raising $a_1, a_2, \dots, a_n$ to a power produces a higher exponent on the x but not on the y:

$$\varphi(a_2^5) = \varphi(a_2)^5 = (y^2x^2y^{-2}) \dots (y^2x^2y^{-2}) = y^2x^{10}y^{-2}. \qquad \square$$

5. Consider the nonnegative coordinate axes of $\mathbb{R}^3$, i.e., the subset

$$Y = \{(x, 0, 0) \in \mathbb{R}^3; x \geq 0\} \cup \{(0, y, 0) \in \mathbb{R}^3; y \geq 0\} \cup \{(0, 0, z) \in \mathbb{R}^3; z \geq 0\}.$$

Calculate the fundamental group of $X = \mathbb{R}^3 \setminus Y$.

Solution. This is Munkres Ex. 56.2(e). Consider that any loop in this space is homotopic to some combination of α, β, γ (and their inverses), as depicted in Figure 2. Note that $\alpha * \beta = \gamma$, so that the fundamental group can be generated without γ. Also, note that $\beta * \alpha \simeq \overline{\gamma} \neq \gamma$, so fundamental group is not abelian.

We have a group with two generators and it is clear by inspection that there is no relation between the two generators; i.e.,

$$\pi_1(X) = F(\alpha, \beta). \qquad \square$$

6. Suppose that X has universal covering space $\tilde{X}$, and $\tilde{X}$ is compact. Show that the fundamental group of X is finite.

Solution. (See also June 2004 #7, and 1998 #4).
By Thm. 54.4, the lifting correspondence gives a surjection

$$\varphi : \pi_1(X, b) \to p^{-1}(b).$$

This is a bijection since $\tilde{X}$, as a universal covering space, must be simply connected. Thus it suffices to show $p^{-1}(b)$ is finite.

Take an open covering $\{U\}$ of X consisting of neighbourhoods which are evenly covered by p. Every point of X has a neighbourhood open neighbourhood which is evenly covered, so this is clearly possible. Then $\{p^{-1}(U)\}$ is an open covering of $\tilde{X}$. We can write each of these sets as

$$p^{-1}(U) = \bigsqcup_{\alpha} V_\alpha,$$

and obtain an open covering $\{V_\alpha\}$ of $\tilde{X}$ which (by compactness) must have a finite subcover $\{V_{\alpha_i}\}_{i=1}^n$. Now suppose $b \in U_b$, so

$$p^{-1}(b) \subseteq p^{-1}(U_b) = \bigsqcup_{\alpha} V_{b\alpha} = \bigsqcup_{j=1}^{J} V_{b\alpha_j},$$

where the $V_{b\alpha_j}$ are from the finite subcover. Since each $V_{b\alpha_j}$ is homeomorphic to U_b (and thus intersects $p^{-1}(b)$ in exactly one point), and since the $V_{b\alpha_j}$ are disjoint, $|p^{-1}(b)| < \infty$. □

Part III.

7. Suppose that M is a smooth n-manifold and that

$$\pi : M' \to M$$

is a covering map. Show that M' has a unique smooth structure relative to which π is locally a diffeomorphism.

Solution. As a covering map, π is locally a homeomorphism, i.e., for $x \in M$, there is a neighbourhood U of x for which $\pi^{-1}(U) = \sqcup V_x$, V_x open in M', and $\pi\big|_{V_x}: V_x \to U$ is a homeomorphism. So we use this to pull the smooth structure of M back to M'; M already has some smooth structure $\mathcal{U} = \{(U_\alpha, \varphi_a)\}$.

For any $x \in U_\alpha$, use π to find a neighbourhood U_x of x so that $\pi^{-1}(U_x \cap U_\alpha) = \sqcup V_{x\alpha}$ and $\pi\big|_{V_{x\alpha}}$ is a homeomorphism.

Claim: $\mathcal{V} = \left\{(V_{x\alpha}, \varphi \circ \pi\big|_{V_{x\alpha}}\right\}$ is a smooth structure.

Proof. We need to check compatibility of the covering. Recall that (V_1, ψ_1) and (V_2, ψ_2) are *compatible* iff $V_1 \cap V_2 \neq 0$ implies that $\psi_2 \circ \psi_1^{-1}$ and $\psi_2 \circ \psi_1^{-1}$ are diffeomorphisms of the open sets $\psi_1(U \cap V)$ and $\psi_2(U \cap V)$.

Let V_1, V_2 be two intersecting sets of $\mathcal{V}$ and check that

$$\psi_2 \circ \psi^{-1} : \psi_1(V_1 \cap V_2) \to \psi_2(V_1 \cap V_2)$$

is a diffeomorphism. First, note that for any $y \in V_1 \cap V_2$, $\pi(y) = x \in \pi(V_1) \cap (V_2)$. Hence, for $V_1 = U_{1\alpha}$ and $V_2 = U_{2\beta}$ we can take $\alpha = \beta$, i.e., we can choose sets from the "same level" in the pancake stack so that $y \in U_{1\alpha} \cap U_{2\alpha}$. Then

$\psi_2 \circ \psi_1^{-1}(\psi_1(V_1 \cap V_2)) = \psi_2(V_1 \cap V_2)$	ψ_i are diffeo	
$= \varphi_2 \circ \pi\big|_{V_2}(V_1 \cap V_2)$	def ψ_2	
$= \varphi_2(\pi(U_{1\alpha} \cap U_{2\alpha}))$	$\alpha = \beta$	
$= \varphi_2(U_1 \cap U_2)$	def π	
$= \varphi_1(U_1 \cap U_2)$	smooth str of M	
$= \psi_1 \circ \psi_2^{-1}(\psi_2(V_1 \cap V_2))$	BSA (in reverse)	□

The uniqueness of this smooth structure comes from the covering map. □

8. Prove every smooth manifold admits a Riemannian metric.

Solution. By the Whitney Embedding Theorem, any smooth manifold M of dimension n may be smoothly embedded into R^{2n+1} as a closed submanifold. The restriction of the standard Riemannian metric on R^{2n+1} to M is a Riemannian metric on M. □

9. To define the Hopf fibration $h : S^3 \to S^2$, we think of S^3 as the unit sphere in $\mathbb{C} \oplus \mathbb{C}$ and S^2 as the unit sphere in $\mathbb{C} \oplus \mathbb{R}$. With respect to these coordinates, the formula for h is

$$h : (a, c) \mapsto \left(2a\bar{c}, |a|^2 - |c|^2\right).$$

(a) Show that the image of h is indeed contained in S^2.

Solution by Wayne Lam. For $(a, c) \in S^3$ we have $|a|^2 + |c|^2 = 1$. Then

$$\begin{aligned} \|h(a,c)\|^2_{\mathbb{C}\oplus\mathbb{R}} &= |2a\bar{c}|^2 + \left||a|^2 - |c|^2\right|^2 \\ &= 4|a|^2|c|^2 + |a|^4 - 2|a|^2|c|^2 + |c|^4 \\ &= |a|^4 + 2|a|^2|c|^2 + |c|^4 \\ &= \left(|a|^2 + |c|^2\right)^2 \\ &= 1. \end{aligned}$$

□

(b) Show that h is a quotient map.

Solution. First note that h is continuous (as a composition of continuous maps), S^3 is compact and S^2 is Hausdorff; thus h is a closed map. A surjective closed map is always a quotient map, since a quotient map only requires that *saturated* closed sets have closed images. Hence we only need to show h is surjective. Pick $(z, x) \in S^2$. Then let

$$a = \sqrt{\tfrac{1+x}{2}}e^{i \arg z} \qquad c = \sqrt{\tfrac{1-x}{2}}.$$

Now we have

$$\begin{aligned} h(a,c) &= h\left(\sqrt{\tfrac{1+x}{2}}e^{i\arg z}, \sqrt{\tfrac{1-x}{2}}\right) \\ &= \left(2\sqrt{\tfrac{1+x}{2}\tfrac{1-x}{2}}e^{i\arg z}, \tfrac{1+x}{2} - \tfrac{1-x}{2}\right) \\ &= \left(\sqrt{1-x^2}e^{i\arg z}, x\right) \\ &= \left(|z|e^{i\arg z}, x\right) \\ &= (z, x). \end{aligned}$$

□

(c) Identify S^1 with the unit circle in $\mathbb{C}$. Consider the S^1-action on S^3,

$$H : S^1 \times S^3 \to S^3,$$
$$H : (\omega; (a, c)) \to (\omega a, \omega c),$$

where the multiplication takes place in $\mathbb{C}$. Show that the orbits of this circle action coincide with the fibers of h.

Solution by Wayne Lam. First, we pick a point in the orbit of (a, c) and show it is in the same fiber; so let $(a, c) \in S^3 \subseteq \mathbb{C} \oplus \mathbb{C}$. Then

$$\begin{aligned} h(\omega a, \omega c) &= \left(2\omega a\overline{\omega c}, |\omega a|^2 - |\omega c|^2\right) \\ &= \left(2a\bar{c}, |a|^2 - |c|^2\right) && \omega\bar{\omega} = |\omega|^2 = 1 \\ &= h(a, c), \end{aligned}$$

so $(\omega a, \omega c) \in [(a, c)]$, $\forall \omega$.
Next, we pick two points in the same fiber and show that one can be written as the image of the other under the action; so suppose $h(a_1, c_1) = h(a_2, c_2)$. By the formula for h, this gives the equalities

$$a_1\overline{c_1} = a_2\overline{c_2} \tag{9.1}$$

and

$$2|a_1|^2 - 1 = 2|a_2|^2 - 1 \quad \Longrightarrow \quad a_2 = a_1 e^{i\theta}. \tag{9.2}$$

Thus for $a_1 \neq 0$ we have

$$\begin{aligned} a_1\overline{c_1} &= a_1 e^{i\theta}\overline{c_2} \\ \overline{c_1} &= \overline{e^{-i\theta}c_2} \\ c_1 &= e^{-i\theta}c_2 \\ c_2 &= e^{i\theta}c_1. \end{aligned}$$

In case $a_1 = 0$, we have $a_2 = 0$ by (9.1) and then $|c_1|^2 = |c_2|^2$ by (9.2), which implies $c_2 = c_1 e^{i\theta}$.
Either way, $(a_2, c_2) = e^{i\theta}(a_2, c_2)$. □

. June 2004

Part I.

1. (a) Suppose that X is a set and $\mathcal{U}$ and $\mathcal{V}$ are topologies for X. Prove that $\mathcal{U} \cap \mathcal{V}$ is also a topology for X.

Solution by Wayne Lam.

(i) It is clear that $\varnothing, X \in \mathcal{U} \cap \mathcal{V}$.

(ii) Suppose $\{O_\alpha\} \subseteq \mathcal{U} \cap \mathcal{V}$. Then $\{O_\alpha\} \subseteq \mathcal{U}$ and hence $\bigcup_\alpha O_\alpha \in \mathcal{U}$, since $\mathcal{U}$ is a topology. Similarly, $\bigcup_\alpha O_\alpha \in \mathcal{V}$, whence $\bigcup_\alpha O_\alpha \in \mathcal{U} \cap \mathcal{V}$.

(iii) Suppose $\{O_n\}_{n=1}^N \subseteq \mathcal{U} \cap \mathcal{V}$. Then $\{O_n\} \subseteq \mathcal{U}$ and hence $\bigcap_{n=1}^N O_n \in \mathcal{U}$, since $\mathcal{U}$ is a topology. Similarly, $\bigcap_{n=1}^N O_n \in \mathcal{V}$, whence $\bigcap_{n=1}^N O_n \in \mathcal{U} \cap \mathcal{V}$.

□

(b) Suppose that $\sim$ is the binary relation on $\mathbb{R}$ such that $x \sim y$ iff y is a positive multiple of x. Determine whether the quotient space is Hausdorff and prove your conclusion is correct.

Solution by Wayne Lam. The quotient space is not Hausdorff; we denote it by

$$\mathbb{R}/\sim \quad = \quad \{[-1], [0], [1]\}.$$

U is open in $\mathbb{R}/\sim$ iff $p^{-1}(U)$ is open in $\mathbb{R}$, so let U be any open neighbourhood of 0. Then $p^{-1}(U)$ contains an open interval about 0, i.e., $\exists (a, b) \subseteq p^{-1}(U)$ for $a < 0 < b$. Then U contains $[-1]$ and $[1]$. Hence the quotient space is not even T_1, let alone Hausdorff. □

2. (a) Let X and Y be topological spaces, let $a \in X$ and $b \in Y$, and let A and B be the connected components of a and b in X and Y respectively. Prove that $A \times B$ is the connected component of (a, b) in $X \times Y$.

Solution. Recall that a component is defined to be an element of the partition associated to the equivalence relation

$$x \sim y \quad \text{iff} \quad \exists W \subseteq X \text{ connected, with } x, y \in W.$$

Also recall the following theorem: The components $\{C_i\}$ form a partition of X such that each connected nonempty $W \subseteq X$ intersects only one C_i.
To see that $A \times B$ is the component of (a, b), we show

$$A \times B \nsubseteq C \implies C \text{ not connected.}$$

Project C to the axes: $F_X = \pi_X(C), F_Y = \pi_Y(C)$. Then lift back to the product space: $C \subseteq \pi_X^{-1}(F_X) \times \pi_Y^{-1}(F_Y) = D$. Let $\{D_i\}$ be the connected components of D.
Since $A \times B \subseteq C \subseteq D$, let D_1 be the component containing $A \times B$.
Claim: $D_1 = A \times B$.

Proof. Suppose $A \nsubseteq \pi_X(D_1)$. Then $\pi_X(D_1)$ is not connected, by the theorem cited initially. Whence D_1 is not connected, by Thm. 23.5 (a contin image of connected is connected). The same holds for $B = \pi_Y(D_1)$. □

Now since $C \subseteq D$ and $D_1 = A \times B$, any pt of C which is not in $A \times B$ is in some *other* connected component of C, hence C is not connected by the disjointness of partitions (by the thm again). □

(b) Let X be a topological space and suppose that $x \in X$ is **not** a limit point of X. Prove that the one point subset $\{x\}$ is open in X.

Solution. If x is not a limit point, we can find an open neighbourhood U of x that does not intersect X in any point other than x, i.e.,

$$U = U \cap X = \{x\}.$$

Since U is open, we are done. □

3. Let X be a topological space such that every point in X lies in a maximal compact subset. Prove that X is compact.

Solution. We will use Zorn's Lemma. Using inclusion for an order, define

$$\mathcal{A} = \{\text{nonempty compact subsets of } X\}.$$

Pick a chain $A = \{A_i\}$, so $A_i \subseteq A_j$ for $i < j$. Suppose A has no upper bound. Then we can find a sequence

$$\{A_{j_k}\}_{k=1}^{\infty}, \quad A_{j_k} \subseteq A_{j_{k+1}}, \tag{3.1}$$

and for any $M \in \mathcal{A}$, $\exists K$ such that $A_{j_K} \nsubseteq M$. But choose $x \in A_{j_1}$. There is no maximal subset containing x, since such a set would contradict (3.1). $\lightning$

Hence $\mathcal{A}$ has a maximal element B by Zorn. To see that $B = X$, suppose not. Then we can find $y \in X \setminus B$ and consider $C = B \sqcup \{y\} \not\supseteq B$. C can easily be seen to be compact by using the finite subcover definition of compactness. The compactness of X contradicts the maximality of B. So $B = X$. □

4. (a) Suppose that X is a separable metric space and A is a subspace of X. Prove that A is also separable.

Solution. We begin by proving the following useful fact.
Claim: When X is metrizable, separable is equivalent to second countable.

Proof. ($\Rightarrow$) (Munkres Ex30.5(a)) Let $W = \{w_n\}_{n=1}^{\infty}$ be a countable dense subspace of X. For each w_n, consider the family of metric balls $\mathcal{V} = \{V_{nk}\}_{k=1}^{\infty}$, where

$$V_{nk} = B(w_n, 1/k) = \{x \in X : d(x, w_n) < 1/k\}.$$

Evidently, $\mathcal{V}$ is a countable basis. ($\Leftarrow$) (Munkres Thm 30.3) From each nonempty basis element B_n, choose a point x_n. Then $D = \{x_n\}_{n=1}^{\infty}$ is dense in X: given any point $x \in X$, every basis element containing x intersects D, so $x \in \overline{D}$. □

Now note that a subspace of a second-countable space is also second-countable by Thm 30.2: If $\mathcal{B}$ is a countable basis for X, then $\{B \cap A : B \in \mathcal{B}\}$ is a countable basis for the subspace A of X. □

(b) Give an example of a topological space that is metrizable but does not have a complete metric, and give reasons for your answer. [*Hint*: Look at dense subsets of the real numbers.]

Solution. If you have studied Real Analysis, you can use an L^p space with the L^q metric. Otherwise, follow the hint and consider $\mathbb{Q} \subseteq \mathbb{R}$. $\mathbb{Q}$ has the metric it inherits as a subspace of $\mathbb{R}$, so is clearly metrizable. To see that it is not complete, choose a Cauchy sequence of rationals which converges to π, like

$$\begin{aligned} q_1 &= 3 \\ q_2 &= 3.1 \\ q_3 &= 3.14 \\ q_4 &= 3.141 \\ &\vdots \\ q_{20} &= 3.1415926535897932384 \\ &\vdots \end{aligned}$$

$Q = \{q_k\}_{k=1}^{\infty}$ is clearly a sequence in $\mathbb{Q}$ which converges to $\pi \in \mathbb{R} \setminus \mathbb{Q}$. To see Q is Cauchy, note that it is convergent in $\mathbb{R}$, hence Cauchy in $\mathbb{R}$ and also in $\mathbb{Q}$. Alternatively, put $M = \min\{m, n\} - 1$, note that

$$|q_m - q_n| \leq 10^{-M},$$

which is less than any fixed ε for sufficiently large M. □

Part II.

5. Let $B^3 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$ be the 3-ball. Let H be the solid torus in the interior of B^3 obtained by revolving the disk $D = \{(x, 0, z) \in \mathbb{R}^3 : (x - 1/2)^2 + z^2 \leq 1/9\}$ about the z-axis. Let $X = \overline{B^3 \setminus H}$. Calculate the fundamental group $\pi_1(X)$.

Solution. If we take S^1 to be the unit circle in the xy-plane, then X is the deformation retract of $Y = \mathbb{R}^3 \setminus S^1$, as seen by the following homotopy:

$$F(\bar{x}, t) = \begin{cases} (1-t)\bar{x} + t\bar{x}/\|\bar{x}\| & \|\bar{x}\| \geq 1, \\ \bar{x} & \bar{x} \in X, \\ (1-t)\bar{x} + ty & \bar{x} \in B^3 \setminus X, \end{cases}$$

where y is the metric projection of $\bar{x}$ to X, i.e., the closest point of X to $\bar{x}$ (which is unique because the circle S^1 has been removed). It is easily seen that this is continuous, and that it fixes each point of X as t changes from 0 to 1.

Thus, $\pi_1(X) = \pi_1(Y)$. The fundamental group of Y is

$$\pi_1(Y) = \langle [\gamma] \rangle \cong \mathbb{Z}, \tag{5.1}$$

where γ is the curve going from $(2,0,0)$ to $(0,0,2)$ to $(0,0,-2)$ to $(2,0,0)$. (Y is path-connected, so we have arbitrarily chosen the base point $x_0 = (2,0,0)$.) To see why, let α be any curve with "winding number" 0, i.e., which is homotopic to some curve β which does not pass through the disk $D^2 = hull(S^1)$. Such a curve may be disentangled (if necessary) from D^2 as follows:

(i) Let C_ε be a cylinder of very small radius ε around the x-axis.

(ii) For any segment of β which passes through C_ε, replace this segment by a geodesic across C_ε which connects the entry and exit points of β. This process can be done, even if β passes through C_ε infinitely many times. Call the result $\dot{\beta}$.

(iii) Make a homotopy from $\dot{\beta}$ by expanding ε until $\varepsilon = 2$, and call the result $\ddot{\beta}$.

(iv) Make a straight line homotopy from $\ddot{\beta}$ to the projection of $\ddot{\beta}$ in the xz-plane. This resulting curve is straight-line homotopic to the constant map at x_0, by the contractibility of the xz-plane.

Thus, $[\alpha] = [\gamma]$. Hence, any loop can be decomposed as a concatenation of trivial loops and powers of γ, and (5.1) is valid. □

Steps (i)–(iv) are necessary in case the image of γ is dense in some neighbourhood of D^2; see Ex. 59.2 and consider Thm. 44.1.

6. Let $F(a,b,c)$ be the free group of rank 3 generated by a,b,c and $F(x,y)$ be the free group of rank 2 generated by x,y. Find an explicit injective homomorphism $\varphi : F(a,b,c) \to F(x,y)$. Also describe a complete list of cosets of the subgroup $\varphi(F(a,b,c))$ of $F(x,y)$.

Solution. (See also Jan 2005 #4).
We define the homomorphism on the generators by

$$\begin{aligned} a &\mapsto yxy^{-1} \\ b &\mapsto y^2x^2y^{-2} \\ c &\mapsto y^3x^3y^{-3}. \end{aligned}$$

Now consider that an element in the image of φ has the form

$$\varphi(\alpha) = y^{\delta_1}x^{\varepsilon_1}y^{\delta_2}x^{\varepsilon_2}y^{\delta_3}x^{\varepsilon_3}\dots y^{\delta_n}x^{\varepsilon_n}y^{\delta_{n+1}}.$$

We can recover α via the following decoding algorithm:

(i) $\delta_1 = 1,2,3$ indicates whether the first letter of α is an a, b, or c, respectively.

(ii) ε_1/δ_1 is the exponent of the first letter of α.

(iii) $\delta_2' = \delta_1 + \delta_2$ indicates whether the second letter of α is an a, b, or c, as in (i).

(iv) ε_2/δ_2' is the exponent of the second letter of α.

$\vdots$

Since this procedure for constructing the inverse exists, φ is injective. This procedure works because raising a, b, or c to a power produces a higher exponent on the x but not on the y:

$$\varphi(b^5) = \varphi(b)^5 = (y^2x^2y^{-2})\dots(y^2x^2y^{-2}) = y^2x^{10}y^{-2}.$$

The cosets of $H = \varphi(F(a, b, c))$ are sets wH. Here, $u \in wH$ iff $u = wh$ where $h \in H$. I.e., h begins with y^ε, $\varepsilon = 1, 2, 3$ and ends with y^ε for $\varepsilon = -1, -2, -3$, and has y^ε for $\varepsilon = \pm1, \pm2$ for any other factor of y which appears in h. Of course, the exponents on the factors of x must also make sense. □

7. Suppose that X is a compact topological space, and $p : \tilde{X} \to X$ is the universal space of X. Show that $\tilde{X}$ is compact iff the fundamental group of X is finite.

Solution. (See also Jan 2005 #6, and 1998 #4).
($\Rightarrow$) This was #6 on the Jan 2005 qualifier.
($\Leftarrow$) By Thm. 54.4, the lifting correspondence gives a surjection

$$\varphi : \pi_1(X, x_0) \to p^{-1}(x_0).$$

This is a bijection since $\tilde{X}$, as a universal covering space, must be simply connected. Thus we know $|p^{-1}(x_0)| = n < \infty$.

Let $\mathcal{C} = \{C\}$ be an open covering of $\tilde{X}$. For any $x \in X$, we have

$$p^{-1}(x) = \{b_1(x), \dots, b_n(x)\}.$$

Thus, for each $x \in X$, we may choose sets from $\mathcal{O}$ which are neighbourhoods of the points in its preimage, i.e., pick sets

$$C_1(x), \dots, C_n(x) \in \mathcal{C} \quad \text{with} \quad b_i(x) \in C_i(x) \in \mathcal{C}.$$

The sets $C_i(x)$ are depicted as rectangles in Figure 3.

Also, we have that p is a covering map, so we also have an evenly covered neighbourhood $U(x)$ of each point x, i.e., an open set $U(x)$ such that $x \in U(x)$ and

$$p^{-1}(U(x)) = \bigsqcup_{i=1}^{n} V_i(x).$$

$U(x)$ and each of the sets $V_i(x)$ is depicted as an ellipse in Figure 3.

Each set $V_i(x) \cap C_i(x)$ is a neighbourhood of $b_i(x)$ which is homeomorphic by p to a subset of $U(x)$. We intersect these to obtain a new neighbourhood of x:

$$W(x) = \bigcap_{i=1}^{n} p\left(V_i(x) \cap C_i(x)\right).$$

Now $\{W(x)\}_{x\in X}$ is an open cover of the compact space X, so we take a finite subcover $\{W(x_j)\}_{j=1}^{J}$ of X. Looking back to $\mathcal{C}$, we see that the collection

$$\{C_i(x_j) \in \mathcal{C} : i = 1, \dots, n; j = 1, \dots, J\}$$

is a finite covering of $\tilde{X}$. To check this, note that $\{U(x_j)\}$ is a finite cover of X, so $\{V_i(x_j) \cap C_i(x_j)\}$ is a covering of $\tilde{X}$, and each of these sets is contained in $C_i(x_j)$. □

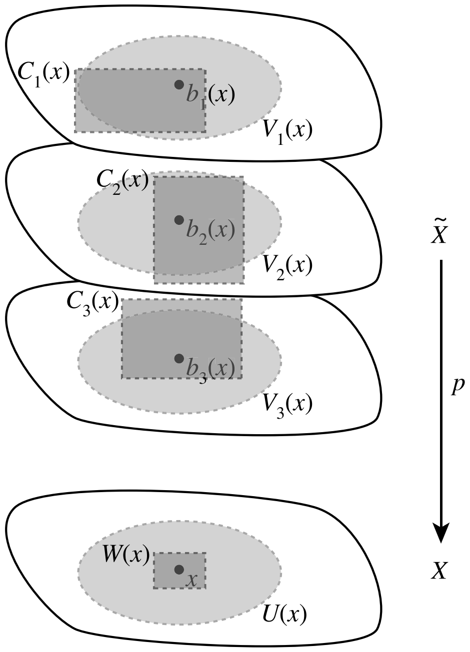

Figure 3. Constructing a finite subcover for $\tilde{X}$, the universal covering space of the compact simply connected space X.

Part III.

8. Show there is no C^∞ injection $g : S^2 \to S^1$.

Solution. Since $S^1 \subseteq \mathbb{R}^2$, Borsuk-Ulam (Thm. 57.3) says that there is some $x \in S^2$ for which $g(x) = g(-x)$. Hence, there can be no injection $S^2 \to S^1$, let alone a C^∞ injection. $\square$

Alternative proof using differentiability:

Solution by Alissa Crans.
Assume g is injective and pick $y \in \text{Im}(f)$. If y is a regular value, then $g^{-1}(y)$ is a smooth 1-manifold. But since g is injective, $g^{-1}(y)$ contains only a single point, so it is a 0-manifold. $\lightning$

So each $y \in \text{Im}(g)$ is a critical value, not a regular value. Thus, $g^{-1}(y)$ contains at least one critical point. But by injectivity, $g^{-1}(y)$ is a singleton and hence *is* that critical point. Thus the domain of g consists entirely of critical point, i.e., $dg_x = 0$, $\forall x \in S^2$. Since the domain is connected, this implies g is constant. But S^2 is more than a single point, so g is clearly not injective. $\lightning$ $\square$

9. Let U be an open subset of euclidean space and let X and Y be smooth vector fields on U. Show that

$$[X, Y] \equiv 0$$

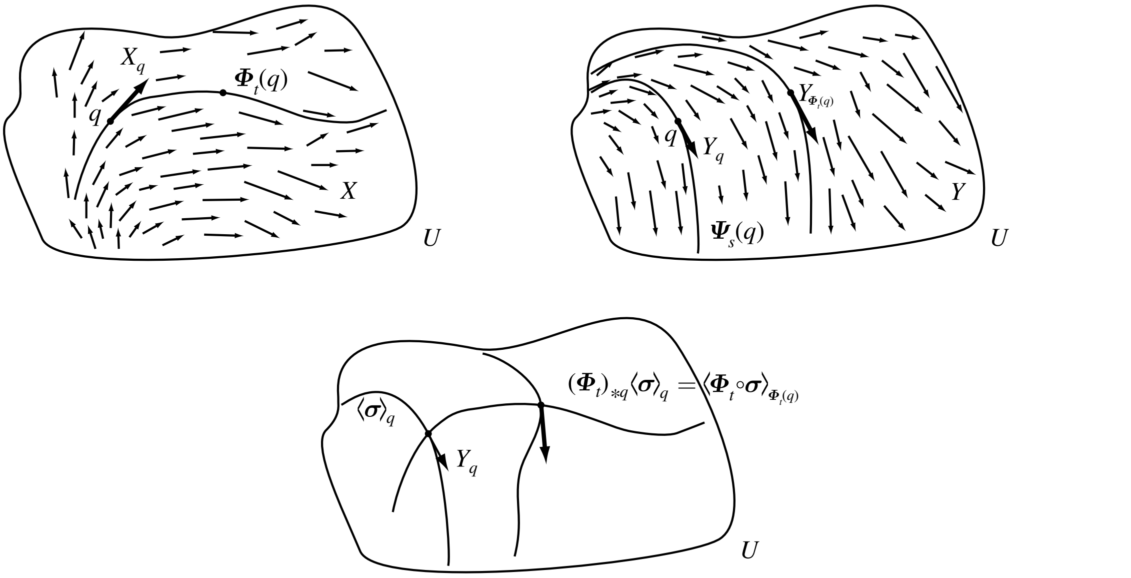

FIGURE 4. What it means for flows to commute. Recall that for any smooth map $\Phi : U \to V$, the differential or *push-forward* is defined on infinitesimal curves via $\Phi_{*q}\langle\sigma\rangle_q = \langle\Phi \circ \sigma\rangle_{\Phi(q)}$. Here we use the smooth map $\Phi_t : U \to U$ and the curve $\sigma(s) = \Psi_s(q)$. The left side of (9.1) is $(\Phi_t)_{*q}(Y_q)$, depicted in the bottom diagram as a tangent vector to the image of the flow line $\sigma(s)$, after evolving it in time by the flow Φ of X. The right side of (9.1) is the tangent vector depicted at top right, obtained by letting q flow along Φ by time t, and evaluating the field Y at that point.

iff the local flows generated by X and Y commute.

Solution from Conlon p.78.
If the local flows commute on U, then by definition we have

$$\Phi_t\Psi_s(q) = \Psi_s\Phi_t(q), \quad -\delta_q < s, t < \delta_q, \quad \forall q \in U.$$

This means that for $-\delta_q < t < \delta_q$, Φ_t carries the flow line $\{\Psi_s(q) \,\vdots\, -\delta_q < s < \delta_q\}$ of Ψ onto another flow line of Ψ. By taking the infinitesimal curve point of view, we see that evaluating a vector field at a point and then mapping it by the differential of the flow gives the same result as evolving the point along the flow, and then evaluating the vector field. In symbols,

$$(\Phi_t)_{*q}(Y_q) = Y_{\Phi_t(q)}. \tag{9.1}$$

Putting this into the expression for the bracket, we have

$$\begin{aligned}
[X, Y] &= \mathcal{L}_X(Y) && \text{Thm. 2.8.16} \\
&= \lim_{t\to 0} \frac{\varphi_{-t*}(Y) - Y}{t} && \text{def of } L_X(Y) \\
&= \lim_{t\to 0} \frac{Y - Y}{t} && \text{above remark} \\
&= 0
\end{aligned}$$

throughout U.

For the converse, assume $[X, Y] \equiv 0$ on U. Let $q \in U$, fix $s \in (-\delta_q, \delta_q)$ and let $q' = \Psi_s(q)$ be a the point of U to which q has flowed at time s, under Ψ. We define a mapping into the tangent space at this new point q' as

$$v : (-\delta_q, \delta_q) \to T_{q'}(U) \quad \text{by} \quad v(t) = \Phi_{-t*}(Y_{\Phi_t(q')}).$$

Then $v(t)$ is a differentiable curve in $T_{q'}(U) = \mathbb{R}^n$, i.e., a smoothly varying family of vectors. Thus we may differentiate

$$\begin{aligned}
\frac{dv}{dt} &= \lim_{h\to 0} \frac{v(t+h) - v(t)}{h} \\
&= \lim_{h\to 0} \frac{(\Phi_{-t-h})_* \left(Y_{\Phi_{t+h}(q')}\right) - \Phi_{-t*}\left(Y_{\Phi_t(q')}\right)}{h} && \text{def } v \\
&= \lim_{h\to 0} \Phi_{-t*}\left(\frac{(\Phi_{-h})_* \left(Y_{\Phi_{t+h}(q')}\right) - \left(Y_{\Phi_t(q')}\right)}{h}\right) && \text{linearity} \\
&= \Phi_{-t*}\left(\lim_{h\to 0} \frac{(\Phi_{-h})_* \left(Y_{\Phi_t(\Phi_h(q'))}\right) - \left(Y_{\Phi_t(q')}\right)}{h}\right) && \text{continuity} \\
&= \Phi_{-t*}[X, Y]_{\Phi_t(q')} && \text{Thm. 2.8.16} \\
&= 0 && \text{hyp } [X, Y] \equiv 0.
\end{aligned}$$

Whence $\frac{dv}{dt} = 0$ on $-\delta_q < t < \delta_q$ implies that $v(t)$ is constant on this interval, i.e.

$$v(t) = \Phi_{-t*}\left(Y_{\Phi_t(q')}\right) = \Phi_{0*}\left(Y_{\Phi_0(q')}\right) = Y_{q'}, \quad -\delta_q < t < \delta_q. \tag{9.2}$$

But this is true for any q' on the integral curve to Y given by $\sigma(s) = \Psi_s(q)$. Thus,

$$\dot{\sigma}(s) = Y_{\Psi_s(q)} = Y_{\sigma(s)} = Y_{q'} \qquad \text{by Def. 2.8.3,}$$

Whence (9.2) becomes

$$\dot{\sigma}(s) = \Phi_{-t*}\left(Y_{\Phi_t(q')}\right),$$

and applying the flow Φ_{t*} gives

$$\Phi_{t*}(\dot{\sigma}(s)) = \Phi_{t*}\Phi_{-t*}\left(Y_{\Phi_t(q')}\right) = Y_{\Phi_t(q')} = Y_{\Phi_t(\Psi_s(q))}, \tag{9.3}$$

as s and t range independently over $(-\delta_q, \delta_q)$. Now (9.3) states that $\Phi_t \circ \sigma$ is also an integral curve to Y with initial conditions $\Phi_t(\sigma(0)) = \Phi_t(q)$. But then

$$\Phi_t \Psi_s(q) = \Psi_s \Phi_t(q), \quad -\delta_q < s, t < \delta_q, \quad \forall q \in U.$$

by the uniqueness of solutions to ODE. □

10. To define the Hopf fibration $h : S^3 \to S^2$, we think of S^3 as the unit sphere in $\mathbb{C} \oplus \mathbb{C}$ and S^2 as the unit sphere in $\mathbb{C} \oplus \mathbb{R}$. With respect to these coordinates, the formula for h is

$$h : (a, c) \mapsto \left(2a\overline{c}, |a|^2 - |c|^2\right).$$

(a) Show that the image of h is indeed contained in S^2.

Solution by Wayne Lam. For $(a, c) \in S^3$ we have $|a|^2 + |c|^2 = 1$. Then

$$\begin{aligned}\|h(a,c)\|^2_{\mathbb{C}\oplus\mathbb{R}} &= |2a\overline{c}|^2 + \left||a|^2 - |c|^2\right|^2 \\ &= 4|a|^2|c|^2 + |a|^4 - 2|a|^2|c|^2 + |c|^4 \\ &= |a|^4 + 2|a|^2|c|^2 + |c|^4 \\ &= \left(|a|^2 + |c|^2\right)^2 \\ &= 1.\end{aligned}$$

□

(b) Show that h is a quotient map.

Solution. First note that h is continuous (as a composition of continuous maps), S^3 is compact and S^2 is Hausdorff; thus h is a closed map. A surjective closed map is always a quotient map, since a quotient map only requires that *saturated* closed sets have closed images. Hence we only need to show h is surjective. Pick $(z, x) \in S^2$. Then let

$$a = \sqrt{\tfrac{1+x}{2}}e^{i\arg z} \qquad c = \sqrt{\tfrac{1-x}{2}}.$$

Now we have

$$\begin{aligned}h(a,c) &= h\left(\sqrt{\tfrac{1+x}{2}}e^{i\arg z}, \sqrt{\tfrac{1-x}{2}}\right) \\ &= \left(2\sqrt{\tfrac{1+x}{2}\tfrac{1-x}{2}}e^{i\arg z}, \tfrac{1+x}{2} - \tfrac{1-x}{2}\right) \\ &= \left(\sqrt{1-x^2}e^{i\arg z}, x\right) \\ &= \left(|z|e^{i\arg z}, x\right) \\ &= (z, x).\end{aligned}$$

□

. January 2004

Part I.

1. Let A and B by subsets of a topological space X and let C be a subset of $A \cap B$ that is closed in each of A and B with respect to the subspace topologies. Prove that C is a closed subset of $A \cup B$.

   Solution by Wayne Lam.
   By the hypotheses and the nature of subspace topology, we know that
   $$C = C_A \cap A, \quad \text{and} \quad C = C_B \cap B,$$
   for C_A, C_B closed in X. Then $C_A \cap C_B$ is closed in X and
   $$\begin{aligned}(C_A \cap C_B) \cap (A \cup B) &= (C_A \cap C_B \cap A) \cup (C_A \cap C_B \cup B) \\ &= C \cup C \\ &= C.\end{aligned}$$
   This shows C as an intersection of $A \cup B$ with a closed subset of X; hence C is closed in $A \cup B$. □

2. (a) Let f and g be continuous functions from a topological space X to a Hausdorff space Y. Prove that the set of all points $x \in X$ such that $f(x) = g(x)$ is a closed subset of X.

   Solution. Y is Hausdorff iff the diagonal $\Delta \subseteq Y \times T$ is closed. The mapping
   $$\Phi : X \to Y \times Y, \quad \text{by } \Phi(x) = (f(x), g(x))$$
   is continuous, by Thm. 18.4 (maps into products). Then the preimage of the closed set Δ must be closed in X. But
   $$\Phi^{-1}(\Delta) = \{x \in X \vdots f(x) = g(x)\}.$$
   □

   (b) Let A be a subset of the Hausdorff space X and let $r : X \to A$ be a retraction. Prove that A is a closed subset of X.

   Solution. Consider r as a map into X, whose image is A, so that $r : X \to X$; and write $id : X \to X$ for the identity map. Now if $r(x) = id(x)$, then $\operatorname{Im} r \subseteq A$ implies $x \in A$. Conversely, if $x \in A$, then $r(x) = a = id(x)$ by the defn of r. We have just shown
   $$A = \{x \vdots r(x) = id(x)\},$$
   so A is closed by (a). □

3. Let (X, d) be a metric space, and $A \subseteq X$. Show that the function $d_A : X \to \mathbb{R}$ defined by $d_A(x) = \inf\{d(x,a) : a \in A\}$ is continuous.

Solution by Zhi Yao.
Let (r, s) be a basic open set, i.e., an open interval, in $\mathbb{R}$. Then

$$d_A^{-1}(r,s) = \{x \in X : r < \inf_{a \in A}\{d(x,a)\} < s\}.$$

To show this set is open, we pick $x \in d_A^{-1}(r,s)$ and find a metric ball $B(x,\varepsilon)$ about x, which is contained in $d_A^{-1}(r,s)$. Denote $d := d_A(x)$ so that $r < d < s$. Since the inequalities are strict, we can find r', s' such that

$$r < r' < r'' < d < s' < s.$$

Now define

$$\delta_1 := r'' - r' \quad \text{and} \quad \delta_2 := s - s'.$$

For $\varepsilon := \min\{\delta_1, \delta_2\}$, we will show that $B(x,\varepsilon) \subseteq d_A^{-1}(r,s)$. Pick $y \in B(x,\varepsilon)$. Then

$$\begin{aligned} d(y,a) &\leq d(y,x) + d(x,a) && \text{triangle ineq} \\ &< (s - s') + s' && \text{def } \delta_2 \\ &= s, \end{aligned}$$

so $\inf_{a \in A}\{d(y,a)\} < s$. Also,

$$\begin{aligned} d(x,a) &\leq d(x,y) + d(y,a) && \text{triangle ineq} \\ r'' &< (r'' - r') + d(y,a) && \text{def } \delta_1 \\ r' &< d(y,a). \end{aligned}$$

This gives $r' \leq \inf_{a \in A}\{d(y,a)\}$, and hence $r < \inf_{a \in A}\{d(y,a)\}$. (This extra step is why we require the r'' for this direction; to obtain a strict inequality after taking the infimum.) Thus we have $r < d_A(y) < s$. By the arbitrariness of y, this puts $B(x,\varepsilon) \subseteq d_A^{-1}(r,s)$. □

4. Recall that if (X, d) is a metric space, then a contraction mapping $\varphi : X \to X$ is any map that satisfies

$$d(\varphi(x), \varphi(y)) \leq k \cdot d(x,y),$$

for all $x, y \in X$ and some $k \in [0,1)$. Show that any contraction mapping from a compact metric space to itself has a unique fixed point. (*Hint*: Let x_0 be any point in X and consider the recursively defined sequence $x_{n+1} = \varphi(x_n)$. Show that $\{x_{n+1}\}_{n=1}^{\infty}$ is Cauchy, and that its limit is the desired fixed point.)

Solution. Following the hint, choose any $x_0 \in X$ and define a sequence by $x_{n+1} = \varphi(x_n)$. Applying φ repeatedly, we see

$$d(\varphi^n(x), \varphi^n(y)) \leq k^n d(x,y).$$

Using $\varphi^{n+m}(x_0) = \varphi^n(\varphi^m(x_0)$, we obtain

$$d(x_n, x_{n+m}) \leq k^n d(x_0, \varphi^m(x_0)). \tag{4.1}$$

Note that if $k = 0$, there is nothing to prove, so assume $k > 0$. Hence

$$\begin{aligned} d(x_0, \varphi^m(x_0)) &\leq d(x_0, \varphi(x_0)) + d(\varphi(x_0), \varphi^2(x_0)) + \\ &\qquad \cdots + d(\varphi^{m-1}(x_0), \varphi^m(x_0)) && \text{by the triangle ineq} \\ &\leq (1 + k + k^2 + \cdots + k^{m-1} d(x_0, \varphi(x_0)) && \text{by applying (4.1)} \\ &\leq \frac{1}{1-k} d(x_0, \varphi(x_0)) && \text{geometric series.} \end{aligned}$$

Thus

$$d(x_n, x_{n+m}) \leq \frac{k^n}{1-k} d(x_0, \varphi(x_0)) \xrightarrow{n,m\to\infty} 0.$$

We have just shown $\{x_n\}$ to be a Cauchy sequence. Compactness is equivalent to being complete and totally bounded, and the completeness of X gives a point y such that $\lim_{n\to\infty} x_n = y$. Since $\{x_{n+1}\}$ obviously has the same limit, we see that

$$\begin{aligned} d(y, \varphi(y)) &= \lim d(x_n, \varphi(x_n)) && \text{continuity of } d \\ &= \lim d(x_n, x_{n+1}) && \varphi(x_n) = x_{n+1} \\ &= 0, \end{aligned}$$

so $\varphi(y) = y$ and y is a fixed point on φ.

Suppose there were another fixed point z. Then

$$d(y, z) = d(\varphi(y), \varphi(z)) \leq k d(y, z),$$

contradicting the fact that $k < 1$. $\lightning$ □

5. Let ω be a C^∞ differential k-form on $\mathbb{R}^n$ such that

$$\int_M \omega = 0$$

for every compact oriented smooth k-manifold $M \subseteq \mathbb{R}^n$ with $\partial M = \varnothing$. Use Stokes' Theorem to show that ω is closed, that is, $d\omega = 0$. (*Hint*: To show that $d\omega$ is 0 at a point p, let M be a very small sphere whose center is p.)

Solution by Shilong Kuang. Suppose $d\omega \neq 0$. For definiteness, let $d\omega > 0$ at p. Then by continuity, $d\omega$ is positive in some small open neighbourhood of p, and hence in U_ε, a ball of radius ε centered at p. Then we have

$$\begin{aligned} 0 &= \int_{\partial U_\varepsilon} w && \text{by hypothesis} \\ &= \int_{U_\varepsilon} d\omega && \text{Stokes' Thm} \\ &> 0 && d\omega > 0 \text{ on } U_\varepsilon. \end{aligned}$$

This contradiction shows that we must have $d\omega(p) = 0$, and hence $d\omega = 0$. □

6. Let $p \in \mathbb{R}^n$, and let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-mapping such that df_p is injective. Show that there is an $\varepsilon > 0$ such that if $g : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1-mapping with $\|dg_p\| < \varepsilon$, then $h \equiv f + g$ is injective in a neighbourhood of p.

Solution by Shilong Kuang. Note that by the Inverse Function Theorem,

$$\|dh_p\| \neq 0 \qquad \Longleftrightarrow \qquad h \text{ is locally injective at } p.$$

By the given condition on f, we may assume $\|df_p\| > \varepsilon_0$. Since $h = f + g$, linearity of d gives $dh = d(f+g) = df + dg$ and evaluation at p gives $dh_p = df_p + dg_p$. Then

$$\begin{aligned} \|dh_p\| &= \|df_p + dg_p\| \\ &\geq \|df_p\| - \|dg_p\| && \text{triangle ineq} \\ &> \varepsilon_0 - \varepsilon \\ &> 0, \end{aligned}$$

where the last line follows by choosing $\varepsilon = \varepsilon_0$. Thus, we find the required ε. ☐

Part II.

7. Let $p : E \to B$ be a covering map and B path connected. Show the the sets $p^{-1}(b), b \in B$ all have the same cardinality.

Solution. Suppose $|p^{-1}(b_0)| = k$. Then we can find U such that

$$p^{-1}(U) = \bigsqcup_{i=1}^{k} V_i \qquad \text{and} \qquad p\big|_{V_i} = p_i : V_i \to U \text{ is a homeomorphism}, \forall i.$$

Assume that $\exists b_1$ such that $|p^{-1}(b_1)| = j \neq k$; we will contradict the fact that B is connected. Define

$$C = \{b : |p^{-1}(b)| = k\} \quad \text{and} \quad D = \{b : |p^{-1}(b)| \neq k\}.$$

We have $b_0 \in C$, so $C \neq \varnothing$, and $b_1 \in D$, so $D \neq \varnothing$. Also, it is clear that $C \cap D = \varnothing$ and $C \cup D = B$. So it just remains to show C, D are open.

For $b \in C$, we can find U_b such that $p^{-1}(U_b) = \bigsqcup_{i=1}^{k} V_i$, where the V_i are open. Thus for $x \in U_b$, $|p^{-1}(x)| = k$. Hence $b \in U_b \subseteq C$ shows C is open. Similarly, D is open. This gives C, D as a disconnection of B. ↯ Hence no such b_1 exists. ☐

8. Let $p_i = (i, 0) \in \mathbb{R}^2, i = 1, 2, \ldots, n$, and $X = \mathbb{R}^2 \setminus \{p_1, p_2, \ldots, p_n\}$. Use the Seifert-van Kampen Theorem to prove that the fundamental group $\pi_1(X)$ is a free group of rank n.

Solution. First, we consider the case $n = 2$, so X is a plane punctured at $(1, 0)$ and $(2, 0)$. Let

$$U = \{(x, y) \in X : x < 2\} \quad \text{and} \quad V = \{(x, y) \in X : x > 1\}.$$

Then we have

$$U \cup V = X \quad \text{and} \quad U \cap V = \{(x, y) \in X : 1 < x < 2\},$$

so that U, V are open and path-connected, and $U \cap V$ is simply connected. Since $\pi_1(U \cap V) = 0$, the Seifert-van Kampen Theorem gives an isomorphism

$$k : \pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0).$$

Since each of U, V has a circle as a deformation retract, we know

$$\pi_1(U, x_0) = F(a_1) \quad \text{and} \quad \pi_1(V, x_0) = F(a_2),$$

are each a free group with one generator. By Thm. 69.2, the free product of two free groups is just the free group whose generators are the union of the generators of the two factor groups, i.e.,

$$\pi_1(U, x_0) * \pi_1(V, x_0) = F(a_1, a_2).$$

Hence $\pi_1(X, x_0) \cong F(a_1, a_2)$.

Now we use induction. At stage k, we have

$$U = \{(x, y) \in X : x < k\} \quad \text{and} \quad V = \{(x, y) \in X : x > k - 1\},$$

so $U \cap V$ is simply connected. Then

$$\pi_1(U, x_0) = F(a_1, a_2, \dots, a_{k-1}) \quad \text{and} \quad \pi_1(V, x_0) = F(a_k),$$

whence by Thm. 69.2 we have

$$\pi_1(U \cup V, x_0) = F(a_1, a_2, \dots, a_k).$$

□

9. (a) Let X be a topological space. Let $f, g : X \to S^1$ be two continuous maps. Show that if $f(x)$ and $g(x)$ are not antipodal to each other for every $x \in X$, then f and g are homotopic.

   Solution.

   `SOLUTION NEEDED!!!`

   □

   (b) Find two non-homotopic continuous maps $f, g : S^1 \to S^1$ such that there is exactly one point $x_0 \in S^1$ where $f(x_0) = -g(x_0)$.

   Solution.

   `SOLUTION NEEDED!!!`

   □

Part III.

10. Identify $\mathbb{R}^4$ with $\mathbb{C}^2$ so that $(x, y, u, v) \in \mathbb{R}^4$ corresponds to $(z_1, z_2) \in \mathbb{C}^2$ for $z_1 = x + iy$ and $z_2 = u + iv$. Let V be the zero locus of the polynomial $P(z_1, z_2) = z_1^3 + z_2^2$, i.e.,

$$V = \{(z_1, z_2) \in \mathbb{C}^2 : z_1^3 + z_2^2 = 0\} \subseteq \mathbb{R}^4.$$

Show that $V \setminus \{(0, 0)\}$ is a 2-dimensional smooth manifold.

Solution by Alissa Crans.
First, we compute

$$\begin{aligned} z_1^3 + z_2^2 &= (x+iy)^3 + (u+iv)^2 \\ &= x^3 + 3x^2 iy + 3xi^2y^2 + i^3y^3 + u^2 + 2uiv + i^2v^2 \\ &= 0 + 0i \qquad \text{(by hypothesis).} \end{aligned}$$

Thus we may define $f : \mathbb{R}^4 \setminus \{(0,0,0,0)\} \to \mathbb{R}^2$ by

$$f(x,y,u,v) = (x^3 - 3xy^2 + u^2 - v^2, 3x^2y - y^3 + 2uv)$$

so that $V \setminus \{(0,0)\} = f^{-1}((0,0))$. We use the following theorem:

Theorem. If $f : M^m \to N^n$ is a smooth map, $m \geq n$, and $y \in N$ is a regular value, then $f^{-1}(y) \subseteq M$ is a smooth manifold of dimension $(m-n)$.

To see that the origin is a regular point of f, we must show the differential matrix has rank 2. Using row reduction,

$$\begin{aligned} df &= \begin{bmatrix} 3x^2 - 3y^2 & -6xy & 2u & -2v \\ 6xy & 3x^2 - 3y^2 & 2v & 2u \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & \frac{-6xy}{3x^2-3y^2} & \frac{2u}{3x^2-3y^2} & \frac{-2v}{3x^2-3y^2} \\ 6xy & 3x^2-3y^2 & 2v & 2u \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & \frac{-6xy}{3x^2-3y^2} & \frac{2u}{3x^2-3y^2} & \frac{-2v}{3x^2-3y^2} \\ 0 & \frac{(3x^2-3y^2)^2+36x^2y^2}{3x^2-3y^2} & \frac{2v(3x^2-3y^2)-12xyu}{3x^2-3y^2} & \frac{2u(3x^2-3y^2)+12xyv}{3x^2-3y^2} \end{bmatrix} \end{aligned}$$

To show that df has rank 2, it suffices to show $(3x^2 - 3y^2)^2 + 36x^2y^2 > 0$. This is always true unless $x = y = 0$. However, if $x = y = 0$, then

$$f(0,0,u,v) = (u^2 - v^2, 2uv),$$

and for $(0,0,u,v) \in V$, we would need

$$\begin{aligned} u^2 - v^2 &= 0 \\ 2uv &= 0 \end{aligned} \implies \begin{aligned} u &= \pm v \\ uv &= 0 \end{aligned} \implies u = v = 0.$$

Thus, x and y are not both simultaneously 0 on $V \setminus \{(0,0,0,0)\}$, and hence df has rank 2 on $V \setminus \{(0,0,0,0)\}$. Thus $V \setminus \{(0,0,0,0)\}$ is a manifold of dimension $4 - 2 = 2$, by the theorem. □

11. Let M be a smooth manifold. For f, g smooth functions on M and X, Y smooth vector fields on M, we have

$$[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X.$$

Suppose that a smooth function f on M satisfies $[fX, Y] = f[X,Y]$ for all smooth vector fields X and Y on M. What can one say about f?

Solution. Suppose that g is the constant function which takes the value 1 everywhere. Putting this into the first formula above, we obtain

$$\begin{aligned} [fX, gY] &= [fX, Y] \\ &= f1[X, Y] + f(X1)Y - 1(Yf)X && \text{product identity} \\ f(X1)Y &= 1(Yf)X && \text{hypothesis on } f. \end{aligned}$$

But $X1$ is the directional derivative of $g(p)$ in the direction of X_p, which is always 0 because g is constantly equal to 1. Thus we have

$$(Yf)X = 0.$$

This implies that $Yf = 0$ and hence that f must be a constant function. □

12. (a) State what it means for a smooth vector field on a smooth manifold to be complete.

Solution from Conlon Def. 4.1.7.
A smooth vector field $X \in \mathfrak{X}(M)$ is *complete* iff its maximal local flow of contains a global flow. □

Additional background information:
A *local flow on a manifold M* is a system of compatible local flows defined on a covering of M, i.e., a family of smooth maps

$$\{\Phi : (-\varepsilon_\alpha, \varepsilon_\alpha) \times V_\alpha \to U_\alpha \mid \varepsilon_\alpha > 0\}_{\alpha \in A},$$

written $\Phi^\alpha(t, x) = \Phi_t^\alpha(x)$, such that

(i) $V_\alpha \subseteq U_\alpha \subseteq M$ are open sets and $\{V_\alpha\}_{\alpha \in A}$ covers M.
(ii) $\Phi_0^\alpha : V_\alpha \to U_\alpha$ is the inclusion map, $\forall \alpha \in A$.
(iii) $\Phi_{t_1+t_2}^\alpha = \Phi_{t_1}^\beta \circ \Phi_{t_2}^\alpha$, wherever both sides are defined, $\forall \alpha, \beta \in A$.

Ordering local flows by inclusion, we see there exists a unique maximal local flow on M containing Φ: for any chain, just take the union to find an upper bound; then apply Zorn.
To understand the *local flow of a vector field*, observe that every vector field $X \in \mathfrak{X}(M)$ is the infinitesimal generator of a local flow Φ on M, by ODE; i.e., the integral curves of X are flow lines for some local flow Φ that satisfies

$$X_{s_q^\alpha(t)} = \dot{s}_q^\alpha(t), \quad \forall t \in (-\varepsilon_\alpha, \varepsilon_\alpha).$$

A *global flow on a manifold M* is a smooth map

$$\Phi : \mathbb{R} \times M \to M,$$

written $\Phi_t(x) = \Phi(t, x)$, such that

(i) $\Phi_0 = id_M$.
(ii) $\Phi_{t_1+t_2} = \Phi_{t_1} \circ \Phi_{t_2}$, $\forall t_1, t_2 \in \mathbb{R}$.

(b) Give an example of a smooth manifold M_1 such that every vector field on M_1 is complete.

Solution from Conlon 4.1.10–12.
Let $M_1 = S^n$ or T^n be an example. To justify this, we use the following:

Proposition. *If the maximal local flow of $X \in \mathfrak{X}(M)$ contains an element of the form*

$$\Phi : (-\varepsilon, \varepsilon) \times M \to M, \qquad \varepsilon > 0,$$

then X is complete.

Now we show that if $X \in \mathfrak{X}(M)$ has compact support, then X is complete. By compactness, we can find cover $\operatorname{spt}(X)$ with finitely many open sets $U_1, \dots, U_r$ of M such that the local flow of X contains elements

$$\Phi^i : (-\varepsilon_i, \varepsilon_i) \times U_i \to M, \quad 1 \le i \le r.$$

Let $U_0 = M \setminus \operatorname{spt}(X)$, an open set with $X|U_o \equiv 0$. Define

$$\Phi^0 : \mathbb{R} \times U_0 \to M$$

by $\Phi^0_t(x) = x$, $\forall x \in U_0, \forall t \in \mathbb{R}$. Since $\{U_i\}_{i=0}^r$ covers M and the Φ^i agree on overlaps, we have just constructed a local flow on M generated by X. Let $\varepsilon := \min \varepsilon_i$ and fit the elements of the local flow together to get

$$\Phi : (-\varepsilon, \varepsilon) \times M \to M$$

generated by X. From the proposition above, X is complete. Hence, if M_1 is any compact smooth manifold, then every smooth vector field on it is complete. □

(c) Give an example of a smooth manifold M_2 which has vector fields that are not complete.

Solution. Intuitively, a incomplete vector field corresponds to dynamical system which breaks down or blows up in a finite amount of time [Schultz].
[Conlon, Ex. 2.8.13] $\quad e^x \frac{d}{dx} \in \mathfrak{X}(\mathbb{R})$ is not complete.
Let $a \in \mathbb{R}$. We compute the integral curve to X through a.

$$\begin{aligned} \tfrac{d}{dt} x(t) &= e^x \\ \int e^{-x} dx &= \int dt \\ e^{-x} &= c - t \\ x &= -\log(c - t). \end{aligned}$$

Then $x(0) = -\log c = a$ implies $c = e^{-a}$. So $x_a(t) = -\log(e^{-a} - t)$, which is only defined for $-\infty < t < e^{-a}$. Thus $M_2 = \mathbb{R}$ is an example of a manifold which has incomplete vector fields. □