Math 023 - Matrix Algebra for Business
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Contents

I. SYSTEMS OF LINEAR EQUATIONS

I.1. Introduction to Systems of Linear Equations.

I.1.1. *Linear equations.*

Definition 1. A **linear equation** is a sum of variables with coefficients. This is a simple type of equation, the kind with which you have the most familiarity - it is an equation whose graph is straight.

Example 1.
A **linear equation in 2 variables** looks like $ax_1 + bx_2 = d$ where a, b, d are constants and x_1, x_2 are the two variables:

- $x_2 = 2x_1 + 1$
- $3x_1 - x_2 = 4$
- $x_1 + x_2 - 1 = 0$

A **linear equation in 3 variables** looks like $ax_1 + bx_2 + cx_3 = d$ where a, b, c, d are constants and x_1, x_2, x_3 are the three variables:

- $0.5x_1 - 3x_2 + x_3 = 2$
- $x_3 - 2x_2 + 3 = x_1$
- $\frac{2}{3}x_1 - \frac{3}{4}x_2 = x_3$

In general, a **linear equation in n variables** looks like

$$a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n = b$$

where $a_1, a_2, a_3, \ldots, a_n, b$ are constants and $x_1, x_2, x_3, \ldots, x_n$ are n variables.

For contrast, here are some examples of equations that are **not** linear:

$x_1x_2 = 1$**:** The variables are multiplied together.
$\frac{1}{x_3} = x_1$**:** Reciprocals are not linear.
$-2x_1 + 2x_2^2 + 3x_3 = 1$**:** Raising a variable to a power produces a nonlinear eqn.
$x_2 = \sin x_1$**:** Trigonometric functions are not linear.
$e^{x_3} - 3x_2 = 0$**:** The exponential function is not linear.

I.1.2. *Systems of Linear Equations.*

Definition 2. A **system of linear equations** is simply a collection of two or more equations which share the same variables.

Example 2. Suppose you have a collection of dimes and nickels worth 80 cents, and you have 11 coins total. How do you determine how many of each type of coin you have? Let the variable x_1 be the number of dimes and x_2 be the number of nickels. The associated system of linear equations is

$$\begin{aligned} 10x_1 + 5x_2 &= 80 \\ x_1 + x_2 &= 11 \end{aligned}$$

A solution for the system may be found by solving for one variable in one equation, and substituting this relation into the other. For example, the second equation may be rewritten as $x_2 = 11 - x_1$. This new expression for x_2 may be substituted into the first equation to produce

$$10x_1 + 5(11 - x_1) = 80,$$

which then gives

$$5x_1 = 25 \implies x_1 = 5 \text{ and } x_2 = 6.$$

Equivalently, the solution may be found by multiplying one entire equation by a constant, and then adding this new equation to the other one. Multiplying the second equation by -10 gives

$$-10x_1 - 10x_2 = -110,$$

which, when added to the first gives

$$-5x_2 = -30 \implies x_2 = 6 \text{ and } x_1 = 5.$$

Performing either technique gives $x_1 = 5, x_2 = 6$.

Definition 3. A **solution to a system of linear equations** is sequence of numbers $s_1, s_2, \ldots, s_n$ such that the system of eqns is satisfied (i.e., true) when s_i is substituted in for x_i. In the previous example,

$$\begin{aligned} 10 \cdot 5 + 5{\cdot}6 &= 80 \\ 5 + \quad 6 &= 11 \end{aligned}$$

shows that $(x_1, x_2) = (5, 6)$ is a solution to the system. Geometrically, a solution is a point where all the graphs intersect.

A **solution set** for a system of linear equations is the set of all possible solutions for the system.

This last definition might prompt you to ask, "How many solutions can a system of linear eqns have?" Intuitively, you might expect that every system has exactly one solution, but this is not the case. Consider the following systems:

Example 3.

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 2 \end{aligned}$$

This system represents two lines which intersect at the point $(2, 0)$. Hence, it has the unique solution $(2, 0)$.

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 + x_2 &= 1 \end{aligned}$$

This system represents two parallel lines. Since these lines do not intersect, there is no solution (s_1, s_2) which satisfies both equations simultaneously. Such a system is said to be *inconsistent*. More intuitively, think of this system as being impossible to solve because two numbers cannot sum to two different values.

$$\begin{aligned} x_1 + x_2 &= 2 \\ -x_1 - x_2 &= -2 \end{aligned}$$

This system represents the same line two different ways. Since these two lines overlap each other, any point on one line is also on the other. Hence, any point on the line is a solution to the system. Thus, there are an infinite number of solutions to the system (any point on the line will work!).

This example serves to illustrate the general case: for any system of linear equations, it is always the case that there is either **one** unique solution, **no** solution, or an **infinite** number of solutions. In other terminology, the solution set can consist of one point, it can be empty, or it can contain infinitely many points. This is due to the nature of straight lines and the ways they can intersect. For example, it is impossible for two straight lines to intersect in precisely two places (in flat space).

I.1.3. *Solution Techniques.*

Definition 4. A system is in **triangular form** if each successive equation has one less variable than the previous one.

Example 4. Each of these three systems is in triangular form. (Note how the variables are aligned in columns, this will be important later on.)

$$\begin{aligned} x_1 - \tfrac{1}{2}x_2 + \tfrac{3}{2}x_3 - \ \ x_4 &= \tfrac{1}{2} \\ x_2 - 2x_3 + 3x_4 &= 2 \\ x_3 + \tfrac{3}{4}x_4 &= \tfrac{3}{4} \\ x_4 &= 1 \end{aligned} \qquad \begin{aligned} x_1 - 3x_2 &= 1 \\ -2x_2 &= 2 \end{aligned} \qquad \begin{aligned} x_1 - x_2 + 2x_3 &= \ \ 1 \\ x_2 - \ \ x_3 &= -3 \\ 4x_3 &= \ \ 2 \end{aligned}$$

Example 5. None of these three systems is in triangular form.

$$\begin{aligned} x_1 - \tfrac{1}{2}x_2 + \tfrac{3}{2}x_3 - \ \ x_4 &= \tfrac{1}{2} \\ x_2 - 2x_3 + 3x_4 &= 2 \\ -3x_2 + \ \ x_3 + \tfrac{3}{4}x_4 &= \tfrac{3}{4} \\ x_4 &= 1 \end{aligned} \qquad \begin{aligned} -x_1 - 3x_2 &= 1 \\ 3x_1 + \ \ x_2 &= 2 \end{aligned} \qquad \begin{aligned} x_1 - \ \ x_2 + 2x_3 &= \ \ 1 \\ 2x_2 - \ \ x_3 &= -3 \\ 2x_1 + \ \ x_2 + \ \ x_3 &= \ \ 0 \end{aligned}$$

Back substitution

We can solve a system that is in triangular form

$$\begin{aligned} 2x_1 - x_2 + 3x_3 - 2x_4 &= 1 \\ x_2 - 2x_3 + 3x_4 &= 2 \\ 4x_3 + 3x_4 &= 3 \\ 4x_4 &= 4 \end{aligned}$$

using the technique of back substitution as follows:

$$\begin{aligned} 4x_4 = 4 &\Longrightarrow x_4 = \ \ 1 \\ 4x_3 + 3 \cdot 1 = 3 &\Longrightarrow x_3 = \ \ 0 \\ x_2 - 2 \cdot 0 + 3 \cdot 1 = 2 &\Longrightarrow x_2 = -1 \\ 2x_1 - (-1) + 3 \cdot 0 - 2 \cdot 1 = 1 &\Longrightarrow x_1 = \ \ 1 \end{aligned}$$

So the solution to this system is $(1, -1, 0, 1)$.

Elementary operations

If a system is not in triangular form, we can manipulate it until it is, by using certain elementary operations.

Definition 5. The **elementary operations** are:

I. Interchange two equations.
II. Multiply an equation by a nonzero constant.
III. Add a multiple of one equation to another.

Performing elementary operations on a system of linear equations produces an equivalent system.

Definition 6. Two systems of linear equations are called **equivalent** iff they have the same solution.

Example 6. To solve the system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 && (1)\\ 3x_1 - x_2 - 3x_3 &= -1 && (2)\\ 2x_1 + 3x_2 + x_3 &= 4 && (3) \end{aligned}$$

we first need to convert it into triangular form using elementary operations.

Multiply (1) by -3 to get $-3x_1 - 6x_2 - 3x_3 = -9$ and add this to (2) to obtain

$$-7x_2 - 6x_3 = -10.$$

Multiply (1) by -2 to get $-2x_1 - 4x_2 - 2x_3 = -6$ and add this to (3) to obtain

$$-x_2 - x_3 = -2.$$

The new equivalent system is

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 && (4)\\ -7x_2 - 6x_3 &= -10 && (5)\\ -x_2 - x_3 &= -2 && (6) \end{aligned}$$

Multiply (5) by $-\frac{1}{7}$ to obtain $x_2 + \frac{6}{7}x_3 = \frac{10}{7}$.
Add this to (6) to obtain $-\frac{1}{7}x_3 = -\frac{4}{7}$.
Multiplying this by -7, we obtain another equivalent system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3\\ x_2 + \tfrac{6}{7}x_3 &= \tfrac{10}{7}\\ x_3 &= 4 \end{aligned}$$

Now the system is in triangular form and can be solved by back-substitution, starting with $x_3 = 4$:

$$\begin{aligned} x_2 + \tfrac{6}{7}\cdot 4 &= \tfrac{10}{7} \implies x_2 = -2\\ x_1 + 2(-2) + 4 &= 3 \implies x_1 = 3 \end{aligned}$$

Example 7. Consider the linear system

$$x_1 + 2x_2 - 3x_3 = -4 \tag{7}$$

$$2x_1 + x_2 - 3x_3 = 4 \tag{8}$$

What's the first thing you notice about this system? It has two equations, and 3 unknowns. So can we still solve it? Well, mostly ...

Begin by eliminating x_1 by multiplying (7) by -2 and adding it the second equation to obtain

$$-3x_2 + 3x_3 = 12. \tag{9}$$

Now solve 9 for x_2 as

$$x_2 = x_3 - 4. \tag{10}$$

Since this is about as far as we can go in solving this system, we let $x_3 = t$, where t is a *parameter* that can be any number, i.e., $t \in \mathbb{R}$ or $-\infty < t < \infty$ or $t \in (-\infty, \infty)$. Now by substituting $x_3 = t$ into (10), we get $x_2 = t - 4$. Now we rewrite equation (7) as

$$\begin{aligned} x_1 &= -4 - x_2 + 3x_3 \\ &= -4 - 2(t-4) + 3t \\ &= t + 4 \end{aligned}$$

and we obtain the solution set $(t+4, t-4, t)$, where $-\infty < t < \infty$. Note that there are an infinite number of solutions, but not just any three numbers (a, b, c) is a solution of the system. A solution needs to have the specific form $(t+4, t-4, t)$.

Definition 7. A **parameter** is a variable, usually with a specified range, which remains as part of the solution; the solution is said to be "given in terms of the parameter". An infinite solution set which is described in terms of a parameter is called a **parametric representation**. If one of the variables has been set equal to the parameter, then it is called a **free variable**.

A parametric representation is not unique; it can be written many ways. For example, you should check that the parametric solution to the system above may also be written as:

$$\begin{aligned} &(r, r-8, r-4), -\infty < r < \infty && x_1 \text{ is a free variable.} \\ &(s+8, s, s+4), -\infty < s < \infty && x_2 \text{ is a free variable.} \\ &(u+2, u-6, u-2), -\infty < u < \infty && \text{No free variable.} \end{aligned}$$

Homework Assignment:

Read: 1-10
Exercises: 1-6,11-18,39-44
Supplement: *Application to Production Planning.*

I.2. Gaussian Elimination and Gauss-Jordan Elimination.

I.2.1. *Matrices.*

Remark. We begin today by introducing the idea of a matrix. Matrices are essentially just a form of shorthand notation for systems of linear equations. You may remember my previous remark about how important it is to keep the variables aligned in their respective columns. The reason for this is that it leads naturally to the representation of the system in matrix form.

Definition 8. A **matrix** is a rectangular array of numbers. An $m \times n$ matrix is a matrix with m rows and n columns:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Definition 9. Each **entry** a_{ij} in the matrix is a number, where i tells what row the number is on, and j tells which column it is in. For example, a_{23} is the number in the second row and third column of the matrix. The subscripts i, j can be thought of as giving the "address" of an entry within the matrix.

Example 8. The following 4×4 matrix gives the airline distances between the indicate cities (in miles).

	London	Madrid	New York	Tokyo
London	0	785	3469	5959
Madrid	785	0	3593	6706
New York	3469	3593	0	6757
Tokyo	5959	6706	6757	0

Example 9. Suppose a manufacturer has four plants, each of which makes three products. If we let a_{ij} denote the number of units of product i made by plant j in one week, then the 3×4 matrix

	Plant 1	Plant 2	Plant 3	Plant 4
Product 1	560	360	380	0
Product 2	340	450	420	80
Product 3	280	270	210	380

gives the manufacturer's production for the week. For example, Plant 2 makes 270 units of Product 3 in one week.

Definition 10. If we have an $m \times n$ matrix where $m = n$, then it is called a **square** matrix. For a square matrix, the entries $a_{11}, a_{22}, \ldots, a_{nn}$ are called the **main diagonal** or sometimes just the **diagonal**.

Remark. We will discuss how to perform arithmetic operations with matrices shortly, that is, how to add two matrices together or what it might mean to multiply two together. First, however, we will consider the application of matrices that we will be using most often, and develop some motivation for why matrices might be important.

Definition 11. The **coefficient matrix** of a system of linear equations is the matrix whose entries a_{ij} represent the coefficient of the jth unknown in the ith equation.

Example 10. Given the linear system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 3x_1 - x_2 - 3x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 4 \end{aligned}$$

which we solved previously, the coefficient matrix of this system is

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -3 \\ 2 & 3 & 1 \end{bmatrix}$$

Definition 12. The **augmented matrix** of a system of linear equations is like the coefficient matrix, but we include the additional column of constants on the for right side.

Example 11. The augmented matrix of the system given above is

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{bmatrix}.$$

Sometimes augmented matrices are written with a bar to emphasize that they are augmented matrices:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{array}\right].$$

Example 12. Note that any term which is missing from an equation (in a system of linear equations) must be represented by a 0 in the coefficient matrix. From the linear system

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ x_2 - x_3 &= -3 \\ 4x_3 &= 2 \end{aligned}$$

the coefficient matrix would be

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{bmatrix}$$

and the augmented matrix would be

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 4 & 2 \end{bmatrix}.$$

Definition 13. Now, as you might expect, we also have the **elementary row operations** for matrices:

I. Interchange two rows.
II. Multiply an row by a nonzero constant.
III. Add one row to another.

Definition 14. Two matrices are said to be **row-equivalent** iff one can be obtained from the other by a sequence of elementary row operations.

Remark. If we have an augmented matrix corresponding to a system of linear equations, then an elementary row operation on this matrix corresponds to an elementary operation on the original system and the resulting matrix corresponds to the new (but equivalent) system of linear equations.You should check how similar these definitions are to the analogous ones for linear systems.

On first glance, it appears that matrices are merely a shorthand notation for solving systems of linear equations, by not having to write the variable names at each step. While this is partially true, using matrices also allows for much greater and more general analysis.

When using matrices to solve systems, we will frequently find it advantageous to have a matrix which has been converted into an equivalent matrix of a much simpler form.

Definition 15. A matrix in **row-echelon form** is a matrix which has the following properties:

1. The first nonzero entry in each row is a 1.
2. The first 1 of each row appears to the right of the first 1 in the row above it.
3. If any row consists entirely of zeroes, it appears at the bottom of the matrix.

Thus a matrix in row-echelon form corresponds to a triangular system of equations with the additional requirement that the first coefficient be a 1.

Definition 16. A matrix in **reduced row-echelon form** is a matrix in row-echelon form which has the additional requirement that the leading 1 of each row has only zeroes above and below it.

Example 13. Each of these matrices is in row-echelon form

$$\begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

but only the last two are in reduced row-echelon form.

I.2.2. *Gaussian Elimination.*

Remark. We now show how to use matrices in row-echelon form to solve systems of equations.

Definition 17. Gaussian elimination is the following method of solving systems of linear equations:

1. Write the system as an augmented matrix.
2. Use elementary row operations to convert this matrix into an equivalent matrix which is in row-echelon form.
3. Write this new matrix as a system of linear equations.
4. Solve this simplified equivalent system using back-substitution.

Essentially, Gaussian elimination is the same technique we were using previously, and as you work a few exercises, you will see exactly how the two relate.

Example 14. We demonstrate how to use Gaussian elimination to solve one of your homework problems. Consider the following system:

$$\begin{aligned} x_1 - 3x_3 &= -2 \\ 3x_1 - 2x_3 + x_2 &= 5 \\ 2x_1 + 2x_2 + x_3 &= 4 \end{aligned}$$

First, we write the system as an augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 3 & 1 & -2 & 5 \\ 2 & 2 & 1 & 4 \end{array}\right]$$

Second, we perform elementary row operations as follows:

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 2 & 2 & 1 & 4 \end{array}\right] \qquad (-3)R_1 + R_2 \to R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 2 & 7 & 8 \end{array}\right] \qquad (-2)R_1 + R_3 \to R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & -7 & -14 \end{array}\right] \qquad (-2)R_2 + R_3 \to R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad (-\tfrac{1}{7})R_3 \to R_3$$

Third, we write this last matrix as a system of equations:

$$\begin{aligned} x_1 \qquad - \ x_3 &= -2 \\ x_2 + 7x_3 &= 11 \\ x_3 &= 2 \end{aligned}$$

Finally, we use back-substitution to obtain

$$\begin{aligned} x_2 + 7 \cdot 2 &= 11 \implies x_2 = -3 \\ x_1 \qquad - \quad 2 &= 2 \implies x_1 = 4 \end{aligned}$$

Thus, Gaussian elimination yields the solution $(4, -3, 2)$.

Definition 18. Gauss-Jordan elimination is the following method of solving systems of linear equations:

1. Write the system as an augmented matrix.
2. Use elementary row operations to convert this matrix into an equivalent matrix which is in reduced row-echelon form.
3. Write this new matrix as a system of linear equations.
4. Solve this simplified equivalent system using back-substitution.

Gauss-Jordan elimination is just an extension of Gaussian elimination where you convert the matrix all the way to *reduced* row-echelon form before converting back to a system of equations.

Example 15. Continuing from the previous example, we could convert the matrix to reduced row-echelon form as follows:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad (3)R_3 + R_1 \to R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad (-7)R_3 + R_2 \to R_2$$

Now when we convert this matrix back into a linear system, we see that it immediately gives the solution $(4, -3, 2)$.

Remark. Whenever you are working with an augmented matrix and you obtain a row which is all zeroes except for the last, then you have an inconsistent system. That is, if you get a row of the form

$$\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & c \end{array}\right]$$

for $c \neq 0$, then the original system of linear equations has no solution.

Definition 19. One particular important and useful kind of system is one in which all the constant terms are zero. Such a system is called a **homogeneous** system. It is a fact that every homogeneous system is consistent (ie, has at least one solution). One easy way to remember this is to notice that every homogeneous system is satisfied by the **trivial solution**, that is, $x_1, x_2, \ldots, x_n = 0$. When you set all variables to zero, the left side of each equation becomes 0.

Example 16. We can solve the homogeneous system

$$\begin{aligned} x_1 + \ \ x_2 + x_3 + x_4 &= 0 \\ x_1 \qquad\qquad\quad + x_4 &= 0 \\ x_1 + 2x_2 + x_3 \qquad\ &= 0. \end{aligned}$$

by Gauss-Jordan elimination as:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

Then letting $x_4 = t$, back-substitution gives the solution as $(-t, t, -t, t)$. For example, $t = 2$ gives the nontrivial solution $(-2, 2, -2, 2)$.

Homework Assignment:

Read: 13-24
Exercises: 7-12,19-28
Supplement: *Application to Simple Economies*

II. MATRICES

II.1. Operations with Matrices.

II.1.1. *Matrix Algebra.*

Remark. I'd like to recall a couple of definitions we had earlier.

Definition 20. An $m \times n$ **matrix** is a rectangular array of numbers with m rows and n columns:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Definition 21. Each number a_{ij} in the matrix is a called an **entry**.

Definition 22. If $m = n$, the matrix is said to be **square**.

Remark. As we discuss matrices and matrix operations today, it will be a good idea for you to note where the size ($m \times n$) of the matrices discussed comes into play. We first see this come into play with the idea of matrix equality.

Definition 23. Two matrices are **equal** iff they are the same size and their corresponding entries are equal.

Example 17. For example, these two matrices are equal

$$\begin{bmatrix} 1 & 5 \\ a_{21} & 3 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 & b_{12} \\ -1 & 3 \end{bmatrix}$$

iff $a_{21} = -1$ and $b_{12} = 5$.

Definition 24. If two matrices A and B are both of the same size, then we define the **sum of A and B** as follows:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m1} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m1} & \cdots & b_{mn} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

Remark. This is probably a good time to introduce some shorthand notation for matrices. In future, we may write the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

in the abbreviated form

$$A = [a_{ij}] .$$

In this notation, the sum of two matrices $A = [a_{ij}]$, $B = [b_{ij}]$ is written

$$A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$

Example 18. For

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 2 & -1 & 3 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 & 2 & -4 \\ 1 & 3 & 1 \end{bmatrix},$$

the sum is given by

$$A + B = \begin{bmatrix} 1+0 & -2+2 & 4-4 \\ 2+1 & -1+3 & 3+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 4 \end{bmatrix}$$

Note that this definition only makes sense when A and B are the same size. If two matrices are of different size, then their sum is undefined.

Definition 25. Scalar multiplication (or "multiplication by a number", or "multiplication by a constant") of a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m1} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]$$

by a scalar c is defined by

$$cA = c \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m1} & \cdots & ca_{mn} \end{bmatrix} = c\,[a_{ij}] = [ca_{ij}]$$

Example 19. If we have the matrix

$$A = \begin{bmatrix} 4 & 8 & 2 \\ 6 & 8 & 10 \end{bmatrix},$$

then two scalar multiples of it are

$$\tfrac{1}{2}A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 4 & 5 \end{bmatrix} \qquad \text{and} \qquad 3A = \begin{bmatrix} 12 & 24 & 6 \\ 18 & 24 & 30 \end{bmatrix}$$

Definition 26. The **product of two matrices** $A = [a_{ij}]$ and $B = [b_{ij}]$ is only defined when the number of columns of A is equal to the number of rows of B. Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix so that the product AB is well-defined. Then AB is defined as follows:

$$AB = [c_{ij}] \quad \text{where} \quad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

In other our previous notation, this would look like

$$AB = \begin{bmatrix} \sum_{k=1}^{n} a_{1k}b_{k1} & \sum_{k=1}^{n} a_{1k}b_{k2} & \cdots & \sum_{k=1}^{n} a_{1k}b_{kp} \\ \sum_{k=1}^{n} a_{2k}b_{k1} & \sum_{k=1}^{n} a_{2k}b_{k2} & \cdots & \sum_{k=1}^{n} a_{ik}b_{kp} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^{n} a_{mk}b_{k1} & \sum_{k=1}^{n} a_{mk}b_{k2} & \cdots & \sum_{k=1}^{n} a_{mk}b_{kp} \end{bmatrix}$$

so that AB is an $m \times p$ matrix. While this formula is hideous and slightly terrifying, you should not be alarmed. In practice, the entries of a product are not too difficult to compute, and there is a very simple mnemonic for remembering which entries from the factor matrices are used: to find the entry in the ith row and jth column of the product, use the ith row of A and the jth row of B. Using full-blown matrix notation, we have

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{a_{i1}} & \boldsymbol{a_{i2}} & \boldsymbol{a_{i3}} & \cdots & \boldsymbol{a_{in}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & \boldsymbol{b_{1j}} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & \boldsymbol{b_{2j}} & \cdots & b_{2p} \\ b_{31} & b_{32} & \cdots & \boldsymbol{b_{3j}} & \cdots & b_{3p} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & \boldsymbol{b_{nj}} & \cdots & b_{np} \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{i1} & c_{i2} & \cdots & \boldsymbol{c_{ij}} & \cdots & c_{ip} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mj} & \cdots & c_{mp} \end{bmatrix}$$

where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \ldots + a_{in}b_{nj}.$$

You can see why A must have the same number of columns as B has rows - otherwise these numbers would not match up equally, and the product wouldn't be well-defined (ie, make sense).

Example 20. Consider the matrices

$$A = \begin{bmatrix} 3 & -2 \\ 2 & 4 \\ 1 & -3 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} -2 & 1 & 3 \\ 4 & 1 & 6 \end{bmatrix}$$

since A has 2 columns and B has 2 rows, the product of these two matrices is well-defined, and given by

$$\begin{aligned} AB &= \begin{bmatrix} 3 & -2 \\ \mathbf{2} & \mathbf{4} \\ 1 & -3 \end{bmatrix} \begin{bmatrix} -2 & 1 & \mathbf{3} \\ 4 & 1 & \mathbf{6} \end{bmatrix} \\ &= \begin{bmatrix} 3\cdot(-2)-2\cdot 4 & 3\cdot 1-2\cdot 1 & 3\cdot 3-2\cdot 6 \\ 2\cdot(-2)+4\cdot 4 & 2\cdot 1+4\cdot 1 & \mathbf{2\cdot 3+4\cdot 6} \\ 1\cdot(-2)-3\cdot 4 & 1\cdot 1-3\cdot 1 & 1\cdot 3-3\cdot 6 \end{bmatrix} \\ &= \begin{bmatrix} -14 & 1 & -3 \\ 12 & 6 & \mathbf{30} \\ -14 & -2 & -15 \end{bmatrix} \end{aligned}$$

Note that B has 3 columns and A has 3 rows, so the product BA is also defined! We compute this product as

$$\begin{aligned} BA &= \begin{bmatrix} -2 & 1 & 3 \\ 4 & 1 & 6 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 2 & 4 \\ 1 & -3 \end{bmatrix} \\ &= \begin{bmatrix} -2\cdot 3+1\cdot 2+3\cdot 1 & -2\cdot(-2)+1\cdot 4+3\cdot(-3) \\ 4\cdot 3+1\cdot 2+6\cdot 1 & 4\cdot(-2)+1\cdot 4+6\cdot(-3) \end{bmatrix} \\ &= \begin{bmatrix} -1 & -1 \\ 20 & -22 \end{bmatrix} \end{aligned}$$

This example illustrates a very important point: when we multiply matrices, AB is not necessarily equal to BA. In fact, they are usually different, and sometimes only one of them will even be defined! Note that in this example, AB and BA do not even have the same size.

Example 21. Let

$$A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 6 \end{bmatrix}.$$

Then we can find the product

$$BA = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 17 & 26 \\ 15 & 24 \end{bmatrix}$$

because B has 2 columns and A has 2 rows. However, the product AB is not even defined! Note that in general, the product matrix gets its height from the first matrix and its width from the second.

Definition 27. A **vector** is a matrix whose height or width is 1. A matrix with only one column is called a **column vector** and a matrix with only one row is called a **row vector**. A vector with 2 entries is called a **2-vector**, a vector with 3 entries is called a **3-vector**, and so on. In general, an n**-vector** is a vector with n entries.

Remark. Vectors come up a lot and have many different interpretations. For the moment, we will treat them just as we treat matrices, although we do notice a couple of special things that occur for matrices of this special form.

Example 22. The following are all vectors:

$$A = \begin{bmatrix} 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 0 & 3 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \qquad D = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

The first two are row vectors and the second two are column vectors.

Remark. The first interesting thing that we notice about vectors is that when we have a matrix times a vector, it can be written in different way:

$$\begin{bmatrix} 1 & 3 & -1 \\ 5 & -4 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2-3+3 \\ 10+4-6 \end{bmatrix}$$
$$= \begin{bmatrix} 2 \\ 10 \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ -6 \end{bmatrix}$$
$$= 2 \begin{bmatrix} 1 \\ 5 \end{bmatrix} - 1 \begin{bmatrix} 3 \\ -4 \end{bmatrix} - 3 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Definition 28. Earlier, we discussed linear equations, which we can think of as linear combinations of numbers a_1, a_2, etc. Now, we are ready to define **linear combinations of vectors** (v_1, v_2, etc.) as sums of the form

$$x_1\mathbf{v_1} + x_2\mathbf{v_2} + \ldots + x_n\mathbf{v_n} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Example 23. Considering the previously discussed matrices, we have

$$\begin{bmatrix} 1 & 3 & -1 \\ 5 & -4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ -4 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

where $x_1 = 2, x_2 = -1, x_3 = -3$.

Remark. The motivation for the seemingly strange definition of matrix multiplication comes from the applications to systems of linear equations, so we will consider this carefully. If we have a system of one equation in one unknown, it looks like

$$ax = b.$$

We generally think of a, x, and b as scalars, but they can also be considered (somewhat oddly) as 1×1 matrices. Now we wish to generalize this simple equation $ax = b$ so that it represents an entire $m \times n$ linear system by a single matrix equation

$$A\mathbf{x} = \mathbf{b}$$

where A is an $m \times n$ matrix, $\mathbf{x}$ is an n-vector, and $\mathbf{b}$ is an m-vector.

Now an $m \times n$ linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

can be written

$$A\mathbf{x} = \mathbf{b}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

because we have defined the product $A\mathbf{x}$ by

$$A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{11}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{21}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{m1}x_n \end{bmatrix}.$$

So you can see that the system of linear equations is equivalent to the matrix equation. In general, we will be working a lot with equations of the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Pay special attention to page 50 in the reading, as the text shows a full comparison of all the equivalent ways we have for writing this equation. These will come up a lot, and developing a good understanding of them now will help just just about everything we do in the rest of the course!

Example 24. Convince yourself that the following are all the same thing:

(1)

$$\begin{aligned} 2x_1 + \ x_2 - 3x_3 &= 2 \\ x_1 - 2x_2 + \ x_3 &= -1 \\ -3x_1 \qquad + 2x_3 &= 4 \end{aligned}$$

(2)

$$\begin{bmatrix} 2x_1 + 1x_2 - 3x_3 \\ 1x_1 - 2x_2 + 1x_3 \\ -3x_1 + 0x_2 + 2x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

(3)

$$x_1 \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

(4)

$$\begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ -3 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

(5)

$$A\mathbf{x} = \mathbf{b}, \text{ where}$$

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ -3 & 0 & 2 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}.$$

Homework Assignment:

Read: 43-52
Exercises: 3-6,11-15,19-24,31-34
Supplement: *Application to Production Costs*

II.2. **Algebraic Properties of Matrix Operations.**

Remark. Mathematical thought proceeds (like scientific thought in general) by studying certain objects until one can extract the salient features of the system and develop rules about how they interact. This mode of thinking is at the base of physics, chemistry, biology, and the other sciences. In mathematics, the objects we're studying are abstract (like numbers, matrices, etc), but the manner of investigation is the same.

For example, when you first learned basic arithmetic, you learned how to add specific numbers, you learned the times tables by heart, and it was not until later in elementary school that you started to develop the properties of numbers in general. Algebra is the distillation of properties of numbers and how they behave with respect to the operations of addition and multiplication. Linear Algebra is the distillation of properties of matrices and how they behave under addition and multiplication, and, as we will see, other operations unique to matrices.

You have probably already seen these properties several times, but now we are going to pay special attention to their names, so we can see the full parallel between algebra and linear algebra.

II.2.1. *Algebraic Properties of Scalars.*

1. Commutative	$a + b = b + a$	additive
	$ab = ba$	multiplicative
2. Associative	$a + (b + c) = (a + b) + c$	additive
	$a(bc) = (ab)c$	multiplicative
3. Identity	$\exists! b$ s.t. $a + b = a$	additive ($b = 0$)
	$\exists! b$ s.t. $a \cdot b = a$	multiplicative ($b = 1$)
4. Inverses	$\exists b$ s.t. $b + a = 0$	additive ($b = -a$)
	$\exists b$ s.t. $b \cdot a = 1$	multiplicative ($b = \frac{1}{a}$)
5. Distributive	$a(b + c) = ab + ac$	mixed
	$(a + b)c = ac + bc$	mixed
6. Zero	$a \cdot 0 = 0$	mixed
	$ab = 0 \implies a = 0$ or $b = 0$	mixed

Even if the names are not familiar, the properties are. Now contrast this with the rules governing matrices (c, d are scalars, A, B, C are matrices):

II.2.2. *Algebraic Properties of Matrices.*

1. Commutative	$A + B = B + A$	additive (matrix)
	$AB \neq BA$	multiplicative (matrix)
2. Associative	$A + (B + C) = (A + B) + C$	additive (matrix)
	$(cd)A = c(dA)$	multiplicative (scalar)
	$A(BC) = (AB)C$	multiplicative (matrix)

3. Identity	$\exists! B$ s.t. $A + B = A$	additive (matrix)
	$\exists! B$ s.t. $AB = A$ (square)	multiplicative (matrix)
4. Inverses	$\exists B$ s.t. $A + B = 0_{mn}$	additive (matrix)
	$\exists B$ s.t. $AB = I_n$ (sometimes)	multiplicative (matrix)
5. Distributive	$c(A + B) = cA + cB$	mixed
	$(c + d)A = cA + dA$	mixed
	$A(B + C) = AB + AC$	matrix
	$(A + B)C = AC + BC$	matrix
6. Zero	$A0_{mn} = 0_{mn}$	matrix
	$cA = 0_{mn} \implies c = 0$ or $A = 0_{mn}$	mixed
	$AB = 0_{mn} \not\Rightarrow A = 0_{mn}$ or $B = 0_{mn}$	matrix

Note that we now have FOUR operations to worry about: matrix addition and matrix multiplication, but we still also have scalar addition, and scalar multiplication.

II.2.3. *Matrix Identities.* We know what the identities and inverses look like for scalars - what do they look like for matrices?

- Additive identity:

$$0_{mn} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

- Multiplicative identity:

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

- Additive inverses of A: $-A = -1 \cdot A$
- Multiplicative inverse of A: $A^{-1}A = AA^{-1} = I_n$

Note the special cases: matrix multiplication is NOT commutative, multiplicative identity is only defined for SQUARE matrices; multiplicative inverses do NOT always exist, and there ARE zero-divisors.

Remark. By multiplicative associativity for matrices, it makes sense to multiply the same matrix with itself multiple times; in other words, exponents are well defined for matrices and we write A^3 for $A \cdot A \cdot A$ (and so on).

Note however, that $A^k \neq [a_{ij}^k]$, and there is no general explicit formula for A^k - it must be worked out by hand. However, to see how the pattern sort of works, let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then

$$A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1+1 & 1+1 \\ 1+1 & 1+1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

Try computing A^3 as an exercise. *Hint*:

$$A^3 = A^2 A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Also, for

$$B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$

try computing B^2 and B^3.

II.3. **Inverses of Matrices.**

Definition 29. We say that an $n \times n$ matrix A is **invertible** iff there exists an $n \times n$ matrix B such that $AB = BA = I_n$.

Remark. A quick test to see if a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible, check that $ad - bc \neq 0$. Note: this only works for 2×2 matrices. We will learn more about why this works in a moment.

Remark. The most important **Properties of Inverses** are

(1) $(A^{-1})^{-1} = A$
(2) $(A^k)^{-1} = A^{-1}A^{-1} \cdots A^{-1} = (A^{-1})^k$
(3) $(cA)^{-1} = \frac{1}{c}A^{-1}, c \neq 0$
(4) $(AB)^{-1} = B^{-1}A^{-1}$

assuming that both A and B are invertible. Note: this shows that if A and B are invertible, then AB is also invertible.

Finding the Inverse of a Matrix

Let A be an $n \times n$ (square) matrix.

(1) Write the $n \times 2n$ matrix that consists of the given matrix A on the left and the identity matrix of order n on the right, to obtain $[A \vdots I]$.
(2) If possible, convert this new augmented matrix into reduced row-echelon form, by using elementary row operations.
(3) If this is not possible, then A is not invertible.
    If this is possible, then the new matrix is $[I \vdots A^{-1}]$.
(4) Check your work by multiplying to see that $AA^{-1} = A^{-1}A = I$.

Example 25. We will find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix}$$

using this method.

$$\left[\begin{array}{rrr|rrr} 1 & 4 & 3 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{rrr|rrr} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right] \quad \begin{array}{r} R_1 + R_2 \to R_2 \\ (-2)R_1 + R_3 \to R_3 \end{array}$$

$$= \left[\begin{array}{rrr|rrr} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right] \quad 3R_2 + R_3 \to R_3$$

$$= \left[\begin{array}{rrr|rrr} 1 & 4 & 0 & \frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 2 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right] \quad \begin{array}{r} (-\frac{1}{2})R_3 + R_1 \to R_1 \\ (-\frac{1}{2})R_3 + R_2 \to R_2 \end{array}$$

$$= \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 2 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right] \quad (-2)R_2 + R_1 \to R_1$$

$$= \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{array}\right] \quad \begin{array}{r} (\frac{1}{2})R_2 \to R_2 \\ (\frac{1}{6})R_3 \to R_3 \end{array}$$

Leaving us with

$$A^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{bmatrix}$$

Now you can check on your own that

$$\begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 26. Using inverse matrices to solve systems.

If A is an invertible matrix, then the system of linear equations $A\mathbf{x} = \mathbf{b}$ has the unique solution given by

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

For example, to solve the system

$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= 12 \\ -x_1 - 2x_2 \quad\quad &= -12 \\ 2x_1 + 2x_2 + 3x_3 &= 8 \end{aligned}$$

we note that the coefficient matrix of this system is the matrix

$$A = \begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix}$$

of the previous example. Therefore, the solution to the system is given by

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 12 \\ -12 \\ 8 \end{bmatrix} = \begin{bmatrix} -6+6+4 \\ -3+3-2 \\ 2-6+\frac{8}{6} \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ -\frac{8}{3} \end{bmatrix}$$

II.3.1. *Zero Properties of Matrices.*

Example 27. Back when we introduced the Zero properties of matrices, I made the comment that with matrices you occasionally encounter zero-divisors, that is, two matrices which can multiply together to produce the zero matrix. Let's see an example of two matrices which can be multiplied together to produce the zero matrix. Suppose

$$A = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$$

so that we have

$$AB = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_{22}$$

Remark. It is precisely because of this last fact that the familiar Law of Cancellation does NOT hold for matrices. For scalars, we have the Law of Cancellation:

$$ab = ac \implies b = c$$

For matrices, it is not true in general that

$$AB = AC \implies B = C.$$

We do, however, have the following result: if C is a invertible matrix, then

$$\begin{aligned} AC = BC &\implies A = B \qquad \text{and} \\ CA = CB &\implies A = B \end{aligned}$$

Homework Assignment:

Read: 55-63,66-76

Exercises: §2.2 3-5,13-15 §2.3 9-18,25-27

Supplement: *Application to Marital Status Models*

II.4. **Properties of Inverses.**

Last time, we finished by saying that the Law of Cancellation does not hold for matrices in general:

$$AB = AC \not\Longrightarrow B = C.$$

Example 28. Consider the following examples. Let

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$$

Then we have

$$AC = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$$

and

$$BC = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$$

so $AC = BC$ but $A \neq B$:

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}, \text{ but}$$

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}.$$

Last time, we saw a list of the various properties of algebraic operations that can be performed on matrices (and scalars). These tell us how we can do arithmetic with matrices: we can add them, subtract them, multiply them, etc. While we cannot really "divide" by a matrix, we do have the following definition:

Definition 30. We say that an $n \times n$ matrix A is **invertible** iff there exists an $n \times n$ matrix B such that $AB = BA = I_n$.

So if a matrix is invertible, we can essentially "divide by the matrix", by multiplying by its inverse, just as we do with scalars (numbers). This allows us to solve the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

by multiplying both sides by A^{-1} and obtaining

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$$
$$\mathbf{x} = A^{-1}\mathbf{b}$$

Thus we saw that it is possible to solve a matrix equation (and hence the entire associated system of linear equations) by computing the matrix product $A^{-1}\mathbf{b}$. This shows how inverses are useful things - and it is a good idea to know how to obtain them. Hence we have the following method:

Finding the Inverse of a Matrix

Let A be an $n \times n$ (square) matrix.

(1) Write the $n \times 2n$ matrix $[A \,\vdots\, I]$.

(2) Try to convert this matrix into the form $[I \,\vdots\, A^{-1}]$, by using elementary row operations.

(3) If this is not possible, then A is not invertible.
If this is possible, then the righthand half of this matrix is A^{-1}.
(4) Check your work by multiplying to see that $AA^{-1} = A^{-1}A = I$.

However, there is a shortcut available for computing the inverse of 2×2 matrices. Earlier, I said that the 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

is invertible precisely when $ad - bc \neq 0$. Today we see why.

Theorem. If we have a 2×2 matrix A given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then A is invertible if and only if $ad - bc \neq 0$, and in this case, the inverse of A is given by

$$A = \tfrac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Proof. To find the inverse of A, we use the method outlined above and convert the following matrix into reduced row-echelon form:

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & \frac{ad-bc}{a} & -\frac{c}{a} & 1 \end{array}\right] \qquad -\tfrac{c}{a}R_1 + R_2 \to R_2$$

$$= \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{ad-bc}{a} & -\frac{c}{a} & 1 \end{array}\right] \qquad \tfrac{1}{a}R_1 \to R_1$$

$$= \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \qquad \tfrac{a}{ad-bc}R_2 \to R_2$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \qquad -\tfrac{b}{a}R_2 + R_1 \to R_1$$

$$\Longrightarrow A^{-1} = \tfrac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

□

Example 29. To see how this trick makes life easier, we will use it to find the inverses of a couple matrices:

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}.$$

Then by the previous theorem, the formula gives

$$A^{-1} = \tfrac{1}{2} \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}.$$

$$B = \begin{bmatrix} -1 & -5 \\ 2 & 6 \end{bmatrix}$$

Then

$$B^{-1} = \tfrac{1}{4}\begin{bmatrix} 6 & 5 \\ -2 & -1 \end{bmatrix}$$

Example 30. Let's use these examples to review some **Properties of Inverses**:

(1) $(cA)^{-1} = \frac{1}{c}A^{-1}, c \neq 0$:

$$\begin{aligned}(2A)^{-1} &= \left(2\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}\right)^{-1} \\ &= \begin{bmatrix} 6 & -2 \\ 4 & 0 \end{bmatrix}^{-1} \\ &= \tfrac{1}{8}\begin{bmatrix} 0 & 2 \\ -4 & 6 \end{bmatrix} \\ &= \tfrac{1}{4}\left(\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}\right) \\ &= \tfrac{1}{2}A^{-1}\end{aligned}$$

(2) $(A^{-1})^{-1} = A$:

$$\begin{aligned}(A^{-1})^{-1} &= \left(\tfrac{1}{2}\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}\right)^{-1} \\ &= 2\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}^{-1} \\ &= 2\left(\tfrac{1}{2}\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}\right) \\ &= A\end{aligned}$$

(3) $(A^k)^{-1} = A^{-1}A^{-1}\cdots A^{-1} = (A^{-1})^k$ (for $k = 2$):

$$(A^2)^{-1} = \left(\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}^2\right)^{-1} = \begin{bmatrix} 7 & -3 \\ 6 & -2 \end{bmatrix}^{-1} = \tfrac{1}{4}\begin{bmatrix} -2 & 3 \\ -6 & 7 \end{bmatrix}$$

$$(A^{-1})^2 = \left(\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}^{-1}\right)^2 = \left(\tfrac{1}{2}\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}\right)^2 = \tfrac{1}{4}\begin{bmatrix} -2 & 3 \\ -6 & 7 \end{bmatrix}$$

II.4.1. *New Operations.* We also have new operations for matrices that do not have scalar counterparts. The first one we will see is the transpose.

Definition 31. We define the **transpose** of an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

to be the $n \times m$ matrix

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

The transpose is essentially formed by writing the columns of the original matrix as rows in the new matrix. In other notation, $A = [a_{ij}] \implies A^T = [a_{ji}]$.

Example 31.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \implies A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \implies B^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Note that it is precisely the diagonal entries which remain fixed.

$$C = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \implies C^T = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

So it is possible for a matrix to be its own transpose.

Definition 32. For a square matrix A, when $A^T = A$, we say A is **symmetric**. While this definition will not come up much, we will run across it again in the last week of the course, if we have time during our discussion diagonalization and eigenvalues.

Remark. For now, we only concern ourselves with the algebraic properties pertaining to the matrix operation of transposition:

(1) $(A^T)^T = A$
(2) $(A + B)^T = A^T + B^T$
(3) $(cA)^T = c(A^T)$
(4) $(AB)^T = B^T A^T$
(5) $(A^T)^{-1} = (A^{-1})^T$

Homework Assignment:

Read: 61-63
Exercises: §2.2 23-26 §2.3 29-33
Supplement: *none*

II.5. **Elementary Matrices.**

Definition 33. An $n \times n$ matrix is called an elementary matrix if it can be obtained from the identity matrix I_n by a single elementary row operation.

Example 32.

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

E_1 comes from I_3 by an application of the first row operation - interchanging two rows.
E_2 comes from I_3 by an application of the second row operation - multiplying one row by a nonzero constant.
E_3 comes from I_3 by an application of the third row operation - adding a multiple of one row to another.

II.5.1. *Representation of Row Operations.*

Example 33. Suppose we have the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{and} \quad E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so that E_1 is the elementary matrix obtained by swapping the first two rows of I_3. Now we work out the matrix products as

$$E_1 A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

$$AE_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 5 & 4 & 6 \\ 8 & 7 & 9 \end{bmatrix}$$

Conclusion: multiplying by E_1 on the left has the effect of swapping the first two rows of A. Multiplying by E_1 on the right has the effect of swapping the first two columns A. (Take another look at E_1 and notice that it can also be described as the elementary matrix resulting from swapping the first two columns of I_3.)

Compare also

$$E_2 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 21 & 24 & 27 \end{bmatrix}$$

So multiplying on the left by E_2 is the same as multiplying the third row by 3, and recall that this is the same operation by which E_2 was obtained from the identity matrix.

We also have

$$E_3 A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1+14 & 2+16 & 3+18 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

So multiplying on the left by E_3 is the same as adding twice the third row to the first, and recall that this is the same operation by which E_3 was obtained from the identity matrix.

This example serves to demonstrate that row operations correspond to (matrix) multiplication by elementary matrices. Everything that can be performed by row operations can similarly be performed using elementary matrices.

Earlier, we gave the definition for two matrices being row-equivalent as: two matrices are row-equivalent iff there is some sequence of row operations which would convert one into the other. Now we rephrase that definition:

Definition 34. Two matrices A and B are row-equivalent iff there is some sequence of elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_k E_{k-1} \ldots E_1 E_2 A = B.$$

This is the same definition as before, it is just stated in different language.

By the way, since we will not be discussing column operations in this course, we will only multiply by elementary matrices on the LEFT.

II.5.2. *Inverses and Elementary Matrices.*

Remark. If E is an elementary matrix, then E is invertible and its inverse E^{-1} is an elementary matrix of the same type. It is very easy to find the inverse of an elementary matrix E - just take the matrix corresponding to the inverse of operation used to obtain E.

Example 34. Since

$$E = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

comes by $2R_3 + R_1 \to R_1$, we choose the operation that would "undo" this, namely, $(-2)R_3 + R_1 \to R_1$. Then the elementary matrix corresponding to this is

$$E^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Theorem. The following conditions are equivalent (that is, each one implies the others):

(1) A is invertible.
(2) A can be written as the product of elementary matrices.
(3) A is row equivalent to I.
(4) The system of n equations in n unknowns given by $A\mathbf{x} = \mathbf{b}$ has exactly one solution.
(5) The system of n equations in n unknowns given by $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $x_1 = x_2 = \ldots = x_n = 0$.

Homework Assignment:

Read: 79-80,82-84
Exercises: §2.4 1-6
Supplement: *none*

Review

- Is it true that $(A^{-2})^{-1} = A^2$?
  We know that $A^{-2} = (A^2)^{-1} = (A^{-1})^2$, by properties of inverses. Thus:

$$\begin{aligned} (A^{-2})^{-1} &= ((A^2)^{-1})^{-1} && \text{by } A^{-2} = (A^2)^{-1} \\ &= A^2 && \text{by } (A^{-1})^{-1} = A \end{aligned}$$

- The brute-force method of reducing a matrix to row-echelon form.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{b}{a} & \frac{c}{a} \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \tfrac{1}{a}R_1 \to R_1$$

$$\sim \begin{bmatrix} 1 & \frac{b}{a} & \frac{c}{a} \\ 0 & e - d\frac{b}{a} & f - d\frac{c}{a} \\ g & h & i \end{bmatrix} \qquad (-d)R_1 + R_2 \to R_2$$

$$\sim \begin{bmatrix} 1 & \frac{b}{a} & \frac{c}{a} \\ 0 & e - d\frac{b}{a} & f - d\frac{c}{a} \\ 0 & h - g\frac{b}{a} & i - g\frac{c}{a} \end{bmatrix} \qquad (-g)R_1 + R_3 \to R_3$$

$$\sim \begin{bmatrix} 1 & \frac{b}{a} & \frac{c}{a} \\ 0 & 1 & \left(f - d\frac{c}{a}\right) / \left(e - d\frac{b}{a}\right) \\ 0 & h - g\frac{b}{a} & i - g\frac{c}{a} \end{bmatrix} \qquad \tfrac{1}{e-d\frac{b}{a}}R_2 \to R_2$$

$$\vdots$$

But there is usually a better way! Look to see what cancels easily! Look to see what zeroes are already in position. For example, rather than attempting to apply the "brute force" method to this matrix:

$$\begin{bmatrix} 5 & 4 & -13 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

you are better off doing a row swap to obtain

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 5 & 4 & -13 \end{bmatrix}$$

and then subtracting 5 times the first from the third:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & -13 \end{bmatrix}.$$

From here, you can see that it is not much more work to finish reducing this matrix. If you had attempted the brute force method, however, you would have obtained

$$\begin{bmatrix} 1 & \frac{4}{5} & -\frac{13}{5} \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

and nobody wants to do business with $\frac{13}{5}$.

- Why reduced row-echelon form really is more reduced than just row-echelon form. For example, you cannot reduce this matrix any further:

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}.$$

But you *can* reduce this matrix (from Quiz 1) further:

$$B = \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

If we convert B back into a system of linear equations, it is clear that there is still back-substitution to be done before the system is solved. However, once this matrix is fully in reduced row-echelon form,

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right],$$

converting back into equations amounts to just writing down the values for x_1, x_2, x_3. There is no back-substitution to be done.

The only time there is *anything* to be done with a reduced row-echelon matrix (when converted back into equations) is when you need to introduce a parameter. For example, the reduced row-echelon matrix:

$$C = \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

would become the system of equations

$$\begin{aligned} x_1 \quad\; +5x_3 &= 2 \\ x_2 + 3x_3 &= 1 \end{aligned}.$$

Then for $x_3 = t$, we would get $x_2 = 1 - 3t$ and $x_1 = 2 - 5t$.

- It doesn't matter which variable you choose to be the param. Consider §1.1, # 15:

$$\begin{aligned} 5x_1 + 2x_2 + x_3 &= 0 \\ 2x_1 + \;\; x_2 \qquad &= 0 \end{aligned}$$

If we apply back-substitution to this system as it is, then we have the options:

(1) Let $x_1 = t$. Then

$$2t + x_2 = 0 \implies x_2 = -2t.$$

Substituting into the first equation, this gives

$$5t - 4t + x_3 = 0 \implies x_3 = -t,$$

so the solution set is $\{(t, -2t, -t)\}, \forall t \in \mathbb{R}$.

(2) Let $x_2 = s$. Then

$$2x_1 + s = 0 \implies x_1 = -\tfrac{s}{2}.$$

Substituting into the first equation, this gives

$$-5\tfrac{s}{2} + 2s + x_3 = 0 \implies x_3 = \tfrac{s}{2},$$

so the solution set is $\{(-\frac{s}{2}, s, \frac{s}{2})\}, \forall t \in \mathbb{R}$.

Even though these look different, they are actually the same answer. To see this, note that we can obtain the first from the second by letting $s = -2t$. (We can do this, because s can be any real number, just like t.) Then

$$\begin{aligned} \{(-\tfrac{s}{2}, s, \tfrac{s}{2})\} &= \{(-\tfrac{-2t}{2}, -2t, \tfrac{-2t}{2})\} \\ &= \{(t, -2t, -t)\} \end{aligned}$$

So these two solution sets are the same. Note also that when we picked x_1 to be the parameter, we ended up with x_1 as the free variable, and when we picked x_2 to be the parameter, we ended up with x_2 as the free variable.

Any of the following would be correct answers to this problems:

$$\begin{aligned} &\{(-\tfrac{t}{2}, t, \tfrac{t}{2})\} \\ &\{(\tfrac{t}{2}, -t, -\tfrac{t}{2})\} \\ &\{(t, -2t, -t)\} \\ &\{(-t, 2t, t)\} \\ &\{(2t, -4t, -2t)\} \end{aligned}$$

What matters is the relations between the different parts: the third must be the negative of the first, and the second must be twice the third. As long as this is true, the answer is okay. This is the *nonuniqueness of parametric representation.*

Example 35. Following §2.3, #12: use inverses to solve the linear system

$$\begin{aligned} 10x_1 + 5x_2 - 7x_3 &= 2 \\ -5x_1 + 1x_2 + 4x_3 &= 1 \\ 3x_1 + 2x_2 - 2x_3 &= 1 \end{aligned}$$

Since this system is equivalent to the matrix equation

$$\underbrace{\begin{bmatrix} 10 & 5 & -7 \\ -5 & 1 & 4 \\ 3 & 2 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\bar{\mathbf{x}}} = \underbrace{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}}_{\bar{\mathbf{b}}},$$

we can solve $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$ by finding $\bar{\mathbf{x}} = A^{-1}\bar{\mathbf{b}}$:

$$\left[\begin{array}{ccc|ccc} 10 & 5 & -7 & 1 & 0 & 0 \\ -5 & 1 & 4 & 0 & 1 & 0 \\ 3 & 2 & -2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 0 & 7 & 1 & 1 & 2 & 0 \\ -5 & 1 & 4 & 0 & 1 & 0 \\ 3 & 2 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 15 & 10 & -10 & 0 & 0 & 5 \\ -15 & 3 & 12 & 0 & 3 & 0 \\ 0 & 7 & 1 & 1 & 2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & \frac{2}{3} & -\frac{2}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 13 & 2 & 0 & 3 & 5 \\ 0 & 1 & \frac{1}{7} & \frac{1}{7} & \frac{2}{7} & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & \frac{2}{3} & -\frac{2}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{7} & -\frac{13}{7} & -\frac{5}{7} & 5 \\ 0 & 1 & \frac{1}{7} & \frac{1}{7} & \frac{2}{7} & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & \frac{2}{3} & -\frac{2}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{7} & \frac{1}{7} & \frac{2}{7} & 0 \\ 0 & 0 & \frac{1}{7} & -\frac{13}{7} & -\frac{5}{7} & 5 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & \frac{2}{3} & -\frac{2}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & 2 & 1 & -5 \\ 0 & 0 & 1 & -13 & -5 & 35 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{2}{3} & -\frac{4}{3} & -\frac{2}{3} & \frac{11}{3} \\ 0 & 1 & 0 & 2 & 1 & -5 \\ 0 & 0 & 1 & -13 & -5 & 35 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -10 & -4 & 27 \\ 0 & 1 & 0 & 2 & 1 & -5 \\ 0 & 0 & 1 & -13 & -5 & 35 \end{array}\right]$$

$$\implies A^{-1} = \begin{bmatrix} -10 & -4 & 27 \\ 2 & 1 & -5 \\ -13 & -5 & 35 \end{bmatrix},$$

$$\text{so } \bar{\mathbf{x}} = A^{-1}\bar{\mathbf{b}} = \tfrac{1}{47}\begin{bmatrix} -10 & -4 & 27 \\ 2 & 1 & -5 \\ -13 & -5 & 35 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -20-4+27 \\ 4+2-5 \\ -26-5+35 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

Example 36. Following §2.3, #14: use inverses to solve the linear system

$$\begin{aligned} 3x_1 + 2x_2 + 5x_3 &= 1 \\ 2x_1 + 2x_2 + 4x_3 &= 3 \\ -4x_1 + 4x_2 \qquad &= -1 \end{aligned}$$

Since this system is equivalent to the matrix equation

$$\underbrace{\begin{bmatrix} 3 & 2 & 5 \\ 2 & 2 & 4 \\ -4 & 4 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\bar{\mathbf{x}}} = \underbrace{\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}}_{\bar{\mathbf{b}}},$$

we can solve $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$ by finding $\bar{\mathbf{x}} = A^{-1}\bar{\mathbf{b}}$:

$$\left[\begin{array}{ccc|ccc} 3 & 2 & 5 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ -4 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} -2 & 2 & 0 & 0 & 0 & \frac{1}{2} \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 3 & 2 & 5 & 1 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} -1 & 1 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 4 & 4 & 0 & 1 & \frac{1}{2} \\ 3 & 2 & 5 & 1 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} -1 & 1 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 1 & 1 & 0 & \frac{1}{4} & \frac{1}{8} \\ 0 & 5 & 5 & 1 & 0 & \frac{3}{4} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} -1 & 1 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 1 & 1 & 0 & \frac{1}{4} & \frac{1}{8} \\ 0 & 0 & 0 & 1 & -\frac{5}{4} & \frac{1}{8} \end{array}\right]$$

Since we have obtained the those three 0's in the beginning of the third row, we will not be able to get this matrix into the form $[I \mid A^{-1}]$, i.e., A is not invertible. Thus, we cannot use inverses to solve this problem and we must do it the old-fashioned way: put the augmented matrix in reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 3 & 2 & 5 & 1 \\ 2 & 2 & 4 & 3 \\ -4 & 4 & 0 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} -2 & 2 & 0 & -\frac{1}{2} \\ 2 & 2 & 4 & 3 \\ 3 & 2 & 5 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} -1 & 1 & 0 & -\frac{1}{4} \\ 0 & 4 & 4 & \frac{5}{2} \\ 3 & 2 & 5 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} -1 & 1 & 0 & -\frac{1}{4} \\ 0 & 1 & 1 & \frac{5}{8} \\ 0 & 5 & 5 & \frac{1}{4} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} -1 & 1 & 0 & -\frac{1}{4} \\ 0 & 1 & 1 & \frac{5}{8} \\ 0 & 0 & 0 & \frac{27}{8} \end{array}\right]$$

Since we have obtained a matrix with a row of the form

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & \frac{27}{8} \end{array}\right],$$

which corresponds to

$$0x_1 + 0x_2 + 0x_3 = \tfrac{27}{8}, \text{ or } 0 = \tfrac{27}{8}$$

(an obvious falsehood), this system has no solution. It is inconsistent.

Example 37. Following §2.3, #10: use inverses to solve the linear system

$$\begin{aligned} x_1+2x_2+2x_3 &= 2 \\ 3x_1+7x_2+9x_3 &= -1 \\ -x_1-4x_2-7x_3 &= 3 \end{aligned}$$

Since this system is equivalent to the matrix equation

$$\underbrace{\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\bar{\mathbf{x}}} = \underbrace{\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}}_{\bar{\mathbf{b}}},$$

we can solve $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$ by finding $\bar{\mathbf{x}} = A^{-1}\bar{\mathbf{b}}$:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 3 & 7 & 9 & 0 & 1 & 0 \\ -1 & -4 & -7 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 5 & -11 & 0 & 1 & -3 \\ -1 & -4 & -7 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 5 & -11 & 0 & 1 & -3 \\ 0 & -2 & -5 & 1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 10 & -22 & 0 & 2 & -6 \\ 0 & -10 & -25 & 5 & 0 & 5 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 10 & -22 & 0 & 2 & -6 \\ 0 & 0 & -47 & 5 & 2 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{22}{10} & 0 & \frac{1}{5} & -\frac{3}{5} \\ 0 & 0 & 1 & -\frac{5}{47} & -\frac{2}{47} & \frac{1}{47} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{57}{47} & \frac{4}{47} & -\frac{2}{47} \\ 0 & 1 & 0 & \frac{11}{47} & \frac{5}{47} & -\frac{26}{47} \\ 0 & 0 & 1 & -\frac{5}{47} & -\frac{2}{47} & \frac{1}{47} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{35}{47} & -\frac{6}{47} & -\frac{50}{47} \\ 0 & 1 & 0 & \frac{11}{47} & \frac{5}{47} & -\frac{26}{47} \\ 0 & 0 & 1 & -\frac{5}{47} & -\frac{2}{47} & \frac{1}{47} \end{array}\right]$$

$$\implies A^{-1} = \tfrac{1}{47}\begin{bmatrix} 35 & -6 & 50 \\ 11 & 5 & -26 \\ -5 & -2 & 1 \end{bmatrix},$$

$$\text{so } \bar{\mathbf{x}} = A^{-1}\bar{\mathbf{b}} = \tfrac{1}{47}\begin{bmatrix} 35 & -6 & 50 \\ 11 & 5 & -26 \\ -5 & -2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \tfrac{1}{47}\begin{bmatrix} 70+6+150 \\ 22-5-78 \\ -10+2+3 \end{bmatrix} = \tfrac{1}{47}\begin{bmatrix} 226 \\ -61 \\ -5 \end{bmatrix}$$

Example 38. Following §2.3, #16: find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

By inspection, it should be clear that

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}.$$

If this isn't clear immediately, consider what we discussed previously about

(1) elementary matrices representing row operations,
(2) how to find the inverse of an elementary matrix,
(3) how an invertible matrix is the product of elementary matrices.

Example 39. Following §2.3, #18: use inverses to solve the linear system

$$\begin{aligned} x_1 \qquad\qquad &= 1 \\ 3x_1 \qquad\qquad &= 3 \\ 2x_1+5x_2+5x_3 &= 7 \end{aligned}$$

Since this system is equivalent to the matrix equation

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & 5 & 5 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\bar{\mathbf{x}}} = \underbrace{\begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}}_{\bar{\mathbf{b}}},$$

we can solve $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$ by finding $\bar{\mathbf{x}} = A^{-1}\bar{\mathbf{b}}$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 & 0 \\ 2 & 5 & 5 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 5 & 5 & -2 & 0 & 1 \\ 0 & 0 & 0 & -3 & 1 & 0 \end{array}\right]$$

Since we have obtained the those three 0's in the beginning of the third row, we will not be able to get this matrix into the form $[I \,|\, A^{-1}]$, i.e., A is not invertible. Thus, we cannot use inverses to solve this problem and we must do it the old-fashioned way: put the augmented matrix in reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 3 & 0 & 0 & 3 \\ 2 & 5 & 5 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 2 & 5 & 5 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 5 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So if we let $x_2 = s$, then

$$s + x_3 = 1 \qquad \Longrightarrow \qquad x_3 = 1 - s.$$

Thus the solution set is

$$\{(1, s, 1 - s)\}, \ \forall s \in \mathbb{R}.$$

The point of this is example is that just because A is not invertible, it doesn't necessarily mean there is no solution. In the earlier example where A^{-1} didn't exist, the system was inconsistent and there was no solution. In this example, there was a solution - in fact, there was an entire parametric family of solutions given by $\{(1, s, 1 - s)\}$, $\forall s \in \mathbb{R}$.

We saw a theorem that said (among other things):

Theorem. The following conditions are equivalent:

(1) A is invertible.

(2) The system of n equations in n unknowns given by $A\mathbf{x} = \mathbf{b}$ has exactly one solution.

So this can be reinterpreted as saying

Theorem. The following conditions are equivalent:

(1) A is not invertible.

(2) The system of n equations in n unknowns given by $A\mathbf{x} = \mathbf{b}$ has either:

  (a) no solution, **or**

  (b) a parametric family of solutions.

Don't make the mistake of assuming a system has no solution, just because the coefficient matrix is not invertible!!!

II.6. **Stochastic Matrices and Introduction to Markov Processes.**

Definition 35. A **stochastic process** is any sequence of experiments for which the outcome at any point depends on chance. A **stochastic matrix** or **transition matrix** is a square matrix with the properties:

(1) Every entry is a number between 0 and 1 (inclusive).
(2) The sum of the entries from any column is 1.

Note: we need the columns to add to 1, not the rows.

A stochastic matrix represents probabilities. Since each entry is between 0 and 1, it can be thought of as a percentage chance or probability of an event coming to pass. Specifically, an entry a_{ij} represents the probability of something in state j making the transition into state i.

Usually, we talk about stochastic matrices when we have a group which is divided into several subgroups; the primary example for today is a population of people divided into various consumer preference groups.

For example, if you look at last Thursday's application supplement, you will notice that the matrix

$$\begin{array}{cc} & \begin{array}{cc}\text{married} & \text{single}\end{array} \\ A = & \left[\begin{array}{cc} 0.70 & 0.20 \\ 0.30 & 0.80 \end{array}\right] \begin{array}{c}\text{married} \\ \text{single}\end{array} \end{array}$$

representing likelihood of change in marital status, is a stochastic matrix. The first row represents women who are married in 1 year, the second row represents women who are single in 1 year. The first column represents all women who are currently married (all = 100%, so column totals to 1), and the second column represents all women who are currently single (same: all=100%). One way to rephrase this is that the first column is a breakdown of the portion of the population which begins in state 1 (married) and the second column is a breakdown of the portion of the population which begins in state 2 (single). So out of the starting population of married women, 30% will be single next year, or there is a 30% chance that any given married woman will be single next year. There is a 70% chance she will still be married. In the language of the definition, the probability is 0.70 that a married woman will make the transition from married to single, in any given year.

You might ask the question, "Does the portion of married women continue to dwindle as time goes on?" The answer to this is yes and no ... while the percentage of married women gets smaller and smaller, it never gets below 40%, for reasons we will see later. (The progression might look something like 40.1%, 40.01%, 40.001%, ...). Consider that on the first year, $\frac{1}{5}$ of 2000 single women get married, while on the second year, $\frac{1}{5}$ of 4000 women get married, so that more women are actually getting married each year, as the portion of married women drops. Clearly, this is a little more complex than it initially appears, and it requires a little more investigation to really see what's going on.

We can write a stochastic (or transition) matrix as

$$P = \left[\begin{array}{cccc} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{array}\right]$$

where $p_{ij} \in [0,1]$ is a number indicating the probability of a member of the jth state being a member of the ith state at the next stage (or next step). For anyone who's studied probability, $0 \leqslant p_{ij} \leqslant 1$ because a probability can be no less than 0% and no greater than 100%.

Example 40. (Consumer Preference Models) Suppose that only two mobile phone carriers (AT&T and Sprint) service a particular area. Every year, Sprint keeps $\frac{1}{4}$ of its customers while $\frac{3}{4}$ switch to AT&T. Every year, AT&T keeps $\frac{2}{3}$ of its customers while $\frac{1}{3}$. This information can be displayed in matrix form as

$$\begin{array}{cc} & \begin{array}{cc}\text{Sprint} & \text{AT\&T}\end{array} \\ A = & \left[\begin{array}{cc} \frac{1}{4} & \frac{1}{3} \\ \frac{3}{4} & \frac{2}{3} \end{array}\right] \begin{array}{l}\text{Sprint} \\ \text{AT\&T}\end{array}\end{array}$$

We can equivalently interpret this matrix as probabilities; for any Sprint customer, there is a 1 in 4 chance they will still be a Sprint customer next year. For any AT&T customer, there is a 2 in 3 chance they will still be an AT&T customer next year. When we begin the market analysis, Sprint has $\frac{3}{5}$ of the market (market = total no. of customers) and AT&T has $\frac{2}{5}$ of the market. Therefore, we can denote the initial distribution of the market by

$$\bar{\mathbf{x}}^{(0)} = \left[\begin{array}{c} \frac{3}{5} \\ \frac{2}{5} \end{array}\right].$$

One year later, the distribution of the market will be given by

$$\begin{aligned} \bar{\mathbf{x}}^{(1)} = A\bar{\mathbf{x}}^{(0)} &= \left[\begin{array}{cc} \frac{1}{4} & \frac{1}{3} \\ \frac{3}{4} & \frac{2}{3} \end{array}\right] \left[\begin{array}{c} \frac{3}{5} \\ \frac{2}{5} \end{array}\right] \\ &= \left[\begin{array}{c} \frac{1}{4}\left(\frac{3}{5}\right) + \frac{1}{3}\left(\frac{2}{5}\right) \\ \frac{3}{4}\left(\frac{3}{5}\right) + \frac{2}{3}\left(\frac{2}{5}\right) \end{array}\right] = \left[\begin{array}{c} \frac{3}{20} + \frac{2}{15} \\ \frac{9}{20} + \frac{4}{15} \end{array}\right] = \left[\begin{array}{c} \frac{9+8}{60} \\ \frac{27+16}{60} \end{array}\right] = \left[\begin{array}{c} \frac{17}{60} \\ \frac{43}{60} \end{array}\right] \end{aligned}$$

This can be readily seen as follows, suppose the initial market consists of $m = 12,000$ people, and no change in this number occurs with time. Initially, Sprint has $\frac{3}{5}m = 7200$ customers and AT&T has $\frac{2}{5}m = 4800$. At the end of the first year, Sprint keeps $\frac{1}{4}$ of its original customers and gains $\frac{1}{3}$ of AT&T's customers. Thus Sprint has

$$\frac{1}{4}\left(\frac{3}{5}m\right) + \frac{1}{3}\left(\frac{2}{5}m\right) = \left[\frac{1}{4}\left(\frac{3}{5}\right) + \frac{1}{3}\left(\frac{2}{5}\right)\right] m = \frac{17}{60}m = 3400$$

after 1 year has passed. Similarly, at the end of the first year, AT&T keeps $\frac{2}{3}$ of its customers and gains $\frac{3}{4}$ of Sprint's customers. Thus AT&T has

$$\frac{3}{4}\left(\frac{3}{5}m\right) + \frac{2}{3}\left(\frac{2}{5}m\right) = \left[\frac{3}{4}\left(\frac{3}{5}\right) + \frac{2}{3}\left(\frac{2}{5}\right)\right] m = \frac{43}{60}m = 8600$$

Similarly, at the end of 2 years, the distribution of the market will be given by

$$\mathbf{x}^{(2)} = A\mathbf{x}^{(1)} = A\left(A\mathbf{x}^{(0)}\right) = A^2\mathbf{x}^{(0)}$$

Example 41. Now, suppose we are given the matrix

$$A = \begin{bmatrix} \frac{1}{4} & \frac{1}{3} \\ \frac{3}{4} & \frac{2}{3} \end{bmatrix}$$

and the initial distribution of the market is denoted by

$$\mathbf{x}^{(0)} = \begin{bmatrix} a \\ b \end{bmatrix}. \qquad (a \text{ and } b \text{ are percentages})$$

Can we determine a and b so that the distribution will be the same from year to year? When this happens, the distribution of the market is said to be **stable**.

Since Sprint and AT&T control the entire market, we must have

$$a + b = 1.$$

We also want the distribution to remain unchanged after the first year, so we require

$$A\mathbf{x}^{(0)} = \mathbf{x}^{(0)}$$

or

$$\overbrace{\begin{bmatrix} \frac{1}{4} & \frac{1}{3} \\ \frac{3}{4} & \frac{2}{3} \end{bmatrix}}^{A} \overbrace{\begin{bmatrix} a \\ b \end{bmatrix}}^{\bar{\mathbf{x}}^{(0)}} = \overbrace{\begin{bmatrix} a \\ b \end{bmatrix}}^{\bar{\mathbf{x}}^{(0)}}$$

so that we get

$$\begin{aligned} \tfrac{1}{4}a + \tfrac{1}{3}b &= a \\ \tfrac{3}{4}a + \tfrac{2}{3}b &= b \end{aligned}$$

or

$$\begin{aligned} -\tfrac{3}{4}a + \tfrac{1}{3}b &= 0 \\ \tfrac{3}{4}a - \tfrac{1}{3}b &= 0 \end{aligned}$$

Since these last two equations are the same,

$$\tfrac{3}{4}a = \tfrac{1}{3}b \implies a = \tfrac{4}{9}b \implies \tfrac{4}{9}b + b = \tfrac{13}{9}b = 1 \implies b = \tfrac{9}{13}, a = \tfrac{4}{13}$$

This problem is an example of a **Markov process**.

II.6.1. *Markov Processes.*

Consider a system that is, at any one time, in one and only one of a finite number of states. For example, the weather in a certain area is either rainy or dry; a person is either a smoker or a nonsmoker; a person either goes or does not go to college; we live in an urban, suburban, or rural area; we are in the lower, middle, or upper income brackets; or we buy a Chevrolet, Ford, or other make of car. As time goes by, the system may move from one state to another, and we assume that the state of the system is observed at fixed time intervals (every day, every year, etc). In many applications, we know the present state of the system and we wish to know the state at the next, or some other future observation period. We often predict the probability of the system being in a particular state at a future observation period from its past history.

Definition 36. A **Markov process** is a process in which

(1) The probability of the system being in a particular state at a given point in time depends only on its state at the immediately preceding observation period.
(2) The probabilities are constant over time (for example, in the case of the marital status example, the probabilities 0.70, 0.30, 0.20, and 0.80 remain the same from year to year).
(3) The set of possible states/outcomes is finite.

Suppose a system has n possible states. For each $i = 1, 2, 3, \ldots, n$, $j = 1, 2, 3, \ldots, n$, let p_{ij} be the probability that if part of the system is in state j at the current time period, then it will be in state i at the next.

Definition 37. A **transition probability** is an entry p_{ij} in a stochastic/transition matrix. That is, it is a number representing the chance that something in state j right now will be in state i at the next time interval.

Example 42. Coca-cola is testing a new diet version of their best-selling soft drink, in a small town in California. They poll shoppers once per month to determine what customers think of the new product. Suppose they find that every month, $\frac{1}{3}$ of the people who bought the diet version decide to switch back to regular, and $\frac{1}{2}$ the people who bought diet decide to switch to the new diet version. Let D denote diet soda buyers, and let R be regular soda buyers. Then the transition matrix of this Markov process is

$$\begin{array}{cc} & \begin{matrix} \text{D} & \text{R} \end{matrix} \\ P = & \begin{bmatrix} \frac{2}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} \end{bmatrix} \begin{matrix} \text{D} \\ \text{R} \end{matrix} \end{array}$$

In this matrix, note that the entries are transition probabilities: $\frac{1}{3}$ represents the probability of a shopper making the transition from Diet to Regular, and $\frac{2}{3}$ represents the probability of a shopper making the transition from Diet to Diet (they can stay the same).

Example 43. A market research organization is studying a large group of caffeine addicts who buy a can of coffee each week. It is found that 50% of those presently using Starbuck's will again buy Starbuck's brand next week, 25% will switch to Peet's, and 25% will switch to some brand. Of those buying Peet's now, 30% will again buy Peet's next week, 60% will switch to Starbuck's, and 10% will switch to another brand. Of those using another brand now, 40% will switch to Starbuck's and 30% will switch to Peet's in the next week. Let S, P, and O denote Starbuck's, Peet's and Other, respectively. The probability that a person presently using S will switch to P is 0.25, the probability that a person presently using P will again buy P is 0.3, and so on. Thus, the transition matrix of this Markov process is

$$P = \begin{array}{c} \begin{array}{ccc} \text{S} & \text{P} & \text{O} \end{array} \\ \begin{bmatrix} 0.50 & 0.60 & 0.40 \\ 0.25 & 0.30 & 0.30 \\ 0.25 & 0.10 & 0.30 \end{bmatrix} \end{array} \begin{array}{c} \\ \text{S} \\ \text{P} \\ \text{O} \end{array}$$

Definition 38. A **probability vector** is a vector

$$\bar{\mathbf{x}} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$$

(1) whose entries p_i are between 0 and 1: $0 \leqslant p_i \leqslant 1$, and
(2) whose entries p_i sum to 1: $p_1 + p_2 + \ldots + p_n = \sum_{i=1}^{n} p_i = 1$

Example 44. Each column of the previous (coffee) transition matrix is a probability vector:

$$\bar{\mathbf{x}}_{\text{S}} = \begin{bmatrix} 0.50 \\ 0.25 \\ 0.25 \end{bmatrix} \qquad \bar{\mathbf{x}}_{\text{P}} = \begin{bmatrix} 0.60 \\ 0.30 \\ 0.10 \end{bmatrix} \qquad \bar{\mathbf{x}}_{\text{O}} = \begin{bmatrix} 0.40 \\ 0.30 \\ 0.30 \end{bmatrix}$$

Each column of the mobile phone matrix from the very beginning is also a probability vector:

$$\bar{\mathbf{x}}_{\text{S}} = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix} \qquad \bar{\mathbf{x}}_{\text{A}} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

Definition 39. The **state vector** of a Markov process at step k is a probability vector

$$\bar{\mathbf{x}}^{(k)} = \begin{bmatrix} p_1^{(k)} \\ p_2^{(k)} \\ \vdots \\ p_n^{(k)} \end{bmatrix}$$

which gives the breakdown of the population at step k. The state vector $\bar{\mathbf{x}}^{(0)}$ is the **initial state vector**.

Example 45. The initial state vector of the mobile phone example was given as

$$\bar{\mathbf{x}}^{(0)} = \begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix}$$

and we computed the state vector after 1 year to be

$$\bar{\mathbf{x}}^{(1)} = \begin{bmatrix} \frac{17}{60} \\ \frac{43}{60} \end{bmatrix}.$$

Back on the application regarding marital status, the initial state vector would be

$$\bar{\mathbf{x}}^{(0)} = \begin{bmatrix} .8 \\ .2 \end{bmatrix}$$

and the state vector after 1 year would be

$$\bar{\mathbf{x}}^{(1)} = \begin{bmatrix} .6 \\ .4 \end{bmatrix}.$$

We will now use the transition matrix of a Markov process to determine the probability of the system being in any of the n states at future times. The following theorem was alluded to in the marital status assignment:

Theorem. If P is the transition matrix of a Markov process, then the state vector $\bar{\mathbf{x}}^{(k+1)}$ at the $(k+1)^{\text{th}}$ step, can be determined by the previous state vector $\bar{\mathbf{x}}^{(k)}$ by

$$\bar{\mathbf{x}}^{(k+1)} = P\bar{\mathbf{x}}^{(k)}$$

So we have:

$$\begin{aligned}
\bar{\mathbf{x}}^{(1)} &= P\bar{\mathbf{x}}^{(0)} \\
\bar{\mathbf{x}}^{(2)} &= P\bar{\mathbf{x}}^{(1)} = P\left(P\bar{\mathbf{x}}^{(0)}\right) = P^2\bar{\mathbf{x}}^{(0)} \\
\bar{\mathbf{x}}^{(3)} &= P\bar{\mathbf{x}}^{(2)} = P\left(P^2\bar{\mathbf{x}}^{(0)}\right) = P^3\bar{\mathbf{x}}^{(0)} \\
&\vdots \\
\bar{\mathbf{x}}^{(k)} &= P^k\bar{\mathbf{x}}^{(0)}
\end{aligned}$$

This means that in general, we can determine the state of the system at the k^{th} step by computing the k^{th} power of the transition matrix. Recall, this is the solution to the third problem on the marital status homework assignment. (In practice, however, it is usually easier to iterate, that is, we use the output from the k^{th} step to find the input for the $(k+1)^{\text{th}}$, rather than computing P^k each time.)

Homework Assignment:

Read: 90-92 "Stochastic Matrices"
Exercises: §2.5 1-12
Supplement: *Markov Processes (exercises)*

II.7. **Markov Process Concepts: Equilibrium and Regularity.** We begin today by recalling some definitions from yesterday.

Definition 40. A **Markov process** is a process in which

(1) The probability of the system being in a particular state at a given point in time depends only on its state at the immediately preceding observation period.
(2) The probabilities are constant over time.
(3) The set of possible states/outcomes is finite.

Definition 41. A **transition probability** is an entry p_{ij} in a stochastic/transition matrix. That is, it is a number representing the chance that something in state j right now will be in state i at the next time interval.

Definition 42. A **probability vector** is a vector $\bar{\mathbf{x}} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$

(1) whose entries p_i are between 0 and 1: $0 \leqslant p_i \leqslant 1$, and
(2) whose entries p_i sum to 1: $p_1 + p_2 + \ldots + p_n = \sum_{i=1}^{n} p_i = 1$

Definition 43. The **state vector** of a Markov process at step k is a probability vector

$$\bar{\mathbf{x}}^{(k)} = \begin{bmatrix} p_1^{(k)} \\ p_2^{(k)} \\ \vdots \\ p_n^{(k)} \end{bmatrix}$$

which gives the breakdown of the population at step k. The state vector $\bar{\mathbf{x}}^{(0)}$ is the **initial state vector**.

Theorem. If P is the transition matrix of a Markov process, then the state vector $\bar{\mathbf{x}}^{(k+1)}$ at the $(k+1)^{\text{th}}$ step, can be determined by the previous state vector $\bar{\mathbf{x}}^{(k)}$ by

$$\bar{\mathbf{x}}^{(k+1)} = P\bar{\mathbf{x}}^{(k)} = P^{k+1}\bar{\mathbf{x}}^{(0)}.$$

Example 46. Let's consider the Coca-cola example again. Suppose that when we begin market observations, the initial state vector is

$$\bar{\mathbf{x}}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

because Coca-Cola is giving away free samples of their product to everyone. (This vector corresponds to 100% of the people getting the diet version.) Then on month 1 (one month after the product launch), the state vector is

$$\bar{\mathbf{x}}^{(1)} = P\bar{\mathbf{x}}^{(0)} = \begin{bmatrix} \frac{2}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

That is, one-third of the people switched back to the regular version immediately. Now in the continuing months,

$$\bar{\mathbf{x}}^{(2)} = P\bar{\mathbf{x}}^{(1)} = \begin{bmatrix} \frac{2}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3}\cdot\frac{2}{3} + \frac{1}{2}\cdot\frac{1}{3} \\ \frac{1}{3}\cdot\frac{2}{3} + \frac{1}{2}\cdot\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{11}{18} \\ \frac{7}{18} \end{bmatrix} \approx \begin{bmatrix} 0.611 \\ 0.389 \end{bmatrix}$$

$$\bar{\mathbf{x}}^{(3)} = P\bar{\mathbf{x}}^{(2)} = \begin{bmatrix} \frac{2}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{11}{18} \\ \frac{7}{18} \end{bmatrix} = \begin{bmatrix} \frac{2}{3}\cdot\frac{11}{18} + \frac{1}{2}\cdot\frac{7}{18} \\ \frac{1}{3}\cdot\frac{11}{18} + \frac{1}{2}\cdot\frac{7}{18} \end{bmatrix} = \begin{bmatrix} \frac{65}{108} \\ \frac{43}{108} \end{bmatrix} \approx \begin{bmatrix} 0.602 \\ 0.398 \end{bmatrix}$$

$$\bar{\mathbf{x}}^{(4)} = P\bar{\mathbf{x}}^{(3)} = \begin{bmatrix} \frac{2}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{65}{108} \\ \frac{43}{108} \end{bmatrix} = \begin{bmatrix} \frac{2}{3}\cdot\frac{65}{108} + \frac{1}{2}\cdot\frac{43}{108} \\ \frac{1}{3}\cdot\frac{65}{108} + \frac{1}{2}\cdot\frac{43}{108} \end{bmatrix} = \begin{bmatrix} \frac{389}{648} \\ \frac{259}{648} \end{bmatrix} \approx \begin{bmatrix} 0.600 \\ 0.400 \end{bmatrix}$$

From the fourth day on, the state vector of the system only gets closer to [0.60 0.40]. A very practical application of this technique might be to answer the question, "If we want our new product to eventually retain x% of the market share, what portion of the population must we initially introduce to the product?" In other words, "How much do we need to give away now in order to make a profit later?"

Example 47. Consider the coffee example again. Suppose that when the survey begins, we find that Starbuck's has 20% of the market, Peet's has 20% of the market, and the other brands have 60% of the market. Then the initial state vector is

$$\bar{\mathbf{x}}^{(0)} = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.6 \end{bmatrix}.$$

The state vector after the first week is

$$\bar{\mathbf{x}}^{(1)} = P\bar{\mathbf{x}}^{(0)} = \begin{bmatrix} 0.50 & 0.60 & 0.40 \\ 0.25 & 0.30 & 0.30 \\ 0.25 & 0.10 & 0.30 \end{bmatrix} \begin{bmatrix} 0.20 \\ 0.20 \\ 0.60 \end{bmatrix} = \begin{bmatrix} 0.4600 \\ 0.2900 \\ 0.2500 \end{bmatrix}$$

Similarly,

$$\bar{\mathbf{x}}^{(2)} = P\bar{\mathbf{x}}^{(1)} = \begin{bmatrix} 0.50 & 0.60 & 0.40 \\ 0.25 & 0.30 & 0.30 \\ 0.25 & 0.10 & 0.30 \end{bmatrix} \begin{bmatrix} 0.4600 \\ 0.2900 \\ 0.2500 \end{bmatrix} = \begin{bmatrix} 0.5040 \\ 0.2770 \\ 0.2190 \end{bmatrix}$$

$$\bar{\mathbf{x}}^{(3)} = P\bar{\mathbf{x}}^{(2)} = \begin{bmatrix} 0.50 & 0.60 & 0.40 \\ 0.25 & 0.30 & 0.30 \\ 0.25 & 0.10 & 0.30 \end{bmatrix} \begin{bmatrix} 0.5040 \\ 0.2770 \\ 0.2190 \end{bmatrix} = \begin{bmatrix} 0.5058 \\ 0.2748 \\ 0.2194 \end{bmatrix}$$

$$\bar{\mathbf{x}}^{(4)} = P\bar{\mathbf{x}}^{(3)} = \begin{bmatrix} 0.50 & 0.60 & 0.40 \\ 0.25 & 0.30 & 0.30 \\ 0.25 & 0.10 & 0.30 \end{bmatrix} \begin{bmatrix} 0.5058 \\ 0.2748 \\ 0.2194 \end{bmatrix} = \begin{bmatrix} 0.5055 \\ 0.2747 \\ 0.2198 \end{bmatrix}$$

$$\bar{\mathbf{x}}^{(5)} = P\bar{\mathbf{x}}^{(4)} = \begin{bmatrix} 0.50 & 0.60 & 0.40 \\ 0.25 & 0.30 & 0.30 \\ 0.25 & 0.10 & 0.30 \end{bmatrix} \begin{bmatrix} 0.5055 \\ 0.2747 \\ 0.2198 \end{bmatrix} = \begin{bmatrix} 0.5055 \\ 0.2747 \\ 0.2198 \end{bmatrix}$$

So as k increases (that is, as time passes), we see that the state vector approaches the fixed vector

$$\bar{\mathbf{x}} = \begin{bmatrix} 0.5055 \\ 0.2747 \\ 0.2198 \end{bmatrix}.$$

This means that in the long run, Starbuck's will command about 51% of the market, Peet's will retain about 27%, and the other brands will have about 22%.

Definition 44. In the last two examples, we have seen that as the number of observation periods increases, the state vectors converge to a fixed vector. In this case, we say that the Markov process has reached **equilibrium**. The fixed vector is called the **steady-state vector**.

Remark. Markov processes are generally used to determine the behavior of a system in the long run; for example, the share of the market that a certain manufacturer can expect to retain on a somewhat permanent basis. Thus the question of whether or not a Markov process reaches equilibrium is of paramount importance.

Example 48. The following example shows that not every Markov process reaches an equilibrium. Let

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad \bar{\mathbf{x}}^{(0)} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{2} \end{bmatrix}$$

Then

$$\bar{\mathbf{x}}^{(1)} = P\bar{\mathbf{x}}^{(0)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 0 + \frac{2}{3} \\ \frac{1}{3} + 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$\bar{\mathbf{x}}^{(2)} = P\bar{\mathbf{x}}^{(1)} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \quad \text{and} \quad \bar{\mathbf{x}}^{(3)} = P\bar{\mathbf{x}}^{(2)} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

Thus the state vector oscillates between the vectors

$$\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

and does not converge to a fixed vector.

However, if we demand that the transition matrix of a Markov process satisfies a rather reasonable property, then we obtain a large class of Markov processes, many of which arise in applications, which *do* reach equilibrium.

Definition 45. A transition matrix P of a Markov process is called **regular** iff all the entries in some power of P are positive; that is, strictly larger than 0. A Markov process is called **regular** iff its transition matrix is regular.

Example 49. The Markov processes in the soda and coffee examples are regular, since all the entries in the transition matrices themselves are regular. That is, if P is the transition matrix, then the first power of P has all nonzero entries.

To see an example which starts out having an entry of 0, but is still regular, consider

$$P = \begin{bmatrix} 0.2 & 1 \\ 0.8 & 0 \end{bmatrix}$$

This transition matrix is regular because

$$P^2 = \begin{bmatrix} 0.2 & 1 \\ 0.8 & 0 \end{bmatrix} \begin{bmatrix} 0.2 & 1 \\ 0.8 & 0 \end{bmatrix} = \begin{bmatrix} 0.04 + 0.8 & 0.2 + 0 \\ 0.16 + 0 & 0.8 + 0 \end{bmatrix} = \begin{bmatrix} 0.84 & 0.2 \\ 0.16 & 0.8 \end{bmatrix}.$$

From here on (i.e., all successive powers of P), P will have strictly positive entries. Why? Notice that subtraction does not appear in the definition of matrix multiplication, and there are no negative numbers in a transition matrix (by definition). So once the numbers are greater than 0, they can never go back to being 0. This brings us to our first "deep" result.

Theorem. If P is the transition matrix of a regular Markov process, then

(1) As k increases, P^k approaches a matrix

$$P^k \xrightarrow{k \to \infty} B = \begin{bmatrix} b_1 & b_1 & \dots & b_1 \\ b_2 & b_2 & \dots & b_2 \\ \vdots & \vdots & \ddots & \vdots \\ b_n & b_n & \dots & b_n \end{bmatrix}$$

all of whose columns are identical. This expression is read "as k increases (as time passes), P^k gets closer and closer to B".

(2) Every column

$$\bar{\mathbf{b}} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

of B is a probability vector whose entries are all positive (nonzero). That is, $b_i > 0$ and $b_1 + b_2 + \ldots + b_n = 1$.

(3) For any probability vector $\bar{\mathbf{x}}$,

$$P^k\bar{\mathbf{x}} \xrightarrow{k\to\infty} \bar{\mathbf{b}}$$

so that $\bar{\mathbf{b}}$ is a steady-state vector. This means that no matter what initial state we begin with, the system will tend toward a distribution of

$$\bar{\mathbf{b}} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

as time goes on.

(4) The steady state vector is the unique probability vector satisfying the matrix equation

$$P\bar{\mathbf{b}} = \bar{\mathbf{b}}.$$

How can we make use of these results? The last part of the theorem actually gives us a great method for determining the steady-state vector of a system. In the Coca-Cola and Coffee examples, we looked for the steady-state vectors by computing the powers $P^k\bar{\mathbf{x}}$. An alternative way of finding the steady-state vector is as follows. First, recall that the theorem tells us that the steady-state vector $\bar{\mathbf{b}}$ satisfies the matrix equation

$$P\bar{\mathbf{b}} = \bar{\mathbf{b}},$$

so we can rewrite this equation as

$$P\bar{\mathbf{b}} = I_n\bar{\mathbf{b}},$$

or

$$(I_n - P)\bar{\mathbf{b}} = \mathbf{0},$$

using the distributivity property for matrix arithmetic. This last equation is a homogeneous system, so we know it is consistent. Moreover, the theorem tells us that it will have a *unique* solution which is a probability vector, i.e., such that

$$b_1 + b_2 + \ldots + b_n = 1.$$

Summary: How to find the steady-state vector of a regular transition matrix.

(1) The first method is iterative:
  (a) Compute the powers $P^k\bar{\mathbf{x}}$ where $\bar{\mathbf{x}}$ is any probability vector.

   (b) Try to determine $\bar{\mathbf{b}}$ as the limit of the $P^k\bar{\mathbf{x}}$: multiply $\bar{\mathbf{x}}$ by P, then multiply the result by P, then multiply the result of *that* by P, and so on.
(2) The second method is:
   (a) Solve the homogeneous system $(I_n - P)\bar{\mathbf{b}} = \mathbf{0}$.
   (b) From the infinitely many solutions obtained this way, determine the unique solution whose components satisfy $b_1 + b_2 + \ldots + b_n = 1$.

Remark. The first method usually requires more work.

Example 50. To see how this works, let's return to the marital status example. For

$$P = \begin{bmatrix} 0.70 & 0.20 \\ 0.30 & 0.80 \end{bmatrix},$$

we have

$$(I_n - P) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.70 & 0.20 \\ 0.30 & 0.80 \end{bmatrix} = \begin{bmatrix} 0.30 & -0.20 \\ -0.30 & 0.20 \end{bmatrix}$$

This gives the homogeneous system

$$\begin{bmatrix} 0.30 & -0.20 \\ -0.30 & 0.20 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

And the reduced row-echelon form of the augmented matrix for this system is

$$\left[\begin{array}{cc|c} 0.30 & -0.20 & 0 \\ -0.30 & 0.20 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 0.30 & -0.20 & 0 \\ 0.00 & 0.00 & 0 \end{array}\right]$$

So if we put $b_2 = t$, the solution set for the system is given by $(\frac{2}{3}t, t)$. The interpretation of this result is that in the long run, the ratio of married to single women will be about $2 : 3$.

Example 51. Now let's return to the coffee example. For

$$P = \begin{bmatrix} 0.50 & 0.60 & 0.40 \\ 0.25 & 0.30 & 0.30 \\ 0.25 & 0.10 & 0.30 \end{bmatrix},$$

we have

$$(I_n - P) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.50 & 0.60 & 0.40 \\ 0.25 & 0.30 & 0.30 \\ 0.25 & 0.10 & 0.30 \end{bmatrix} = \begin{bmatrix} 0.50 & -0.60 & -0.40 \\ -0.25 & 0.70 & -0.30 \\ -0.25 & -0.10 & 0.70 \end{bmatrix}$$

This gives the homogeneous system

$$\begin{bmatrix} 0.50 & -0.60 & -0.40 \\ -0.25 & 0.70 & -0.30 \\ -0.25 & -0.10 & 0.70 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

And the reduced row-echelon form of the augmented matrix for this system is

$$\left[\begin{array}{rrr|r} 0.50 & -0.60 & -0.40 & 0 \\ -0.25 & 0.70 & -0.30 & 0 \\ -0.25 & -0.10 & 0.70 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & -1.20 & -0.80 & 0 \\ -0.25 & 0.70 & -0.30 & 0 \\ 0.25 & 0.10 & -0.70 & 0 \end{array}\right] \begin{array}{l} 2R_1 \to R_1 \\ \\ -R_3 \to R_3 \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & -1.20 & -0.80 & 0 \\ -0.25 & 0.70 & -0.30 & 0 \\ 0 & 0.80 & -1.00 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_2 + R_3 \to R_3 \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & -1.20 & -0.80 & 0 \\ -1 & 2.80 & -1.20 & 0 \\ 0 & 0.80 & -1.00 & 0 \end{array}\right] \; 4R_2 \to R_2$$

$$\sim \left[\begin{array}{rrr|r} 1 & -1.20 & -0.80 & 0 \\ 0 & 1.60 & -2.00 & 0 \\ 0 & 1.60 & -2.00 & 0 \end{array}\right] \begin{array}{c} \\ R_1 + R_2 \to R_2 \\ 2R_3 \to R_3 \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & -1.20 & -0.80 & 0 \\ 0 & 1 & -1.25 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \\ \frac{1}{1.60} R_2 \to R_2 \\ R_3 - R_2 \to R_3 \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & -2.30 & 0 \\ 0 & 1 & -1.25 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1 + 1.2R_2 \to R_2 \\ \\ \\ \end{array}$$

So if we put $b_3 = t$, the solution set for this system is given by $(2.3t, 1.25t, t)$, where t is any real number. To find the steady-state vector, all we do now is pick the solution whose components sum to 1, i.e., let

$$2.3t + 1.25t + t = 1.$$

Then

$$4.55t = 1 \implies t = \tfrac{1}{4.55} \approx 0.2198$$

gives the vector

$$\bar{\mathbf{b}} = \begin{bmatrix} 0.5055 \\ 0.2747 \\ 0.2198 \end{bmatrix}.$$

Note that these numbers agree with our previous calculations of $P^k\bar{\mathbf{x}}$. The interpretation of this result is that in the long run, Starbuck's will retain about 51% of the market, and Peet's will retain about 27%.

Homework Assignment:

Read: none

Exercises: none

Supplement: *Markov Processes (word problems)*

II.8. **The Closed Leontief Input-Output Model.**

Everything we discuss today will be in reference to the Leontief *Closed* Model, and will largely be a review of the second application supplement.

Suppose an economic system has several different industries, each of which has certain input requirements, as well as some sort of product or output. In an application supplement from first week, we saw an example of this concerning Farmers, Manufacturers and Clothing Producers. Let's reconsider this example with some different numbers; and let's change Manufacturers to Carpenters (Carpenters build Housing) and Clothing Producers to Tailors, for brevity. Assume for convenience that each group produces 1 unit per year of whatever they produce. Suppose that during the year, the portion of each commodity consumed by each group is given by

		Goods Produced by:		
		Farmers	Carpenters	Tailors
Goods Consumed by:	Farmers	$\frac{7}{16}$	$\frac{1}{2}$	$\frac{3}{16}$
	Carpenters	$\frac{5}{16}$	$\frac{1}{6}$	$\frac{5}{16}$
	Tailors	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$

Thus the farmers consume $\frac{7}{16}$ of their own produce while the carpenters consume $\frac{5}{16}$ of the farmers' produce and $\frac{5}{16}$ of the tailors' produce, etc. Let p_1 be the price of food, p_2 be the price of tools, and p_3 the price of clothes. (Each is price per unit). We assume everyone pays the same amount for each commodity, so the farmers pay just as much for food as the tailors, even though they grew it. We are interested in determining the prices p_1, p_2, p_3 so that we have a state of equilibrium, that is, no one makes money and no one loses money. Apparently, we are in a communist state.

The farmers' total consumption (or expenditure) is

$$\tfrac{7}{16}p_1 + \tfrac{1}{2}p_2 + \tfrac{3}{16}p_3,$$

and their income is p_1 because they produce one unit of food. If we have expenditures equal to income, then we equate the values of the different quantities to get

$$\tfrac{7}{16}p_1 + \tfrac{1}{2}p_2 + \tfrac{3}{16}p_3 = p_1.$$

Similarly, for the carpenters we have

$$\tfrac{5}{16}p_1 + \tfrac{1}{6}p_2 + \tfrac{5}{16}p_3 = p_2$$

and for the tailors we have

$$\tfrac{1}{4}p_1 + \tfrac{1}{3}p_2 + \tfrac{1}{2}p_3 = p_3.$$

This system of equations can be written in matrix notation as $E\mathbf{p} = \mathbf{p}$ for

$$E = \begin{bmatrix} \frac{7}{16} & \frac{1}{2} & \frac{3}{16} \\ \frac{5}{16} & \frac{1}{6} & \frac{5}{16} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \end{bmatrix} \text{ and } \mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}, \text{ i.e., } \begin{bmatrix} \frac{7}{16} & \frac{1}{2} & \frac{3}{16} \\ \frac{5}{16} & \frac{1}{6} & \frac{5}{16} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}.$$

Now from $E\mathbf{p} = \mathbf{p}$, we use some matrix algebra to obtain a homogeneous system:

$$\begin{aligned} E\mathbf{p} &= \mathbf{p} \\ E\mathbf{p} &= I\mathbf{p} \\ \mathbf{0} &= I\mathbf{p} - E\mathbf{p} \\ \mathbf{0} &= (I - E)\mathbf{p} \end{aligned}$$

We know that a homogeneous solution always has the trivial solution ($p_1 = p_2 = p_3 = 0$ in this case), but then all prices would be zero, which makes no sense. From the last line above, we obtain

$$\begin{aligned}
(I - E) &= \begin{bmatrix} \frac{7}{16} & \frac{1}{2} & \frac{3}{16} \\ \frac{5}{16} & \frac{1}{6} & \frac{5}{16} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{9}{16} & \frac{1}{2} & \frac{3}{16} \\ \frac{5}{16} & -\frac{5}{6} & \frac{5}{16} \\ \frac{1}{4} & \frac{1}{3} & -\frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{9}{16} & \frac{1}{2} & \frac{3}{16} \\ \frac{9}{16} & -\frac{1}{2} & -\frac{3}{16} \\ \frac{1}{4} & \frac{1}{3} & -\frac{1}{2} \end{bmatrix} \quad R_3 + R_2 \to R_2 \\
&= \begin{bmatrix} -\frac{9}{16} & \frac{1}{2} & \frac{3}{16} \\ 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{3} & -\frac{1}{2} \end{bmatrix} \quad R_1 + R_2 \to R_2 \\
&= \begin{bmatrix} -\frac{9}{16} & \frac{1}{2} & \frac{3}{16} \\ \frac{1}{4} & \frac{1}{3} & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_3 \\
&= \begin{bmatrix} \frac{3}{4} & -\frac{2}{3} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{3} & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \quad \left(-\frac{4}{3}\right) R_1 \to R_1 \\
&= \begin{bmatrix} 0 & -\frac{5}{3} & \frac{5}{4} \\ \frac{1}{4} & \frac{1}{3} & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \quad (-3)\, R_2 + R_1 \to R_1
\end{aligned}$$

$$= \begin{bmatrix} \frac{1}{4} & \frac{1}{3} & -\frac{1}{2} \\ 0 & -\frac{5}{3} & \frac{5}{4} \\ 0 & 0 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$= \begin{bmatrix} 1 & \frac{4}{3} & -2 \\ 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 \end{bmatrix} \quad 4R_1 \to R_1, \left(-\tfrac{3}{5}\right) R_2 \to R_2$$

Now the matrix is in row-echelon form and we can rewrite it as a system of equations:

$$\begin{aligned} p_1 + \tfrac{4}{3}p_2 - 2p_3 &= 0 \\ p_2 - \tfrac{3}{4}p_3 &= 0 \end{aligned}$$

Letting $p_3 = t$, this gives

$$\begin{aligned} p_2 - \tfrac{3}{4}t = 0 &\implies p_2 = \tfrac{3}{4}t \\ p_1 + \tfrac{4}{3}(\tfrac{3}{4}t) - 2t = 0 &\implies p_1 = t \end{aligned}$$

So our solution is $(t, \frac{3}{4}t, t)$ or $(4t, 3t, 4t)$ for any $t \in \mathbb{R}$. This means that the prices should be assigned in the ratio $4 : 3 : 4$. For example, if we let $t = 1000$, then food costs \$4000 per unit, tools cost \$3000 per unit, and clothes cost \$4000 per unit. If we are in Japan, then food costs ¥474640 per unit, and manufactured goods cost ¥355980 per unit. In Britain, if one unit of clothes costs £6039, then one unit of manufactured goods costs £4604, etc.

Definition 46. A matrix A with entries a_{ij} is said to be **nonnegative** iff $a_{ij} \geqslant 0$ for each i and j. A matrix A with entries a_{ij} is said to be **positive** iff $a_{ij} > 0$ for each i and j.

Example 52. The transition matrices we discussed previously are all nonnegative matrices, and we saw that steady-state vector of a regular Markov process is always positive.

Warning! The book presents the following definition in a strange and confusing manner. You will want to refer to your notes for this material, as the book will likely only provide a source of confusion.

For the Leontief Closed model, the **Exchange Matrix**[1] is set up as follows: suppose we have n manufacturers $m_1, m_2, \ldots, m_n$, who produce the goods $g_1, g_2, \ldots, g_n$, respectively. Consider a fixed unit of time (say, a year for now) and suppose that m_i makes exactly x_i units of g_i during this time.

In producing g_i, manufacturer m_i may consume amounts of goods $g_1, g_2, \ldots, g_n$. We set up the input-output matrix by saying that m_i uses e_{ij} of g_j (in the manufacture of one unit

[1]Whenever we discuss the Closed model, we will be talking about the Exchange matrix, and whenever we talk about the Exchange matrix, you may assume we are discussing the Closed model.

of g_i). Thus the first row of the matrix consists of the amounts of $g_1, g_2, \ldots, g_n$ consumed by m_1 in the production of g_1. The full matrix looks like

$$E = \begin{bmatrix} e_{11} & e_{12} & \ldots & e_{1n} \\ e_{21} & e_{22} & \ldots & e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1} & e_{n2} & \ldots & e_{nn} \end{bmatrix}$$

Note that for this model to work, we need $0 \leqslant e_{ij} \leqslant 1$ and that the sum of any column must be less than or equal to 1: $e_{1j} + e_{2j} + \ldots + e_{nj} \leqslant 1$.

Definition 47. An $n \times n$ matrix $E = [e_{ij}]$ is called an **exchange matrix** iff it satisfies the following two properties:

(1) $e_{ij} \geqslant 0$ (the matrix is nonnegative).
(2) $e_{1j} + e_{2j} + \ldots + e_{nj} = 1$ (each column sums to 1).

Note that these two conditions together imply that $e_{ij} \leqslant 1$.

Question: is every transition matrix an exchange matrix? Yes - although the numbers have a different interpretation, any matrix satisfying the conditions for being a transition matrix also satisfies the conditions for being an exchange matrix, and vice versa.

Example 53. The matrix $E = \begin{bmatrix} \frac{7}{16} & \frac{1}{2} & \frac{3}{16} \\ \frac{5}{16} & \frac{1}{6} & \frac{5}{16} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \end{bmatrix}$ is an exchange matrix.

Definition 48. If no goods leave the system, and no goods are introduced from an outside source, then the total consumption must equal its total production, i.e.,

$$\underbrace{e_{i1}x_1 + e_{i2}x_2 + \ldots + e_{in}x_n}_{\text{consumed by } m_i} = \underbrace{x_i}_{\text{produced by } m_i}$$

and we say the system is **closed**.

If the price per unit of g_k is p_k, then manufacturer m_i pays $e_{i1}p_1 + e_{i2}p_2 + \ldots + e_{in}p_n$ for the goods he uses.

The problem is to determine the prices $p_1, p_2, \ldots, p_n$ so that no manufacturer makes money or loses money. That is, so each manufacturer's income will equal his expenses. Since m_i manufactures one unit of g_i, his income is p_i. Then this needs to equal his expenses, giving

$$e_{i1}p_1 + e_{i2}p_2 + \ldots + e_{in}p_n = p_i.$$

In general, we get

$$\begin{aligned} e_{11}p_1 + e_{12}p_2 + \ldots + e_{1n}p_n &= p_1 \\ e_{21}p_1 + e_{22}p_2 + \ldots + e_{2n}p_n &= p_2 \\ &\vdots \\ e_{n1}p_1 + e_{n2}p_2 + \ldots + e_{nn}p_n &= p_n, \end{aligned}$$

which can be written in matrix form as $E\bar{\mathbf{p}} = \bar{\mathbf{p}}$ or $(I_n - E)\,\bar{\mathbf{p}} = \mathbf{0}$ where

$$E = [e_{ij}] \quad \text{and} \quad \bar{\mathbf{p}} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}.$$

So we can rephrase our problem as trying to find a nonnegative vector $\bar{\mathbf{p}}$ with at least one positive entry, which satisfies the equation $(I_n - E)\,\bar{\mathbf{p}} = \mathbf{0}$.

Theorem. Given an exchange matrix E, we can always find a nonnegative vector $\bar{\mathbf{p}}$ with at least one positive entry, such that $(I_n - E)\,\bar{\mathbf{p}} = \mathbf{0}$. The point is: if the matrix is an exchange matrix, a solution exists.

Remark. In our general problem, we required that each manufacturer's income equal his expenses. Instead, we could have required that each manufacturer's income not exceed his expenses. This would have led to

$$E\bar{\mathbf{p}} \leqslant \bar{\mathbf{p}}.$$

However, it can be shown that

$$E\bar{\mathbf{p}} \leqslant \bar{\mathbf{p}} \implies E\bar{\mathbf{p}} = \bar{\mathbf{p}}.$$

Thus, if no manufacturer spends more than he earns, everyone's income equals his expenses. An economic interpretation of this statement is that in the Leontief *closed* model, if some manufacturer is making a profit, then at least one manufacturer is taking a loss.

Example 54. (International Trade) Suppose that n countries $c_1, c_2, \ldots c_n$ are engaged in a trade agreement (eg, NAFTA) with each other, and that a common currency is in use. We assume that prices are fixed throughout this example, and that c_j's income (which we call y_j) comes entirely from selling its goods either internally or to other countries. We also assume that some fractions of c_j's income that is spent on imports from c_i is a fixed number e_{ij} which does not depend on c_j's income. Since the e_{ij} are fractions of y_j, we have

$$\begin{aligned} e_{ij} &\geqslant 0 \\ e_{1j} + e_{2j} + \ldots + e_{nj} &= 1 \end{aligned}$$

so that $E = [e_{ij}]$ is an exchange matrix. We now wish to determine the total income y_i for each country c_i. Since the value of c_i's exports to c_j is $e_{ij}y_j$, the total income of c_i is given by

$$e_{i1}y_1 + e_{i2}y_2 + \ldots + e_{in}y_n.$$

Hence, we must have

$$e_{i1}y_1 + e_{i2}y_2 + \ldots + e_{in}y_n = y_i.$$

In matrix notation, we must find a nonnegative vector

$$\bar{\mathbf{y}} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

where at least one of the y_i's is strictly greater than 0 and $E\bar{\mathbf{y}} = \bar{\mathbf{y}}$, which is our earlier problem.

Example 55. Suppose that Canada, Mexico, and the USA are trading wood, petroleum, and metal, respectively, and that the exchange matrix is given by

$$\begin{array}{cc} & \begin{array}{ccc} \text{Can} & \text{Mex} & \text{USA} \end{array} \\ E = & \begin{bmatrix} 0.1 & 0.4 & 0.0 \\ 0.6 & 0.1 & 0.6 \\ 0.3 & 0.5 & 0.4 \end{bmatrix} \begin{array}{c} \text{Can} \\ \text{Mex} \\ \text{USA} \end{array} \end{array}$$

Then

$$I_3 - E = \begin{bmatrix} 0.9 & -0.4 & 0.0 \\ -0.6 & 0.9 & -0.6 \\ -0.3 & -0.5 & 0.6 \end{bmatrix}$$

is the coefficient matrix of a homogeneous system, and this is row-equivalent to

$$\begin{bmatrix} 0 & -1.9 & 1.8 \\ -0.6 & 0.9 & -0.6 \\ 0.3 & 0.5 & -0.6 \end{bmatrix} \begin{array}{c} 3R_3 + R_1 \to R_1 \\ \\ \\ \end{array} \sim \begin{bmatrix} -0.6 & 0.9 & -0.6 \\ 0 & -1.9 & 1.8 \\ 0.6 & 1.0 & -1.2 \end{bmatrix} \begin{array}{c} R_1 \leftrightarrow R_2 \\ \\ 2R_3 \to R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & -\frac{3}{2} & 1 \\ 0 & -1.9 & 1.8 \\ 0 & 1.9 & -1.8 \end{bmatrix} \begin{array}{c} -\frac{1}{0.6}R_1 \to R_1 \\ \\ R_1 + R_3 \to R_3 \end{array} \sim \begin{bmatrix} 1 & 0 & -\frac{16}{38} \\ 0 & 1 & -\frac{18}{19} \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{c} \frac{3}{2}R_2 + R_1 \to R_1 \\ -\frac{1}{1.9}R_2 \to R_2 \\ R_2 + R_3 \to R_3 \end{array}$$

So that if $y_3 = t$, then the solution set will be given by $(\frac{16}{38}t, \frac{18}{19}t, t)$. In other words, prices should be assigned in the ratio 16 : 36 : 38, or the incomes of the three countries are in the ratio 16 : 36 : 38 (same thing).

Homework Assignment:

Read: notes only

Exercises: none

Supplement: *Leontief Closed Model*

II.9. **The Open Leontief Input-Output Model.** We begin today by recalling some definitions from yesterday.

Definition 49. A matrix A with entries a_{ij} is said to be **nonnegative** iff $a_{ij} \geqslant 0$ for each i and j. A matrix A with entries a_{ij} is said to be **positive** iff $a_{ij} > 0$ for each i and j.

Definition 50. An $n \times n$ matrix is called an **input-output matrix** or an **exchange matrix** iff it satisfies the following two properties:

(1) $a_{ij} \geqslant 0$ (the matrix is nonnegative).
(2) $a_{1j} + a_{2j} + \ldots + a_{nj} = 1$ (each column sums to 1).

The exchange matrix indicates how goods are exchanged within the system.

Definition 51. If no goods leave the system, and no goods are introduced from an outside source, then the total consumption must equal its total production, i.e.,

$$\underbrace{a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n}_{\text{consumed by } m_i} = \underbrace{x_i}_{\text{produced by } m_i}$$

and we say the system is **closed**.

We talked yesterday about the closed model and how the primary problem of the closed model is to determine prices: if each industry consumes some goods in the course of operation, then how should we assign prices so that goods may be traded fairly?

The open model is quite different. We are still talking about goods produced by various sectors of an economy, and we still take into account that some industries consume the goods of the other industries in the course of operation. However, that is about where the similarities end. In the open model, we also take external demand into consideration; that is, how much demand is there for a given industry's goods, above and beyond that which is required for the system to sustain itself? In this case, the primary problem become very different. We no longer care about determining prices; our main concern is trying to determine *how much product* is required to meet the demand, and still sustain the system. We will work with equations that look like

$$\begin{aligned} &(\text{goods produced by the system}) \\ &\qquad = (\text{goods consumed by the system}) + (\text{goods consumed externally}) \end{aligned}$$

which can be paraphrased by saying that of whatever is produced by the system, some will be consumed by the operation of the system, and some (if any is left) will go to satisfy external demand for those goods.

Now to formalize this discussion, suppose that we have n goods $g_1, g_2, \ldots, g_n$, and n manufacturers $m_1, m_2, \ldots, m_n$. We still assume that each manufacturer m_i produces only g_i and that g_i is produced only by m_i.

Definition 52. Let $c_{ij} \geqslant 0$ be the dollar value of g_i that has to be consumed in the production of \$1 worth of g_j. The matrix $C = [c_{ij}]$ is called the **consumption matrix**. Note that c_{ii} may be positive, which means that we may require some amount of g_i to make g_i.

Remark. Note the difference between the consumption matrix of the open model and the exchange matrix of the closed model:

- The exchange matrix expresses portions of one producer's output required for another's operation.
- The consumption matrix expresses dollar values of one producer's output required for the production of a dollar amount of another's.

Note also that the entries of the exchange matrix represent roughly the opposite of the entries in the consumption matrix:

- e_{ij} is the amount of g_j used in the manufacture of one unit of g_i.
- c_{ij} is the dollar value of g_i consumed in the production of \$1 worth of g_j.

Definition 53. Let x_i be the dollar value of g_i (that is, the *amount* of x_i, given in dollars) produced in a fixed period of time, say 1 year. The vector

$$\bar{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad (x_i \geqslant 0)$$

is called the **production vector** because it tells how much of each good is produced.

Remark. Note that we are using x's to represent the production vector, because it is now "the amount to be produced" which is our unknown.

Now if we wish to determine the total value of the product g_i that is consumed[2] we have that the expression

$$c_{i1}x_1 + c_{i2}x_2 + \ldots + c_{in}x_n.$$

Observe that the above expression is the i^{th} entry of the matrix product $C\bar{\mathbf{x}}$:

$$C\bar{\mathbf{x}} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_{11}x_1 + c_{12}x_2 + \ldots + c_{1n}x_n \\ c_{21}x_1 + c_{22}x_2 + \ldots + c_{2n}x_n \\ \vdots \\ c_{n1}x_1 + c_{n2}x_2 + \ldots + c_{nn}x_n \end{bmatrix}$$

[2]As determined by the production vector, that is, to make \$$x_1$ worth of g_1, \$$x_2$ worth of g_2, and so on.

Definition 54. The difference between the dollar amount of the g_i that is produced and the total dollar value of g_i that is consumed,

$$x_i - (c_{i1}x_1 + c_{i2}x_2 + \ldots + c_{in}x_n),$$

is called the **net production**. Observe that this expression is the i^{th} entry of the matrix

$$\bar{\mathbf{x}} - C\bar{\mathbf{x}} = (I_n - C)\,\bar{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} - \begin{bmatrix} c_{11}x_1 + c_{12}x_2 + \ldots + c_{1n}x_n \\ c_{21}x_1 + c_{22}x_2 + \ldots + c_{2n}x_n \\ \vdots \\ c_{n1}x_1 + c_{n2}x_2 + \ldots + c_{nn}x_n \end{bmatrix}$$

$$= \begin{bmatrix} x_1 - (c_{11}x_1 + c_{12}x_2 + \ldots + c_{1n}x_n) \\ x_2 - (c_{21}x_1 + c_{22}x_2 + \ldots + c_{2n}x_n) \\ \vdots \\ x_n - (c_{n1}x_1 + c_{n2}x_2 + \ldots + c_{nn}x_n) \end{bmatrix}$$

The net production tells how much of a good is produced, beyond that which is required to sustain the system - it indicates how much is available for external demand.

We will return to this matrix in just a moment.

Definition 55. Now suppose that we let d_i represent the dollar value of outside demand for g_i, and let

$$\bar{\mathbf{d}} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} \qquad (d_i \geqslant 0)$$

This vector tells us that outside the system, there is a demand for $\$d_1$ of g_1, $\$d_2$ of g_2, etc.

Definition 56. We can now state the problem of the **Leontief Open Model**: Given a demand vector $\bar{\mathbf{d}} \geqslant \mathbf{0}$, can we find a production vector $\bar{\mathbf{x}}$ such that the outside demand $\bar{\mathbf{d}}$ is met without any surplus? That is, can we find a vector $\bar{\mathbf{x}} \geqslant \mathbf{0}$ such that the following equation is satisfied?

$$(I_n - C)\bar{\mathbf{x}} = \bar{\mathbf{d}}$$

Example 56. Suppose we are considering a simple economic system consisting of Fuel and Machines. Machines are needed to produce fuel, and fuel is needed to run the machines. Suppose the consumption matrix of the system is given by

$$C = \begin{array}{c} \begin{matrix} \text{F} & \text{M} \end{matrix} \\ \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \end{array} \begin{matrix} \text{F} \\ \text{M} \end{matrix} = \begin{array}{c} \begin{matrix} \text{F} & \text{M} \end{matrix} \\ \begin{bmatrix} \$0.25 & \$0.50 \\ \$0.67 & \$0.33 \end{bmatrix} \end{array} \begin{matrix} \text{F} \\ \text{M} \end{matrix}$$

Then

$$I_2 - C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{2} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

Now the equation $(I_n - C)\bar{\mathbf{x}} = \bar{\mathbf{d}}$ becomes

$$\begin{bmatrix} \frac{3}{4} & -\frac{1}{2} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

which gives us

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{2} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix}^{-1} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}.$$

Now we use our techniques for finding the inverse of a matrix to determine that

$$\begin{bmatrix} \frac{3}{4} & -\frac{1}{2} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix}^{-1} = \begin{bmatrix} 4 & 3 \\ 4 & \frac{9}{2} \end{bmatrix}.$$

So this gives us

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 4 & \frac{9}{2} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \geqslant \mathbf{0},$$

where the inequality ($\geqslant \mathbf{0}$) at the end follows from the fact that $d_1 \geqslant 0$ and $d_2 \geqslant 0$.

Thus, we can obtain a production vector for any given demand vector. If there is a demand for \$8 worth of fuel and \$6 worth of machines, then

$$\bar{\mathbf{d}} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

and the optimal production is

$$\bar{\mathbf{x}} = \begin{bmatrix} 4 & 3 \\ 4 & \frac{9}{2} \end{bmatrix} \begin{bmatrix} 8 \\ 6 \end{bmatrix} = \begin{bmatrix} 32+18 \\ 32+27 \end{bmatrix} = \begin{bmatrix} 50 \\ 59 \end{bmatrix}.$$

Summary. Given a consumption matrix C and a demand vector $\bar{\mathbf{d}}$, we can find the production vector $\bar{\mathbf{x}}$ by evaluating

$$\bar{\mathbf{x}} = (I_n - C)^{-1}\bar{\mathbf{d}}.$$

For $C = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$ and $\bar{\mathbf{d}} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$ we find that $\bar{\mathbf{x}} = \begin{bmatrix} 50 \\ 59 \end{bmatrix}$.

This means that in order to produce \$8 of fuel and \$6 of machines for sale on the free market, the system needs to produce \$50 worth of fuel (because it will use \$42 worth of fuel during production). It also needs to produce \$59 worth of machines, in order to have \$6 worth left over to sell. (Yes, this business does not seem particularly efficient ...)

In general, if $(I_n - C)^{-1}$ exists and is nonnegative, then $\bar{\mathbf{x}} = (I_n - C)^{-1}\bar{\mathbf{d}}$ is a production vector (and hence is also nonnegative) for any given demand vector. However, in general (eg, if these conditions are not satisfied) there may not be a solution to the equation $(I_n - C)\bar{\mathbf{x}} = \bar{\mathbf{d}}$.

Example 57. To see an example of this, consider the consumption matrix

$$C = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

Then

$$I_2 - C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{4} \end{bmatrix}$$

and

$$(I_2 - C)^{-1} = \frac{1}{\frac{1}{8} - \frac{1}{4}} \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = -8 \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ -4 & -4 \end{bmatrix}$$

Now if we are given some demand vector $\bar{\mathbf{d}} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ $(d_i \geqslant 0)$, then

$$(I_2 - C)^{-1}\,\bar{\mathbf{d}} = \begin{bmatrix} -2 & -4 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} -2d_1 - 4d_2 \\ -4d_1 - 4d_2 \end{bmatrix} = \bar{\mathbf{x}}.$$

But this $\bar{\mathbf{x}}$ is not a production vector except for the case when $\bar{\mathbf{d}} = 0$. Thus the problem has no solution. If $\bar{\mathbf{d}} = \mathbf{0}$, then we do have a solution; namely, $\bar{\mathbf{x}} = \mathbf{0}$. The interpretation of this result is: if there is no outside demand, then *nothing* is produced. This example brings us to the idea of when a consumption matrix can be considered worthwhile.

Definition 57. An $n \times n$ consumption matrix C is called **productive** iff $(I_n - C)^{-1}$ exists and $(I_n - C)^{-1} \geqslant \mathbf{0}$. That is, C is productive iff $(I_n - C)$ is invertible and $(I_n - C)^{-1}$ has no negative entries. In this case, we also sometimes say the entire Leontief model is **productive**.

Remark. It follows that if C is productive, then for any demand vector $\bar{\mathbf{d}} \geqslant \mathbf{0}$, the equation $(I_n - C)\bar{\mathbf{x}} = \bar{\mathbf{d}}$ has a unique solution $\bar{\mathbf{x}} \geqslant \mathbf{0}$.

Example 58. Consider the consumption matrix $C = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{3} \end{bmatrix}$ Then

$$I_2 - C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{3} \\ -\frac{1}{4} & \frac{2}{3} \end{bmatrix}, \text{ and}$$

$$(I_2 - C)^{-1} = \frac{1}{\frac{1}{3} - \frac{1}{12}} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} = 4 \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

Thus C is productive. If $\bar{\mathbf{d}} \geqslant \mathbf{0}$ is a demand vector, then the equation $(I_n - C)\bar{\mathbf{x}} = \bar{\mathbf{d}}$ has the unique solution $\bar{\mathbf{x}} = (I_n - C)^{-1}\bar{\mathbf{d}} \geqslant \mathbf{0}$.

Homework Assignment: Leontief Open Model handout

III. DETERMINANTS

III.1. The Determinant of a Matrix.

Every square matrix is associated with a real number called the determinant. The value of this number will tell us if the matrix is invertible.

Definition 58. The **determinant** of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by $|A| = ad - bc$.

Remark. The easy way to remember this formula is as the difference of the products of the diagonals. This formula should be familiar to you - we saw it before in our discussion of inverses. In fact, we can now rephrase our criterion for invertibility as

"A 2×2 matix A is invertible iff $|A| \neq 0$."

The determinant of A is written as $|A|$ or $\det(A)$.

Example 59. Let $A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}$. Then the determinant of A is

$$|A| = \begin{vmatrix} -1 & 0 \\ 2 & 3 \end{vmatrix} = -1(3) + 2(0) = -3.$$

Let $B = \begin{bmatrix} \frac{1}{4} & 2 \\ 3 & -4 \end{bmatrix}$. Then the determinant of B is

$$|B| = \begin{vmatrix} \frac{1}{4} & 2 \\ 3 & -4 \end{vmatrix} = \tfrac{1}{4}(-4) + 3(2) = 5.$$

The determinant of a matrix can be positive, negative, or zero.

Remark. So far, we have only defined determinants for 2×2 matrices. How should they be defined for larger matrices? There is an inductive definition for the determinant of an $n \times n$ matrix that will work for any n; however, in this class we will never need to calculate determinants for matrices larger than 3×3. Fortunately, there is a much easier formula for the determinant of 3×3 matrices.

Definition 59. The **determinant** of a 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is given by

$$\begin{aligned} |A| =& a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12} \end{aligned}$$

which looks ornery, but can easily be remembered by the following diagram:

$$\left| \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix} \right| \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

Example 60. If

$$A = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix},$$

then we can find the determinant $|A|$ as follows:

(1) First write

$$\left| \begin{array}{ccc|cc} 2 & 5 & 4 & 2 & 5 \\ 3 & 1 & 2 & 3 & 1 \\ 5 & 4 & 6 & 5 & 4 \end{array} \right.$$

(2) Then we get

$$\begin{aligned} |A| &= 2 \cdot 1 \cdot 6 \ + \ 5 \cdot 2 \cdot 5 \ + \ 4 \cdot 3 \cdot 4 \quad - \ 5 \cdot 1 \cdot 4 \ - \ 4 \cdot 2 \cdot 2 \ - \ 6 \cdot 3 \cdot 5 \\ &= 12 + 50 + 48 - 20 - 16 - 90 \\ &= -16 \end{aligned}$$

Remark. A moment ago, I mentioned that this is actually a shortcut for finding the determinant of a 3×3 matrix. For completion, I would like to give you an idea of what's going on behind the scenes. The full method uses minors and cofactors, and the book talks about these in detail, but I think it will suffice to illustrate the technique by example. Suppose we want to find the determinant of the same matrix,

$$A = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix}.$$

(1) Begin by picking any row or column. Suppose we choose the first column.
(2) Write down the first entry (2). Now draw a line through the row and the column containing that entry.
(3) Write the remaining (uncrossed) numbers as their own, smaller matrix:

$$A_{11} = \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix}.$$

(4) Proceed to the next number in the column (3). Again, draw a line through the row and the column containing it.
(5) Write the remaining (uncrossed) numbers as their own, smaller matrix:

$$A_{21} = \begin{bmatrix} 5 & 4 \\ 4 & 6 \end{bmatrix}.$$

(6) Proceed to the next number in the column (5). Again, draw a line through the row and the column containing it.
(7) Write the remaining (uncrossed) numbers as their own, smaller matrix:

$$A_{31} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}.$$

(8) Now the determinant can be found as

$$\begin{aligned} |A| &= a_{11}A_{11} - a_{21}A_{21} + a_{31}A_{31} \\ &= 2\begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix} - 3\begin{vmatrix} 5 & 4 \\ 4 & 6 \end{vmatrix} + 5\begin{vmatrix} 5 & 4 \\ 1 & 2 \end{vmatrix} \\ &= 2\,(6-8) - 3\,(30-16) + 5\,(10-4) \\ &= -4 - 42 + 30 \\ &= -16 \end{aligned}$$

This method works for any square matrix, regardless of size, and that is its chief advantage. Clearly, however, it involves much more work, even for the case of 3×3 matrices.

Note: the answer is independent of whichever row or column you choose. Try verifying the determinant of this matrix using a different row or column, to see how this works.

Definition 60. A **triangular** or **upper triangular** matrix is one in which all entries below the main diagonal are 0. A **lower triangular** matrix is one in which all entries above the main diagonal are 0. An example of an upper triangular matrix would be

$$A = \begin{bmatrix} 7 & 1 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

An example of an lower triangular matrix would be

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix}.$$

Definition 61. A **diagonal** matrix is one in which all nonzero entries lie on the main diagonal. An example of a diagonal matrix would be

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

A diagonal matrix is both upper and lower triangular.

Remark. It should be clear by brief inspection that the determinant of a triangular or diagonal matrix is very easy to calculate - it just consists of the product of the entries on the main diagonal. (Add the diagram to the above examples!)

Homework Assignment:

Read: pp. 111-112
Exercises: §3.1 #3-10, 19-22, 25-26 (Not cofactors)
Supplement: *none*

III.2. **Evaluation of a Determinant Using Elementary Operations.**

Example 61. Note that if we have many zeroes in the matrix, then it will be very easy to calculate the determinant. If we consider the matrix

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 4 & 5 & 0 \\ 1 & 0 & 3 \end{bmatrix},$$

Then the diagram

$$\left| \begin{array}{ccc|cc} 2 & 3 & 0 & 2 & 3 \\ 4 & 5 & 0 & 4 & 5 \\ 1 & 0 & 3 & 1 & 0 \end{array} \right.$$

shows that only the products $2 \cdot 5 \cdot 3$ and $3 \cdot 4 \cdot 5$ do not contain a zero. So the determinant is easily calculated as

$$2 \cdot 5 \cdot 3 - 3 \cdot 4 \cdot 5 = 30 - 60 = -30.$$

The conclusion to draw is that we would much rather take the determinant of a matrix which contains many zeroes. Fortunately, we have some tools which allow us to convert matrices without zeroes into matrices with zeroes - elementary row operations. Unfortunately, the row operations change the value of the determinant.

Theorem. Let A and B be square matrices.

(1) If B is obtained from A by interchanging two rows of A, then

$$|B| = -|A|.$$

(2) If B is obtained from A by multiplying a row of A by a nonzero constant c, then

$$|B| = c|A|.$$

(3) If B is obtained from A by adding a multiple of one row of A to another, then

$$|B| = |A|.$$

Example 62. Suppose

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 6 & -3 & 4 \end{bmatrix}$$

and we wish to evaluate the determinant $|A|$. Then

$$\begin{aligned}
\begin{vmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 6 & -3 & 4 \end{vmatrix} &= \begin{vmatrix} 2 & 1 & 3 \\ 0 & 0 & -5 \\ 6 & -3 & 4 \end{vmatrix} \quad -2R_1 + R_2 \to R_2 \\
&= \begin{vmatrix} 2 & 1 & 3 \\ 0 & 0 & -5 \\ 0 & -6 & -5 \end{vmatrix} \quad -3R_1 + R_3 \to R_3 \\
&= (-1) \begin{vmatrix} 2 & 1 & 3 \\ 0 & -6 & -5 \\ 0 & 0 & -5 \end{vmatrix} \quad R_2 \leftrightarrow R_3 \\
&=(-1)(2)(-6)(-5) \\
&= -60
\end{aligned}$$

Theorem. Conditions that yield a zero determinant.
If we are considering a matrix A, then $|A| = 0$ whenever

(1) An entire row or column consists of zeroes.
(2) One row is a multiple of another. In particular, if two rows are equal.

Example 63. Let's practice by evaluating the following determinants by inspection:

(1) $\begin{vmatrix} 0 & 0 & 3 \\ 0 & 4 & 1 \\ 2 & 3 & 1 \end{vmatrix} = (-1) \begin{vmatrix} 2 & 3 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 3 \end{vmatrix} = (-1)(2)(4)(3) = -24$

(2) $\begin{vmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 2 & 2 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{vmatrix} = 2$

(3) $\begin{vmatrix} 0 & 0 & 1 \\ 3 & 0 & 0 \\ 0 & 2 & 0 \end{vmatrix} = (-1) \begin{vmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{vmatrix} = (-1)^2 \begin{vmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 6$

Homework Assignment:

Read: pp. 120-123 (skip column operations), 124-126
Exercises: §3.2 #6,7,9,10,15-22
Supplement: *none*

III.3. **Properties of Determinants.**

Theorem. Let A and B be square matrices. Recall that we had the following results:

(1) If B is obtained from A by interchanging two rows of A, then
$$|B| = -|A|.$$

(2) If B is obtained from A by multiplying **a row of** A by a nonzero constant c, then
$$|B| = c|A|.$$

(3) If B is obtained from A by adding a multiple of one row of A to another, then
$$|B| = |A|.$$

Remark. We will now see some of the algebraic properties of determinants. The chief motivation for this section is the development of some tools which will provide easier ways to compute determinants.

Theorem. If A and B are both $n \times n$ matrices, then $|AB| = |A| \cdot |B|$.

Example 64. Let
$$A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \\ 2 & 1 & 1 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then we have that
$$|A| = \begin{vmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \\ 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -5 & -4 \\ 1 & 4 & 2 \\ 0 & -7 & -3 \end{vmatrix} = (-1)^3 \begin{vmatrix} 1 & 4 & 2 \\ 0 & 5 & 4 \\ 0 & 7 & 3 \end{vmatrix} = (-1)(15-14) = -1$$

and
$$|E| = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (-1) \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1$$

So $|E| \cdot |A| = -1(-1) = 1$. Also,
$$|EA| = \begin{vmatrix} 1 & 4 & 2 \\ 2 & 3 & 0 \\ 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 2 \\ 0 & -5 & -4 \\ 0 & -7 & -3 \end{vmatrix} = (-1)^2 \begin{vmatrix} 1 & 4 & 2 \\ 0 & 5 & 4 \\ 0 & 7 & 3 \end{vmatrix} = 1$$

The point of this example is to illustrate why the formula $|AB| = |A| \cdot |B|$ explains the rules we had previously for how elementary row operations change the determinant of a matrix.

Example 65. Consider the following elementary matrices and their determinants:

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad |E_1| = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{bmatrix} \qquad |E_2| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{vmatrix} = c\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = c$$

$$E_3 = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad |E_3| = \begin{vmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

and recall that elementary matrices represent elementary row operations.

Theorem. If A is an $n \times n$ matrix and c is a scalar, then $|cA| = c^n|A|$.

Example 66. Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 2 \\ 2 & -2 & 0 \end{bmatrix}$ and $c = 5$. Then

$$|cA| = \begin{vmatrix} 5 & 10 & 0 \\ 0 & -5 & 10 \\ 10 & -10 & 0 \end{vmatrix} = 5\begin{vmatrix} 1 & 2 & 0 \\ 0 & -5 & 10 \\ 10 & -10 & 0 \end{vmatrix} = 5^2\begin{vmatrix} 1 & 2 & 0 \\ 0 & -1 & 2 \\ 10 & -10 & 0 \end{vmatrix} = 5^3\begin{vmatrix} 1 & 2 & 0 \\ 0 & -1 & 2 \\ 2 & -2 & 0 \end{vmatrix}$$

Remark. It is worth noting that there is no formula for $|A+B|$ in terms of $|A|$ and $|B|$. To find $|A+B|$, you must compute $A+B$ and then find its determinant.

Example 67. Let $A = \begin{bmatrix} 0 & 3 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 0 & 4 \end{bmatrix}$. Then

$$|A| = 3 \quad \text{and} \quad |B| = 4$$

but

$$|A+B| = \begin{vmatrix} 1 & 2 \\ -1 & 6 \end{vmatrix} = 6 + 2 = 8.$$

Theorem. If A is a square matrix, then $|A| = |A^T|$.

Example 68. To see how this might be useful, recall that the transpose converts rows to columns and vice versa. Suppose you have

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -6 \\ 1 & -1 & -3 \end{bmatrix}$$

and you need to find $|A|$. Notice that the third column of the matrix is a multiple of the second. Using the theorem,

$$|A| = |A^T| = \begin{vmatrix} 2 & 0 & 1 \\ 1 & -2 & -1 \\ 3 & -6 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 1 \\ 1 & -2 & -1 \\ 0 & 0 & 0 \end{vmatrix} = 0.$$

Theorem.

(1) A square matrix A is invertible iff $|A| \neq 0$. We saw this rule earlier for 2×2 matrices. Now we point out that it is true in general (for all square matrices).

(2) If $|A| \neq 0$, that is, if A is invertible, then we have the formula

$$|A^{-1}| = \frac{1}{|A|} \quad \text{or} \quad |A^{-1}| = |A|^{-1}.$$

Remark. In the last section, we had a theorem which gives a couple of conditions for when a determinant of a matrix is zero. In light of this theorem, we can consider these conditions as criteria for determining if a matrix is not invertible. In other words, if

(1) an entire row or column consists of zeroes, or

(2) one row is a multiple of another,

then we know the matrix is not invertible.

Example 69. To see how this theorem can simplify finding a determinant, consider the following matrix A.

$$A = \begin{bmatrix} 2 & 13 & -3 \\ 0 & 2 & -11 \\ 0 & 0 & 7 \end{bmatrix}$$

If you were asked to find $|A^{-1}|$, you could do it two ways.

(1) Compute

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{13}{4} & -\frac{137}{28} \\ 0 & \frac{1}{2} & \frac{11}{14} \\ 0 & 0 & \frac{1}{7} \end{bmatrix}$$

by hand, and find its determinant as $|A^{-1}| = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{7} = \frac{1}{28}$.

(2) Use the theorem $|A^{-1}| = |A|^{-1}$ to get $|A|^{-1} = \frac{1}{|A|} = \frac{1}{2 \cdot 2 \cdot 7} = \frac{1}{28}$.

Theorem. Equivalent conditions for a matrix to have an inverse.
If A is an $n \times n$ matrix, then the following statements are equivalent:

(1) A is invertible.

(2) $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$ has a unique solution for every $n \times 1$ matrix b.

(3) $A\bar{\mathbf{x}} = \mathbf{0}$ has only the trivial solution.

(4) A is row-equivalent to I_n.
(5) A can be written as the product of elementary matrices.
(6) $|A| \neq 0$.

Example 70. Suppose $A = \begin{bmatrix} 2 & 0 & 0 \\ 4 & -2 & 0 \\ 0 & -1 & -3 \end{bmatrix}$ and $\bar{\mathbf{b}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Solve $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$.

We compute

$$|A| = \left|A^T\right| = \begin{vmatrix} 2 & 4 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & -3 \end{vmatrix} = 2\,(-2)\,(-3) = 12 \neq 0$$

and conclude $\bar{\mathbf{x}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.
We have used (6) $\implies$ (3).

Example 71. Suppose you are given the matrices

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then you know $ABC\bar{\mathbf{x}} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$, will have a unique solution. (Why?)
We have used (5) $\implies$ (2).

Homework Assignment:
Read: pp. 129-135
Exercises: §3.3 #1-6, 23-28
Supplement: *Determinants*

III.4. **Applications of Determinants: Cramer's Rule.**

Theorem. Let $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$ be a system of n linear equations in n variables where the coefficient matrix A has a nonzero determinant. Then the solution to the system is given by

$$x_1 = \frac{|A_1|}{|A|}, x_2 = \frac{|A_2|}{|A|}, \ldots, x_n = \frac{|A_n|}{|A|},$$

where A_i is A, but with the i^{th} column replaced by $\bar{\mathbf{b}}$.

Example 72. The following system of equations may be solved using Cramer's rule.

$$\begin{aligned} -2x_1 + 3x_2 - x_3 &= 1 \\ x_1 + 2x_2 - x_3 &= 4 \\ -2x_1 - x_2 + x_3 &= -3 \end{aligned}$$

$$\text{We begin with } A = \begin{bmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{bmatrix} \quad \text{and} \quad \bar{\mathbf{b}} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$$

(1) First, compute $|A|$:

$$|A| = \begin{vmatrix} 0 & 7 & -3 \\ 1 & 2 & -1 \\ 0 & 3 & -1 \end{vmatrix} = (-1) \begin{vmatrix} 1 & 2 & -1 \\ 0 & 7 & -3 \\ 0 & 3 & -1 \end{vmatrix} = (-1)\left((-7) - (-9)\right) = -2$$

Since $|A| \neq 0$, we know we can use Cramer's rule.

(2) Now we compute the Cramer determinants as

$$|A_1| = \begin{vmatrix} 1 & 3 & -1 \\ 4 & 2 & -1 \\ -3 & -1 & 1 \end{vmatrix} = \begin{vmatrix} -2 & 2 & 0 \\ 1 & 1 & 0 \\ -3 & -1 & 1 \end{vmatrix} = -2 - (2) = -4$$

$$|A_2| = \begin{vmatrix} -2 & 1 & -1 \\ 1 & 4 & -1 \\ -2 & -3 & 1 \end{vmatrix} = \begin{vmatrix} -4 & -2 & 0 \\ -1 & 1 & 0 \\ -2 & -3 & 1 \end{vmatrix} = (-4) - (2) = -6$$

$$|A_3| = \begin{vmatrix} -2 & 3 & 1 \\ 1 & 2 & 4 \\ -2 & -1 & -3 \end{vmatrix} = \begin{vmatrix} 0 & 7 & 9 \\ 1 & 2 & 4 \\ 0 & 3 & 5 \end{vmatrix} = 27 - 35 = -8$$

(3) Then the solutions to the original equation are given by

$$\begin{aligned} x_1 &= \frac{|A_1|}{|A|} = \frac{-4}{-2} = 2 \\ x_2 &= \frac{|A_2|}{|A|} = \frac{-6}{-2} = 3 \\ x_3 &= \frac{|A_3|}{|A|} = \frac{-8}{-2} = 4. \end{aligned}$$

Example 73. Consider

$$\begin{aligned} x_1 + 2x_2 + \ \ x_3 &= 5 \\ 2x_1 + 2x_2 + \ \ x_3 &= 6 \\ x_1 + 2x_2 + 3x_3 &= 9 \end{aligned}$$

$$\text{We begin with } A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \quad \text{and} \quad \bar{\mathbf{b}} = \begin{bmatrix} 5 \\ 6 \\ 9 \end{bmatrix}$$

(1) First, we compute

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 2 \end{vmatrix} = -4$$

(2) Now we compute the Cramer determinants as

$$|A_1| = \begin{vmatrix} 5 & 2 & 1 \\ 6 & 2 & 1 \\ 9 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 5 & 2 & 1 \\ 1 & 0 & 0 \\ 4 & 0 & 2 \end{vmatrix} = -4$$

$$|A_2| = \begin{vmatrix} 1 & 5 & 1 \\ 2 & 6 & 1 \\ 1 & 9 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 1 \\ 1 & 1 & 0 \\ 0 & 4 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 4 & 1 \\ 1 & 1 & 0 \\ 0 & 4 & 2 \end{vmatrix} = (-1) \begin{vmatrix} 1 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 1 \end{vmatrix} = -4$$

$$|A_3| = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 2 & 6 \\ 1 & 2 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 5 \\ 1 & 0 & 1 \\ 0 & 0 & 4 \end{vmatrix} = -8$$

(3) Then the solutions to the original equation are given by

$$\begin{aligned} x_1 &= \frac{|A_1|}{|A|} = \frac{-4}{-4} = 1 \\ x_2 &= \frac{|A_2|}{|A|} = \frac{-4}{-4} = 1 \\ x_3 &= \frac{|A_3|}{|A|} = \frac{-8}{-4} = 2 \end{aligned}$$

Homework Assignment:

Read: pp. 143-146 “Cramer’s Rule”
Exercises: §3.4 #29-34,39-44
Supplement: *Determinants*

IV. VECTORS AND VECTOR SPACES

IV.1. Vectors.

Remark. So far, we have discussed vectors mainly using the notation

$$\bar{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Today, I'm going to change the notation to make it more suggestive of something familiar, and we will talk about the interpretations.

We can write a 2-vector $\bar{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ as an ordered pair $\bar{\mathbf{x}} = (x_1, x_2)$ and consider it in either of two equivalent ways:

(1) as the coordinates of a point in the plane, or
(2) as the coordinates of the end of a line segment which begins at the origin.

It is usually easier to use the first interpretation when performing arithmetic operations with vectors. However, the interpretation of a vector as a directed line segment is what makes it an invaluable tool for the study of problems involving forces, velocities, and other directed quantities. Here I use the term "directed" to indicate that there is a definite beginning and end associated with the quantity.

Also, the second interpretation occasionally helps to visualize how components of the vectors are changing, and how vectors relate to each other.

Definition 62. A note on terminology. We sometimes refer to the plane as $\mathbb{R}^{\mathbf{2}}$, that is, 2-dimensional space. The $\mathbb{R}$ refers to "real numbers", indicating that the components of vectors in the plane are positive or negative real numbers (or 0). Although complex or imaginary numbers are an essential and powerful tool for the further study of linear algebra, will not be discussing them in this class.

Definition 63. For vectors, we often refer to the entries as coordinates. For example, if we take a vector $\bar{\mathbf{v}} = (v_1, v_2)$ plotted on the standard coordinate axes, then v_1 is the **first coordinate** or **x_1-coordinate**, and v_2 is the **second coordinate** or **x_2-coordinate**.

Example 74. For example, we can take the vector $\bar{\mathbf{x}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ to be the point $(3, 4)$; the point with x_1-coordinate 3 and x_2-coordinate 4. Since $\bar{\mathbf{x}}$ is a vector with 2 components which are real numbers, we say $\bar{\mathbf{x}}$ is an element/member of $\mathbb{R}^2$, written $\bar{\mathbf{x}} \in \mathbb{R}^2$.

Remark. Recall that we have a couple of operations defined for vectors in $\mathbb{R}^2$:

vector addition: The sum of $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ is the vector given by

$$\bar{\mathbf{u}} + \bar{\mathbf{v}} = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2).$$

scalar multiplication: The multiple of a vector $\bar{\mathbf{v}}$ by the scalar c is the vector

$$c\bar{\mathbf{v}} = c(v_1, v_2) = (cv_1, cv_2)$$

Putting these together with $c = -1$ gives us the **difference** of $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ as

$$\bar{\mathbf{u}} - \bar{\mathbf{v}} = \bar{\mathbf{u}} + (-1)\bar{\mathbf{v}}$$

Now these ideas are perfectly compatible with the geometric interpretation of vectors.

Example 75. Given $\bar{\mathbf{u}} = (4, -1)$ and $\bar{\mathbf{v}} = (1, 3)$, we can interpret the sum of these vectors as

$$\bar{\mathbf{u}} + \bar{\mathbf{v}} = (4, -1) + (1, 3) = (4 + 1, -1 + 3) = (5, 2)$$

by bringing the tail (initial point) of one vector to the head (terminal point) of the other and drawing the resulting triangle. The new vector is the sum.

We can interpret the difference of these vectors as

$$\bar{\mathbf{u}} - \bar{\mathbf{v}} = (4, -1) - (1, 3) = (4, -1) + (-1, -3) = (4 - 1, -1 - 3) = (3, -4)$$

by bringing the head of the second to the head of the first, and drawing the resulting vector $\bar{\mathbf{w}}$ to the tail of $\bar{\mathbf{v}}$. Now $\bar{\mathbf{w}} + \bar{\mathbf{v}} = \bar{\mathbf{u}}$.

Note that we can also take the difference

$$\bar{\mathbf{v}} - \bar{\mathbf{u}} = (1, 3) - (4, -1) = (1, 3) + (-4, 1) = (1 - 4, 3 + 1) = (-3, 4)$$

Evaluated geometrically, we bring the head of the second to the head of the first, and drawing the resulting vector $-\bar{\mathbf{w}}$ to the tail of $\bar{\mathbf{v}}$. Now $-\bar{\mathbf{w}} + \bar{\mathbf{u}} = \bar{\mathbf{v}}$.

Example 76. We've been talking primarily about 2-vectors, but it is important to realize that all these ideas extend to 3-vectors as well.

Given $\bar{\mathbf{u}} = (3, 1, 1)$ and $\bar{\mathbf{v}} = (2, 1, -1)$, we can interpret the sum of these vectors as

$$\bar{\mathbf{u}} + \bar{\mathbf{v}} = (3, 1, 1) + (1, 2, -1) = (3 + 1, 1 + 2, 1 - 1) = (4, 3, 0)$$

by bringing the tail (initial point) of one vector to the head (terminal point) of the other and drawing the resulting triangle. The new vector is the sum.

Similarly, we can talk about the difference of vectors just as before. The only difference is that the resulting triangle no longer lies flat in the plane - it is like a triangular sheet dangling diagonally in 3-space, that is, in $\mathbb{R}^{\mathbf{3}}$. Note: $\mathbb{R}^3$ is just like $\mathbb{R}^2$, but now we consider 3 dimensions instead of just 2.

Definition 64.

$\mathbb{R}^{\mathbf{2}}$ is the set of all vectors with 2 entries, all of which are real numbers, with vector addition and scalar multiplication as defined just above. Some members of $\mathbb{R}^2$ are $(2,0)$, $(-1,3)$, and $(3,3)$.

$\mathbb{R}^{\mathbf{3}}$ is the set of all vectors with 3 entries, all of which are real numbers, with vector addition and scalar multiplication as defined just above. Some members of $\mathbb{R}^3$ are $(1,1,-2)$, $(4,0,-2)$.

Theorem. (Closure) For vectors $\bar{\mathbf{u}}, \bar{\mathbf{v}}, \in \mathbb{R}^n$, and scalar $c \in \mathbb{R}$,

(1) $\bar{\mathbf{u}} + \bar{\mathbf{v}}$ is a vector in $\mathbb{R}^n$, and

(2) $c\bar{\mathbf{u}}$ is a vector in $\mathbb{R}^n$.

Definition 65. Recall that a **linear combination** of the vectors $\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \ldots, \bar{\mathbf{v}}_n$ is a sum of the form

$$\bar{\mathbf{x}} = c_1\bar{\mathbf{v}}_1 + c_2\bar{\mathbf{v}}_2 + \ldots + c_n\bar{\mathbf{v}}_n$$

Here, we have written $\bar{\mathbf{x}}$ as a linear combination of the $\bar{\mathbf{v}}_i$.

Example 77. Given $\bar{\mathbf{x}} = (9,-4,17)$, $\bar{\mathbf{v}}_1 = (1,-1,2)$, $\bar{\mathbf{v}}_2 = (-2,3,-5)$, and $\bar{\mathbf{v}}_3 = (3,0,5)$, find c_1, c_2, c_3 such that

$$c_1\bar{\mathbf{v}}_1 + c_2\bar{\mathbf{v}}_2 + c_3\bar{\mathbf{v}}_3 = \bar{\mathbf{x}}.$$

So what is this question asking? How do we find the coefficients which allow us to write $\bar{\mathbf{x}}$ as a linear combination of these other vectors, the $\bar{\mathbf{v}}_i$? Well, this linear combination looks like

$$c_1(1,-1,2) + c_2(-2,3,-5) + c_3(3,0,5) = (9,-4,17),$$

or in our former notation,

$$c_1\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} -2 \\ 3 \\ -5 \end{bmatrix} + c_3\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 9 \\ -4 \\ 17 \end{bmatrix}.$$

But using our rules for vector arithmetic, this is just

$$\begin{bmatrix} 1c_1 \\ -1c_1 \\ 2c_1 \end{bmatrix} + \begin{bmatrix} -2c_2 \\ 3c_2 \\ -5c_2 \end{bmatrix} + \begin{bmatrix} 3c_3 \\ 0c_3 \\ 5c_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -4 \\ 17 \end{bmatrix}, \text{or}$$

$$\begin{bmatrix} 1c_1 - 2c_2 + 3c_3 \\ -1c_1 + 3c_2 + 0c_3 \\ 2c_1 - 5c_2 + 5c_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -4 \\ 17 \end{bmatrix}, \text{or}$$

$$\begin{aligned} 1c_1 - 2c_2 + 3c_3 &= 9 \\ -1c_1 + 3c_2 + 0c_3 &= -4 \\ 2c_1 - 5c_2 + 5c_3 &= 17 \end{aligned}$$

We can even rewrite this system as

$$\begin{bmatrix} 1 & -2 & 3 \\ -1 & 3 & 0 \\ 2 & 5 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -4 \\ 17 \end{bmatrix}, \text{ and then use the inverse } \quad \frac{1}{2} \begin{bmatrix} 15 & -5 & -9 \\ 5 & -1 & -3 \\ -1 & 1 & 1 \end{bmatrix}$$

$$\text{to get } \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 15 & -5 & -9 \\ 5 & -1 & -3 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ -4 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

so for $c_1 = 1, c_2 = -1, c_3 = 2$, we have

$$c_1(1, -1, 2) + c_2(-2, 3, -5) + c_3(3, 0, 5) = (1, -1, 2) - (-2, 3, -5) + 2(3, 0, 5) = (9, -4, 17).$$

The purpose of this example is to illustrate how problems involving vector arithmetic and linear combinations of vectors can be rewritten and solved using the tools you have already learned.

Example 78. Given $\bar{\mathbf{x}} = (1, 1)$, $\bar{\mathbf{v}}_1 = (1, 2)$, and $\bar{\mathbf{v}}_2 = (2, 4)$, find c_1, c_2 such that

$$c_1\bar{\mathbf{v}}_1 + c_2\bar{\mathbf{v}}_2 = \bar{\mathbf{x}}.$$

Note that any linear combination of these vectors would be of the form

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} c_1 \\ 2c_1 \end{bmatrix} + \begin{bmatrix} 2c_2 \\ 4c_2 \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 \\ 2c_1 + 4c_2 \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 \\ 2\,(c_1 + 2c_2) \end{bmatrix},$$

and hence the second entry is always twice the first. Since this is true for any linear combination, it is impossible for $c_1\bar{\mathbf{v}}_1 + c_2\bar{\mathbf{v}}_2 = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, in other words, this system is inconsistent.

Example 79. (Revenue Monitoring). Computers store lists of information in data structures called "arrays" which are essentially n-vectors, where n is the number of items in the list. While computers will help you manipulate, analyze, and compute using vectors and matrices, the study of linear algebra is what will enable you to know what to ask the computer to do.

Suppose that a store handles 100 different items. The inventory on hand can be described by the inventory vector $\bar{\mathbf{u}}$. $\bar{\mathbf{u}}$ is a 100-vector, i.e., a vector in $\mathbb{R}^{100}$. The number of items sold at the end of the week can be described by the sales vector $\bar{\mathbf{v}}$, which is also a vector in $\mathbb{R}^{100}$ (any item which did not sell at all shows up as a 0). Then the inventory at the end of the week is given by

$$\bar{\mathbf{u}} - \bar{\mathbf{v}}.$$

If the store receives a new shipment of goods, represented by the vector $\bar{\mathbf{w}}$, then its new inventory would be

$$\bar{\mathbf{u}} - \bar{\mathbf{v}} + \bar{\mathbf{w}}.$$

Homework Assignment:

Read: pp. 161-168
Exercises: §4.1 #29-34,39-44
Supplement: *none*

IV.2. **Vector Spaces.**

Definition 66. A **vector space** is any set V satisfying the following axioms for vectors $\bar{\mathbf{u}}, \bar{\mathbf{v}}, \bar{\mathbf{w}}, \in V$, and scalars $c, d \in \mathbb{R}$:

(1) $\bar{\mathbf{u}} + \bar{\mathbf{v}} \in V$ (Closure under vector addition)
(2) $\bar{\mathbf{u}} + \bar{\mathbf{v}} = \bar{\mathbf{v}} + \bar{\mathbf{u}}$
(3) $(\bar{\mathbf{u}} + \bar{\mathbf{v}}) + \bar{\mathbf{w}} = \bar{\mathbf{u}} + (\bar{\mathbf{u}} + \bar{\mathbf{w}})$
(4) V has a zero vector 0 such that $\bar{\mathbf{u}} + \mathbf{0} = \bar{\mathbf{u}}$
(5) For any $\bar{\mathbf{u}} \in V$, there is $\bar{\mathbf{v}} \in V$ such that $\bar{\mathbf{u}} + \bar{\mathbf{v}} = \mathbf{0}$ $(\bar{\mathbf{v}} = -\bar{\mathbf{u}})$
(6) $c\bar{\mathbf{u}} \in V$, for any $c \in \mathbb{R}$ (Closure under scalar multiplication)
(7) $c(\bar{\mathbf{u}} + \bar{\mathbf{v}}) = c\bar{\mathbf{u}} + c\bar{\mathbf{v}}$
(8) $(c + d)\bar{\mathbf{u}} = c\bar{\mathbf{u}} + d\bar{\mathbf{u}}$
(9) $c(d\bar{\mathbf{u}}) = (cd)\bar{\mathbf{u}}$
(10) $1(\bar{\mathbf{u}}) = \bar{\mathbf{u}}$

Example 80. §4.2 #18.

Is $\mathcal{A} = \{(x_1, x_2) \vdots x_1, x_2 \in \mathbb{R}, x_1 \geqslant 0\}$ a vector space under the standard operations of $\mathbb{R}^2$?

No, it fails axiom (5) as follows: Let $\bar{\mathbf{u}} = (1, 4)$. Then $\bar{\mathbf{u}} + \bar{\mathbf{v}} = \mathbf{0} \implies \bar{\mathbf{v}} = (-1, -4)$. But $\bar{\mathbf{v}} = (-1, -4)$ is not a member of $\mathcal{A}$ because $v_1 < 0$.

Note that (5) is actually a special case of (6): the case $c = -1$. Thus $\mathcal{A}$ fails (6), too.

Example 81. §4.2 #25.c)

Is the set $\mathcal{B} = \{(x_1, x_2) \vdots x_1, x_2 \in \mathbb{R}\}$ with the following operations a vector space?

$$\bar{\mathbf{x}} + \bar{\mathbf{y}} = (x_1, x_2) + (y_2, y_2) = (x_1 + y_1, x_2 + y_2)$$
$$c(x_1, x_2) = \left(\sqrt{c}x_1, \sqrt{c}x_2\right)$$

No, it fails axiom (6) of the definition: if $c < 0$, then both coordinates of $c(x_1, x_2) = (\sqrt{c}x_1, \sqrt{c}x_2)$ are *not* in $\mathbb{R}$ because the square root of a negative number is imaginary. Hence the vector $(\sqrt{c}x, \sqrt{c}y)$ is not in $\mathcal{B}$.

Example 82. Is $\mathcal{C} = \{(x_1, x_2, x_3) \vdots x_1, x_2, x_3 \in \mathbb{R}, x_2 = 0\}$ a vector space under the standard operations of $\mathbb{R}^3$?

Yes. I will leave it to you to check that all 10 properties hold, but this should be pretty clear. For example, if $\bar{\mathbf{x}} = (x_1, 0, x_3)$ and $\bar{\mathbf{y}} = (y_1, 0, y_3)$, then

(1) $\bar{\mathbf{x}} + \bar{\mathbf{y}} = (x_1, 0, x_3) + (y_1, 0, y_3) = (x_1 + y_1, 0, x_3 + y_3) \in \mathcal{C}$
(2) $\bar{\mathbf{x}} + \bar{\mathbf{y}} = (x_1 + y_1, 0, x_3 + y_3) = (y_1 + x_1, 0, y_3 + x_3) = \bar{\mathbf{y}} + \bar{\mathbf{x}}$

$\vdots$

Homework Assignment:

Read: pp. 170-171
Exercises: §4.2 1,7,17,18,25,28,29,31,32ac
Supplement: *none*

IV.3. **Subspaces.**

Definition 67. The notion of closure leads naturally to the idea of a **subspace**. A **vector subspace** is a subset of a vector space that is itself a vector space (satisfies the axioms).

Remark. The easiest way to tell if a subset is actually a subspace, is to see if the closure conditions hold. That is, suppose we have some vector space V and it has some subset W and we wish to determine whether or not U is a subspace. You need to answer the questions

(1) If we take any two vectors $\bar{\mathbf{v}}_1$ and $\bar{\mathbf{v}}_2$ in U, is $\bar{\mathbf{v}}_\mathbf{1} + \bar{\mathbf{v}}_\mathbf{2}$ a member of U?
(2) If we take any vector $\bar{\mathbf{v}}$ in U, and any number c, is $c\bar{\mathbf{v}}$ a member of U?

Theorem.

(1) V is a 2-dimensional subspace of $\mathbb{R}^3$ iff V is a plane passing through the origin.
(2) V is a 1-dimensional subspace of $\mathbb{R}^3$ iff V is a line passing through the origin.
(3) V is a 1-dimensional subspace of $\mathbb{R}^2$ iff V is a line passing through the origin.

Example 83. Recall the previous example:

$$\mathcal{C} = \{(x, y, z) \vdots x, y, z \in \mathbb{R}, x_2 = 0\}$$

This is a copy of $\mathbb{R}^2$ that lies inside $\mathbb{R}^3$; it corresponds to the plane defined by $y = 0$, i.e., the xz-plane. Since it is a plane through the origin, it is a subspace of $\mathbb{R}^3$.

Example 84. The xy-plane and yz-plane are also subspaces of $\mathbb{R}^3$. How do we see this? If we choose one of these planes, then any pair of vectors that lie in that plane will add to form another vector which also lies in that plane. Any scalar multiple of those vectors will also still lie in that plane.
This subspace can be thought of as the subspace obtained by setting a particular coordinate to 0. For example, the xy-plane is the subspace consisting of all vectors in $\mathbb{R}^3$ whose third coordinate is 0. The xz-plane is the subspace of all vectors in $\mathbb{R}^3$ whose second coordinate is 0.

Example 85. Consider the set of all vectors $\bar{\mathbf{x}} = (x_1, x_2, x_3)$ such that $2x_1 + 3x_2 + 4x_3 = 0$. This is a plane passing through the origin like so:

Homework Assignment:

Read: pp. 178-179, Example 4 on p.180, and 181-184
Exercises: §4.3 #7, 8, 10, 15-20, 24
Supplement: *none*

Review

Review for Midterm 2

V. VECTOR OPERATIONS

V.1. Magnitude.

Remark. Until now, we have discussed only the operations of vector addition and scalar multiplication. We have not discussed methods of multiplying vectors together, or any other operations unique to vectors.

Definition 68. The **length** or **magnitude** of a vector $\bar{\mathbf{v}} = (v_1, v_2, \ldots, v_n)$ is defined to be

$$\|\bar{\mathbf{v}}\| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$$

While this might initially appear strange, it is exactly in keeping with what we have seen so far.

- The magnitude of a real number x is more commonly called its absolute value, and is written
$$\|x\| = |x| = \sqrt{x^2}$$
- The magnitude of a 2-vector $\bar{\mathbf{v}} = (x, y)$ can be found by the Pythagorean theorem as
$$\|\bar{\mathbf{v}}\| = \sqrt{x^2 + y^2}$$
- The magnitude of a 3-vector $\bar{\mathbf{v}} = (x, y, z)$ can be found by repeated applications of the Pythagorean theorem as
$$\begin{aligned}\|\bar{\mathbf{v}}\| &= \sqrt{\sqrt{x^2 + y^2}^2 + z^2} \\ &= \sqrt{x^2 + y^2 + z^2}\end{aligned}$$

In general, this shows how to find the length of a vector with n components. This definition also shows that the length of a vector is never negative, $\|\bar{\mathbf{v}}\| \geqslant 0$, and that $\|\bar{\mathbf{v}}\| = 0$ iff $\bar{\mathbf{v}} = \mathbf{0}$.

Example 86. $\bar{\mathbf{u}} = (2, 2, 0, -1)$ is a vector in $\mathbb{R}^4$. What is the length of $\bar{\mathbf{u}}$?

$$\|\bar{\mathbf{u}}\| = \sqrt{2^2 + 2^2 + 0^2 + (-1)^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$$

Theorem. Geometry of 2×2 determinants. The absolute value of the determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

is the area of the parallelogram whose adjacent sides are the vectors $\bar{\mathbf{u}}_\mathbf{1} = (a, b)$ and $\bar{\mathbf{u}}_\mathbf{2} = (c, d)$. Note that since $|A| = |A^T|$, we could just as easily have used the columns instead of the rows and considered the parallelogram formed by $\bar{\mathbf{v}}_\mathbf{1} = (a, c)$ and $\bar{\mathbf{v}}_\mathbf{2} = (b, d)$.

Example 87. What is the area of the parallelogram whose sides are the vectors $\bar{\mathbf{u}} = (-2, 3)$ and $\bar{\mathbf{v}} = (1, 1)$?

$$\begin{vmatrix} -2 & 3 \\ 1 & 1 \end{vmatrix} = -2\,(1) - 1\,(3) = -5$$

So the area of the parallelogram is 5.

Theorem. Geometry of 3×3 determinants. The absolute value of the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

is the area of the parallelepiped whose adjacent sides are the vectors $\bar{\mathbf{a}} = (a_1, a_2, a_3)$, $\bar{\mathbf{b}} = (b_1, b_2, b_3)$, and $\bar{\mathbf{c}} = (c_1, c_2, c_3)$.

Remark. In light of this, we can interpret $|A|$ as an indication of the size of A, just as we interpret $\|\bar{\mathbf{v}}\|$ as the size of $\bar{\mathbf{v}}$, and the following theorem should come as no great surprise.

Theorem. Let $\bar{\mathbf{v}} = (v_1, v_2, \ldots, v_n)$ be a vector in $\mathbb{R}^n$ and let c be a scalar. Then

$$\|c\bar{\mathbf{v}}\| = |c| \cdot \|\bar{\mathbf{v}}\|\,.$$

We prove this as follows:

$$\begin{aligned}
\|c\bar{\mathbf{v}}\| &= \|(cv_1, cv_2, \ldots, cv_n)\| && \text{def of scalar multiplication} \\
&= \sqrt{(cv_1)^2 + (cv_2)^2 + \ldots + (cv_n)^2} && \text{def of } \|\cdot\| \\
&= \sqrt{c^2v_1^2 + c^2v_2^2 + \ldots + c^2v_n^2} && (ab)^2 = a^2b^2 \\
&= \sqrt{c^2\left(v_1^2 + v_2^2 + \ldots + v_n^2\right)} && \text{factor out the } c^2 \\
&= \sqrt{c^2}\sqrt{v_1^2 + v_2^2 + \ldots + v_n^2} && \sqrt{ab} = \sqrt{a}\sqrt{b} \\
&= |c| \cdot \|\bar{\mathbf{v}}\| && \text{def of } |c| \text{ and } \|\bar{\mathbf{v}}\|
\end{aligned}$$

Definition 69. If $\|\bar{\mathbf{v}}\| = 1$, then we say that $\bar{\mathbf{v}}$ is a **unit vector**.

Theorem. If $\bar{\mathbf{v}}$ is a nonzero vector in $\mathbb{R}^n$, then the vector

$$\bar{\mathbf{u}} = \frac{\bar{\mathbf{v}}}{\|\bar{\mathbf{v}}\|}$$

has length 1 and has the same direction as $\bar{\mathbf{v}}$. Consequently, it is called the **unit vector in the direction of $\bar{\mathbf{v}}$**.

Example 88. Find the unit vector in the direction of $\bar{\mathbf{v}} = (-2, 2, 1)$.

$$\frac{\bar{\mathbf{v}}}{\|\bar{\mathbf{v}}\|} = \frac{(-2,2,1)}{\sqrt{(-2)^2 + 2^2 + 1^2}} = \frac{(-2,2,1)}{\sqrt{4+4+1}} = \frac{1}{\sqrt{9}}(-2,2,1) = \left(\frac{-2}{3}, \frac{2}{3}, \frac{1}{3}\right)$$

Definition 70. What is the distance between two vectors $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$? If this question seems strange to you, remember that we can consider vectors to be points in $\mathbb{R}^n$. In this way, we can rephrase the question as, what is the distance between two points $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$?

We define the **distance from $\bar{\mathbf{u}}$ to $\bar{\mathbf{v}}$** as $d(\bar{\mathbf{u}}, \bar{\mathbf{v}}) = \|\bar{\mathbf{u}} - \bar{\mathbf{v}}\|$.

Example 89. Let $\bar{\mathbf{u}} = (0, 3, 1, 1)$ and $\bar{\mathbf{v}} = (-1, 2, 0, 2)$ be vectors in $\mathbb{R}^4$. Then the distance from $\bar{\mathbf{u}}$ to $\bar{\mathbf{v}}$ is

$$\begin{aligned} d(\bar{\mathbf{u}}, \bar{\mathbf{v}}) &= \|\bar{\mathbf{u}} - \bar{\mathbf{v}}\| \\ &= \|(0,3,1,1) - (-1,2,0,2)\| \\ &= \|(1,1,1,-1)\| \\ &= \sqrt{1^2 + 1^2 + 1^2 + (-1)^2} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

V.2. **Dot Product.**

Definition 71. The **dot product** of $\bar{\mathbf{u}} = (u_1, u_2, \ldots, u_n)$ and $\bar{\mathbf{v}} = (v_1, v_2, \ldots, v_n)$ is

$$\bar{\mathbf{u}} \cdot \bar{\mathbf{v}} = u_1 v_1 + u_2 v_2 \ldots, u_n v_n$$

Example 90. Let $\bar{\mathbf{u}} = (0, 3, 1, 1)$ and $\bar{\mathbf{v}} = (-1, 2, 0, 2)$ be vectors in $\mathbb{R}^4$. Then the dot product of $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ is given by

$$\begin{aligned} \bar{\mathbf{u}} \cdot \bar{\mathbf{v}} &= (0,3,1,1) \cdot (-1,2,0,2) \\ &= 0(-1) + 3(3) + 1(0) + 1(2) \\ &= 9 + 2 \\ &= 11 \end{aligned}$$

Remark. The dot product should seem very familiar to you. Since the vector $\bar{\mathbf{u}} = (u_1, u_2, \ldots u_n)$ can also be represented as

$$\bar{\mathbf{u}} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix},$$

we can also define the dot product of $\bar{\mathbf{u}} = (u_1, u_2, \dots u_n)$ and $\bar{\mathbf{v}} = (v_1, v_2, \dots v_n)$ as

$$\bar{\mathbf{u}} \cdot \bar{\mathbf{v}} = \bar{\mathbf{u}}^T \bar{\mathbf{v}} = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 v_1 + u_2 v_2 + \dots + u_n v_n \end{bmatrix}.$$

In fact, if we consider the matrix

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{a}} \\ \bar{\mathbf{b}} \\ \bar{\mathbf{c}} \end{bmatrix}$$

as a column of row vectors (so $\bar{\mathbf{a}} = (a_1, a_2, a_3)$ is considered as a row vector, for example), then for

$$\bar{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

the matrix product can be written as

$$A\bar{\mathbf{x}} = \begin{bmatrix} \bar{\mathbf{a}} \cdot \bar{\mathbf{x}} \\ \bar{\mathbf{b}} \cdot \bar{\mathbf{x}} \\ \bar{\mathbf{c}} \cdot \bar{\mathbf{x}} \end{bmatrix},$$

and in fact this is the definition some books use for matrix multiplication.

Example 91. Recall the store that we talked about earlier, where the sales vector $\bar{\mathbf{v}} \in \mathbb{R}^{100}$ describes the number of each item sold at the end of the week. Suppose that $\bar{\mathbf{p}}$ is a vector in $\mathbb{R}^{100}$ which gives the price of each of the 100 items. Then the dot product

$$\bar{\mathbf{v}} \cdot \bar{\mathbf{p}}$$

gives the total revenue received at the end of the week.

Example 92. A large steel manufacturer, who has 2000 employees, lists each employee's salary as a component of a vector $\bar{\mathbf{u}} \in \mathbb{R}^{2000}$. If an 8% across-the-board salary increase has been approved, find an expression in terms of $\bar{\mathbf{u}}$ that gives all the new salaries.

$$1.08\bar{\mathbf{u}}$$

Example 93. The vector $\bar{\mathbf{u}} = (20, 30, 80, 10)$ gives the number of receivers, CD players, speakers, and cassette recorders that are on hand in a stereo shop. The vector $\bar{\mathbf{v}} = (200, 120, 80, 70)$ gives the price (in dollars) of each receiver, CD player, speaker, and cassette recorder, respectively. What does $\bar{\mathbf{u}} \cdot \bar{\mathbf{v}}$ represent?

The dollar value of the merchandise in stock.

Example 94. A brokerage firm records the high and low values of the price of IBM stock each day. The information for a given week is presented in two vectors, $\bar{\mathbf{t}}$ and $\bar{\mathbf{b}}$ in $\mathbb{R}^5$, giving the high and low values, respectively. What expression gives the average daily values of the price of IBM stock for the entire 5-day week? That is, what vector gives the daily averages?

$$\tfrac{1}{2}\left(\bar{\mathbf{t}}+\bar{\mathbf{b}}\right)$$

Theorem. Properties of the Dot Product.
Let c be a scalar, and $\bar{\mathbf{u}}, \bar{\mathbf{v}}, \bar{\mathbf{w}}$ be vectors in $\mathbb{R}^n$.

(1) $\bar{\mathbf{u}} \cdot \bar{\mathbf{v}} = \bar{\mathbf{v}} \cdot \bar{\mathbf{u}}$
(2) $\bar{\mathbf{u}} \cdot (\bar{\mathbf{v}} + \bar{\mathbf{w}}) = \bar{\mathbf{u}} \cdot \bar{\mathbf{v}} + \bar{\mathbf{u}} \cdot \bar{\mathbf{w}}$
(3) $c\,(\bar{\mathbf{u}} \cdot \bar{\mathbf{v}}) = (c\bar{\mathbf{u}}) \cdot \bar{\mathbf{v}} = \bar{\mathbf{u}} \cdot (c\bar{\mathbf{v}})$
(4) $\bar{\mathbf{v}} \cdot \bar{\mathbf{v}} = \|\bar{\mathbf{v}}\|^2$
(5) $\bar{\mathbf{v}} \cdot \bar{\mathbf{v}} \geqslant 0$, and $\bar{\mathbf{v}} \cdot \bar{\mathbf{v}} = 0$ iff $\bar{\mathbf{v}} = 0$.
(6) $|\bar{\mathbf{u}} \cdot \bar{\mathbf{v}}| \leqslant \|\bar{\mathbf{u}}\| \cdot \|\bar{\mathbf{v}}\|$
(7) $\bar{\mathbf{u}} \cdot \bar{\mathbf{v}} = 0$ iff $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ are orthogonal (perpendicular).
(8) The angle between $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ is given by

$$\theta = \arccos \tfrac{\bar{\mathbf{u}} \cdot \bar{\mathbf{v}}}{\|\bar{\mathbf{u}}\| \cdot \|\bar{\mathbf{v}}\|} = \cos^{-1} \tfrac{\bar{\mathbf{u}} \cdot \bar{\mathbf{v}}}{\|\bar{\mathbf{u}}\| \cdot \|\bar{\mathbf{v}}\|}$$

(9) $\|\bar{\mathbf{u}} + \bar{\mathbf{v}}\| \leqslant \|\bar{\mathbf{u}}\| + \|\bar{\mathbf{v}}\|$
(10) $\|\bar{\mathbf{u}} + \bar{\mathbf{v}}\|^2 = \|\bar{\mathbf{u}}\|^2 + \|\bar{\mathbf{v}}\|^2$ iff $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ are orthogonal.

The first three properties are just the same as for matrix multiplication, the fourth is repeated here for completeness. The fifth tells about the dot product of a vector with itself and doesn't come up much, it's also just listed here for completeness. The sixth and ninth are important, but more for physics and math - their use to you is primarily as an estimate, and they are not integral to this course.

(7) is actually a special case of (8), and both of these are very useful. And what about (10)? This is the Pythagorean Theorem.

Homework Assignment:
Read: pp. 250-261
Exercises: §5.1 #5-8, 13-16, 19-20, 25-28, 31-32, 43, 45, 55-58
Supplement: *none*

VI. Linear Transformations

VI.1. **Introduction to Linear Transformations.**

Remark. For what we are going to discuss next, it will be important to understand how matrices correspond to transformations of space.

Definition 72. A **function** or **mapping** T from the set X to the set Y is a rule whereby every element of X is assigned to exactly one element of Y. This is denoted by:

$$T : X \to Y$$
$$T : x \mapsto y$$
$$T(x) = y$$

The set X is called the **domain** and the set Y is called the **codomain** or **range**. We refer to the subset $T(X)$ of Y as the **image** of X under T. Similarly, if $T(x) = y$, then we refer to y as the **image** of x under T. In general, for any subset $A \subseteq X$, we call $T(A)$ the **image** of A under f.

The inverse idea of image is **preimage**. For a point $y \in Y$, the **preimage of y** under T is the set of all points in X that get mapped to y, and it is denoted:

$$T^{-1}(y) = \{x \in X \text{ such that } T(x) = y\}.$$

In general, for any subset $B \subseteq Y$, the **preimage of B** is the set of all points X that get mapped into B:

$$T^{-1}(B) = \{x \in X \text{ such that } T(x) \in B\}.$$

Remark. You may be wondering why I'm using the letter T instead of the letter f to talk about functions. The reason is that we are going to be discussing a special type of function called a linear transform. In mathematics, we frequently use the term "space" to refer to a set with a certain kind of structure. In this class, we are primarily concerned with sets whose elements are n-tuples like $\bar{\mathbf{v}} = (v_1, v_2, \ldots, v_n)$, and where the structure is given by vector addition and scalar multiplication. These two operations put structure on the set, a kind of structure we call a "vector space". Thus, when we study functions on vector spaces, it makes sense to study those functions which preserve the properties of vector spaces. A linear transformation is this kind of function.

Definition 73. For vector spaces V and W, a **linear transformation** is a function $T : V \to W$ which satisfies

(1) $T(\bar{\mathbf{u}} + \bar{\mathbf{v}}) = T(\bar{\mathbf{u}}) + T(\bar{\mathbf{v}})$
(2) $T(c\bar{\mathbf{u}}) = cT(\bar{\mathbf{u}}$

for all vectors $\bar{\mathbf{u}}, \bar{\mathbf{v}} \in V$ and all scalars $c \in \mathbb{R}$. A linear transformation is a function that preserves the properties of a vector space. In fact, if V is a vector space and T is a linear transformation, then the image $(T(V))$ of V is also a vector space.

Example 95. Suppose we wish to determine whether $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T(x_1, x_2) = (x_2, x_1 - x_2, 2x_1 + x_2)$$

is a linear transformation. To test for preservation of addition (property 1), we let $\bar{\mathbf{u}} = (u_1, u_2)$ and $\bar{\mathbf{v}} = (v_1, v_2)$, and compute:

$$\begin{aligned} T(\bar{\mathbf{u}} + \bar{\mathbf{v}}) &= T\left(u_1 + v_1, u_2 + v_2\right) \\ &= (u_2 + v_2, (u_1 + v_1) - (u_2 + v_2), 2(u_1 + v_1) + (u_2 + v_2)) \\ &= (u_2 + v_2, (u_1 - u_2) + (v_1 - v_2), (2u_1 + u_2) + (2v_1 + v_2)) \\ &= (u_2, u_1 - u_2, 2u_1 + u_2) + (v_2, v_1 - v_2, 2v_1 + v_2) \\ &= T(\bar{\mathbf{u}}) + T(\bar{\mathbf{v}}) \end{aligned}$$

So vector addition is preserved. Now we let $c \in \mathbb{R}$ and check scalar multiplication:

$$\begin{aligned} T(c\bar{\mathbf{u}}) &= T\left(cu_1, cu_2\right) \\ &= (cu_2, cu_1 - cu_2, 2cu_1 + cu_2) \\ &= c\left(u_2, u_1 - u_2, 2u_1 + u_2\right) \\ &= cT(\bar{\mathbf{u}}) \end{aligned}$$

So scalar multiplication is also preserved and T is in fact a linear transformation.

Theorem. For a linear transformation $T : V \to W$ of vector spaces:

(1) $T(\bar{\mathbf{0}}) = \bar{\mathbf{0}}$
(2) $T(-\bar{\mathbf{v}}) = -T(\bar{\mathbf{v}})$
(3) $T(\bar{\mathbf{u}} - \bar{\mathbf{v}}) = T(\bar{\mathbf{u}}) - T(\bar{\mathbf{v}})$
(4) $T(c_1\bar{\mathbf{v}}_\mathbf{1} + c_2\bar{\mathbf{v}}_\mathbf{2} + \ldots + c_n\bar{\mathbf{v}}_\mathbf{n}) = T(c_1\bar{\mathbf{v}}_\mathbf{1}) + T(c_2\bar{\mathbf{v}}_\mathbf{2}) + \ldots + T(c_n\bar{\mathbf{v}}_\mathbf{n})$.

These properties follow immediately from the definition of linear transformation.

The next theorem shows the close connection between linear transformations and matrices.

Theorem. Let A be an $m \times n$ matrix. Then the function T defined by

$$T(\bar{\mathbf{v}}) = A\bar{\mathbf{v}}$$

is a linear transformation from $\mathbb{R}^n$ into $\mathbb{R}^m$. Conversely, if $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation of $\mathbb{R}^n$ into $\mathbb{R}^m$, then there exists a unique matrix A such that

$$T(\bar{\mathbf{v}}) = A\bar{\mathbf{v}},$$

for every $\bar{\mathbf{v}} \in \mathbb{R}^n$.

Example 96. §6.1, 20. For the linear transformation given by

$$A = \begin{bmatrix} 0 & 1 & -2 & 1 \\ -4 & 4 & 5 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix},$$

find

a) $T(1,0,2,3) = \begin{bmatrix} 0 & 1 & -2 & 1 \\ -4 & 4 & 5 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0+0-4+3 \\ -4+0+10+0 \\ 0+0+6+3 \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \\ 9 \end{bmatrix}$

b) The preimage of $(0,0,0)$.
So we need to find $\bar{\mathbf{x}} = (x_1, x_2, x_3, x_4)$ such that $T(\bar{\mathbf{x}}) = A\bar{\mathbf{x}} = (0,0,0)$:

$$T(\bar{\mathbf{x}}) = \begin{bmatrix} 0 & 1 & -2 & 1 \\ -4 & 4 & 5 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2-2x_3+x_4 \\ -4x_1+4x_2+5x_3 \\ x_2+3x_3+x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus we can find the preimage of $(0,0,0)$ by solving this homogeneous system (i.e., by putting A into RRE):

$$\begin{bmatrix} 0 & 1 & -2 & 1 \\ -4 & 4 & 5 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} -4 & 4 & 5 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{matrix} R_1 \leftrightarrow R_3 \\ -R_2 + R_3 \to R_3 \\ \frac{1}{5}R_3 \to R_3 \end{matrix}$$

$$\sim \begin{bmatrix} -4 & 4 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{matrix} 2R_3 + R_2 \to R_2 \\ -5R_3 + R_1 \to R_1 \\ -4R_2 + R_1 \to R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{matrix} -4R_2 + R_1 \to R_1 \\ -\frac{1}{4}R_1 \to R_1 \end{matrix}$$

So $x_3 = 0$. Let $x_4 = t$. Then $x_1, x_2 = -t$ and the preimage of $(0,0,0)$ is the set

$$T^{-1} = \{(-t,-t,0,t) \vdots t \in \mathbb{R}\} = \{(-t,-t,0,t)\}, \forall t \in \mathbb{R}.$$

Example 97. §6.5, 1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a reflection in the x-axis. Find the images of the following vectors.

a) $(1,2)$. (Sketch). Then $T(1,2) = (1,-2)$.

b) $(-2,-2)$. (Sketch). Then $T(-2,-2) = (-2,2)$.

c) $(4,0)$. (Sketch). Then $T(4,0) = (4,0)$.

So reflection in the x-axis amounts to changing the sign of the y-component (with no change if the y-component is 0).

Example 98. §6.5, 2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a reflection in the line $y = x$. Find the images of the following vectors.

a) $(1,2)$. (Sketch). Then $T(1,2) = (2,1)$.

b) $(-2,-2)$. (Sketch). Then $T(-2,-2)=(-2,-2)$.

c) $(4,0)$. (Sketch). Then $T(4,0)=(0,4)$.

So reflection in the line $y=x$ amounts to swapping the x and y components (with no change if they are equal)).

Example 99. The linear transformation $T:\mathbb{R}^2\to\mathbb{R}^2$ given by the matrix

$$A=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

has the effect of rotating every vector in $\mathbb{R}^2$ counterclockwise about the origin, by the angle θ. Let us consider the angle $\theta=\frac{\pi}{4}=90°$. Then

$$A=\begin{bmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}=\frac{\sqrt{2}}{2}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$$

Find the images of the following vectors.

a) $(1,2)$. (Sketch). Then

$$T(1,2)=\frac{\sqrt{2}}{2}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1\\ 2\end{bmatrix}=\frac{\sqrt{2}}{2}\begin{bmatrix}-1\\ 3\end{bmatrix}.$$

Note: $\|T(1,2)\|=\sqrt{\left(-\frac{\sqrt{2}}{2}\right)^2+\left(\frac{3\sqrt{2}}{2}\right)^2}=\sqrt{\frac{20}{4}}=\sqrt{5}=\sqrt{1+2^2}=\|(1,2)\|$

b) $(-2,-2)$. (Sketch). Then

$$T(-2,-2)=\frac{\sqrt{2}}{2}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}-2\\ -2\end{bmatrix}=\frac{\sqrt{2}}{2}\begin{bmatrix}0\\ -4\end{bmatrix}=\begin{bmatrix}0\\ -2\sqrt{2}\end{bmatrix}.$$

Note: $\|T(-2,-2)\|=\sqrt{\left(-2\sqrt{2}\right)^2}=\sqrt{8}=\sqrt{4+4}=\sqrt{(-2)^2+(-2)^2}=\|(-2,-2)\|$

c) $(4,0)$. (Sketch). Then

$$T(4,0)=\frac{\sqrt{2}}{2}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}4\\ 0\end{bmatrix}=\frac{\sqrt{2}}{2}\begin{bmatrix}4\\ 4\end{bmatrix}=\begin{bmatrix}2\sqrt{2}\\ 2\sqrt{2}\end{bmatrix}.$$

Note: $\|T(4,0)\|=\sqrt{\left(2\sqrt{2}\right)^2+\left(2\sqrt{2}\right)^2}=\sqrt{8+8}=\sqrt{16}=4=\|(4,0)\|$

Rotations in $\mathbb{R}^n$ preserve vector length as well as the angle between any two vectors: $T(\bar{\mathbf{u}})\angle T(\bar{\mathbf{v}})=\bar{\mathbf{u}}\angle\bar{\mathbf{v}}$.

What would be the effect (on any vector) of the linear transformation given by the matrix

$$A=\sqrt{2}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}?\quad \text{Note: } B=2A.$$

Homework Assignment:
Read: pp. 325-333(skip examples 7,10,11)
Exercises: §6.1 1-9,20-24
Supplement: *none*

VI.2. The Geometry of Linear Transformations in the Plane.

VI.2.1. *The Geometry of Elementary Matrices.*

We consider the geometric effect that the elementary matrices have on the test vectors $\bar{\mathbf{u}} = (0,1), \bar{\mathbf{v}} = (1,1), \bar{\mathbf{w}} = (1,0)$.

(1) Row operation 1:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{Reflection in the line } y = x$$

$T(\bar{\mathbf{x}}) = A\bar{\mathbf{x}}$ has the effect of reflecting $\bar{\mathbf{x}}$ in the line $y = x$, just as in §6.5, #2:

$$A\bar{\mathbf{u}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$A\bar{\mathbf{v}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$A\bar{\mathbf{w}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(2) Row operation 2:

$$B = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{Horizontal stretch}$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \qquad \text{Vertical stretch}$$

$$B\bar{\mathbf{u}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B\bar{\mathbf{v}} = \begin{bmatrix} k \\ 1 \end{bmatrix}, B\bar{\mathbf{w}} = \begin{bmatrix} k \\ 0 \end{bmatrix}$$

So for $k = 5$, the square is expanded by a factor of 5 to the right.
For $k = -2$, the square is expanded by a factor of 2 to the left.

Similarly,

$$C\bar{\mathbf{u}} = \begin{bmatrix} 0 \\ k \end{bmatrix} C\bar{\mathbf{v}} = \begin{bmatrix} 1 \\ k \end{bmatrix}, C\bar{\mathbf{w}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

So for $k = 5$, the square is expanded upward by a factor of 5.
For $k = -1$, the square is stretched downward by a factor of 1. This is the same as reflecting downward through the x-axis.

(3) Row operation 3:

$$D = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \qquad \text{Horizontal shear}$$

$$F = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \qquad \text{Vertical shear}$$

$$D\bar{\mathbf{u}} = \begin{bmatrix} k \\ 1 \end{bmatrix}, D\bar{\mathbf{v}} = \begin{bmatrix} 1+k \\ 1 \end{bmatrix}, D\bar{\mathbf{w}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$F\bar{\mathbf{u}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, F\bar{\mathbf{v}} = \begin{bmatrix} 1 \\ 1+k \end{bmatrix}, F\bar{\mathbf{w}} = \begin{bmatrix} 1 \\ k \end{bmatrix}$$

Thus you can see how D shears the square horizontally and F shears the square vertically.

Homework Assignment:

Read: pp. 366-369
Exercises: §6.5 1,2,9-18 (sketch)
Supplement: *none*

VII. EIGENVALUES AND EIGENVECTORS

VII.1. The Eigenvalue Problem.

In each of the previous two examples, we saw that there were some vectors that remained fixed (i.e. $T(\bar{\mathbf{v}}) = \bar{\mathbf{v}}$) during a linear transformation:

(1) For reflection in the x-axis, any vector lying on the x-axis remains fixed.
(2) For reflection in the line $y = x$, any vector lying on the line $y = x$ remains fixed.

Written formally, these were examples of linear transformations T whose matrices A satisfy

$$A\bar{\mathbf{x}} = \bar{\mathbf{x}}.$$

This is the beginnings of one of the most important problems of linear algebra: the eigenvalue problem. Namely, for a matrix A, are there some vectors for which

$$A\bar{\mathbf{x}} = \lambda\bar{\mathbf{x}}$$

for some $\lambda \in \mathbb{R}$?

Note that when $\lambda = 1$, this is just the same as asking if there are any vectors fixed by A. Since it is always the case that $T(\bar{\mathbf{0}}) = \bar{\mathbf{0}}$, for any linear transformation T, we generally ignore this case: we are interested only in nonzero vectors for which $A\bar{\mathbf{x}} = \lambda\bar{\mathbf{x}}$. Also, we will only be talking about $n \times n$ matrices for this section.

Definition 74. Suppose A is an $n \times n$ matrix such that $A\bar{\mathbf{x}} = \lambda\bar{\mathbf{x}}$ for some scalar $\lambda \in \mathbb{R}$. Then we say that $\bar{\mathbf{x}}$ is an *eigenvector* of A and that λ is an *eigenvalue* of A.

An eigenvector of A is a vector whose direction does not change under the linear transformation $T(\bar{\mathbf{x}}) = A\bar{\mathbf{x}}$. Only the length of $\bar{\mathbf{x}}$ changes, and the factor by which it changes is the corresponding eigenvalue λ.

Sometimes, eigenvalues are called *characteristic values* and eigenvectors are called *characteristic vectors.*

Example 100. §7.1, 2.

(1) $A\bar{\mathbf{x}} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4-5 \\ 2-3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Thus $A\bar{\mathbf{x}} = (-1)\bar{\mathbf{x}}$, so $\bar{\mathbf{x}} = (1,1)$ is an eigenvector of A with eigenvalue -1.
(2) $A\bar{\mathbf{x}} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 20-10 \\ 10-6 \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ Thus $A\bar{\mathbf{x}} = (-1)\bar{\mathbf{x}}$, so $\bar{\mathbf{x}} = (5,2)$ is an eigenvector of A with eigenvalue 2.

So it is fairly straightforward to verify that a given vector is an eigenvector of a given matrix, but how does one find the eigenvectors and eigenvalues in the first place?

Definition 75. For an $n \times n$ matrix A, the *eigenvalue problem* is to determine the eigenvalues of A and find their corresponding eigenvectors.

Remark. Just as finding the steady-state vector of a Markov process involved solving the system

$$P\bar{\mathbf{x}} = \bar{\mathbf{x}}$$

by finding a probability vector which satisfied

$$(I_n - P)\bar{\mathbf{x}} = \bar{\mathbf{0}},$$

we find eigenvectors by solving

$$A\bar{\mathbf{x}} = \lambda\bar{\mathbf{x}},$$

that is, by finding vectors $\bar{\mathbf{x}}$ which satisfy

$$(\lambda I_n - A)\bar{\mathbf{x}} = \bar{\mathbf{0}}.$$

Recall that this homogeneous system of equations has nonzero solutions if and only if the coefficient matrix $(\lambda I_n - A)$ is *not* invertible, that is, if and only if $|\lambda I_n - A| = 0$. This is because if $(\lambda I_n - A)$ were invertible, the equation

$$(\lambda I_n - A)\bar{\mathbf{x}} = \bar{\mathbf{0}}$$

would have the unique solution given by

$$\bar{\mathbf{x}} = (\lambda I_n - A)^{-1}\bar{\mathbf{0}} = \bar{\mathbf{0}}.$$

Theorem. Let A be an $n \times n$ matrix. Then

(1) An eigenvalue of A is a scalar λ such that $|\lambda I_n - A| = 0$.
(2) The eigenvectors of A corresponding to λ are the nonzero solutions of $(\lambda I_n - A)\bar{\mathbf{x}} = \bar{\mathbf{0}}$.

This theorem gives us a method for solving the eigenvalue problem!

Definition 76. $|\lambda I_n - A| = \lambda^n + c_{n-1}\lambda^{n-1} + \ldots + c_1\lambda + c_0$ is the *characteristic polynomial* of A. When we set it equal to 0, it is called the *characteristic equation* of A:

$$|\lambda I_n - A| = 0.$$

The eigenvalues of A are the roots of the *characteristic polynomial* of A. Thus, A can have at most n distinct eigenvalues.

Example 101. We will find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -8 & 4 & -6 \\ 8 & 1 & 9 \end{bmatrix}.$$

The characteristic polynomial of A is

$$\begin{aligned} |\lambda I - A| = \begin{vmatrix} \lambda - 1 & 0 & 0 \\ 8 & \lambda - 4 & 6 \\ -8 & -1 & \lambda - 9 \end{vmatrix} &= (\lambda-1)(\lambda-4)(\lambda-9) - (-1)(6)(\lambda-1) \\ &= (\lambda-1)(\lambda^2 - 13\lambda + 42) \\ &= (\lambda-1)(\lambda-6)(\lambda-7) \end{aligned}$$

Thus the eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 6$, and $\lambda_3 = 7$.

For $\lambda_1 = 1$, we have

$$\lambda_1 I - A = \begin{bmatrix} 0 & 0 & 0 \\ 8 & -3 & 6 \\ -8 & -1 & -8 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 2 \\ 8 & 1 & 8 \end{bmatrix} \sim \begin{bmatrix} 8 & 1 & 8 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{8} & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{15}{16} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

So if $x_3 = t$, then we have $x_2 = -\frac{1}{2}t$, and $x_1 = -\frac{15}{16}t$. Thus, the eigenvalue $\lambda_1 = 1$ has eigenvectors given by

$$\bar{\mathbf{x}}_1 = \begin{bmatrix} -\frac{15}{16}t \\ -\frac{1}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} 15 \\ 8 \\ -16 \end{bmatrix}, \forall t \in \mathbb{R}.$$

For $\lambda_2 = 6$, we have

$$\lambda_2 I - A = \begin{bmatrix} 5 & 0 & 0 \\ 8 & 2 & 6 \\ -8 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 6 \\ 0 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

So if $x_3 = s$, then we have $x_2 = -3s$, and $x_1 = 0$. Thus, the eigenvalue $\lambda_2 = 6$ has eigenvectors given by

$$\bar{\mathbf{x}}_2 = \begin{bmatrix} 0 \\ -3s \\ s \end{bmatrix} = s \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}, \forall s \in \mathbb{R}.$$

For $\lambda_3 = 7$, we have

$$\lambda_3 I - A = \begin{bmatrix} 6 & 0 & 0 \\ 8 & 3 & 6 \\ -8 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 6 \\ 0 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

So if $x_3 = r$, then we have $x_2 = -2r$, and $x_1 = 0$. Thus, the eigenvalue $\lambda_3 = 6$ has eigenvectors given by

$$\bar{\mathbf{x}}_3 = \begin{bmatrix} 0 \\ -2r \\ r \end{bmatrix} = r \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}, \forall r \in \mathbb{R}.$$

Example 102. Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 3 & 1 \end{bmatrix}.$$

The characteristic polynomial of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda-2 & -1 & 0 \\ 1 & \lambda & -1 \\ -1 & -3 & \lambda-1 \end{vmatrix} = \begin{vmatrix} \lambda-2 & -1 & 0 \\ 1 & \lambda & -1 \\ 0 & \lambda-3 & \lambda-2 \end{vmatrix}$$

$$= \lambda(\lambda-2)(\lambda-2) - (\lambda-3)(-1)(\lambda-2) - (\lambda-2)(1)(-1)$$

$$= (\lambda-2)\left(\lambda(\lambda-2) + (\lambda-3) + 1\right) \quad = (\lambda-2)\left(\lambda^2 - 2\lambda + \lambda - 2\right)$$

$$= (\lambda-2)\left(\lambda^2 - \lambda - 2\right) \quad = (\lambda-2)(\lambda-2)(\lambda+1)$$

Thus the eigenvalues of A are $\lambda_1 = -1$, $\lambda_2, \lambda_3 = 2$. This is an example of a *repeated eigenvalue* or *eigenvalue of multiplicity 2*.

For $\lambda_1 = -1$, we have

$$\lambda_1 I - A = \begin{bmatrix} -3 & -1 & 0 \\ 1 & -1 & -1 \\ -1 & -3 & -2 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & -4 & -3 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & 1 & \frac{3}{4} \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & -\frac{3}{4} \\ 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{3}{4} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{3}{4} \\ 0 & 0 & 0 \end{bmatrix}$$

So if $x_3 = t$, then we have $x_2 = -\frac{3}{4}t$, and $x_1 = \frac{1}{4}t$. Thus, the eigenvalue $\lambda_1 = -1$ has eigenvectors given by

$$\bar{\mathbf{x}}_1 = \begin{bmatrix} \frac{1}{4}t \\ -\frac{3}{4}t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}, \forall t \in \mathbb{R}.$$

For $\lambda_2 = \lambda_3 = 2$, we have

$$\lambda_2 I - A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 2 & -1 \\ -1 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So $x_2 = 0$, and if $x_3 = s$, then $x_1 = s$ also. Thus, the eigenvalue $\lambda_2 = \lambda_3 = 2$ has eigenvectors given by

$$\bar{\mathbf{x}}_2 = \begin{bmatrix} s \\ 0 \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \forall s \in \mathbb{R}.$$

Homework Assignment:

Read: pp. 379-385
Exercises: §7.1 #1-5,10-11,15-20
Supplement: *none*

VII.2. **Applications of Eigenvalues: Population Growth.**

Remark. We will use eigenvector to study a generalization of Markov processes. Recall that finding the steady-state vector of a Markov process involved solving the system

$$P\bar{\mathbf{x}} = \bar{\mathbf{x}}$$

by finding a probability vector which satisfied

$$(I_n - P)\bar{\mathbf{x}} = \bar{\mathbf{0}}.$$

Now we will be solving systems which look like

$$A\bar{\mathbf{x}} = \lambda\bar{\mathbf{x}}$$

by trying to find vectors which satisfy

$$(\lambda I_n - A)\bar{\mathbf{x}} = \bar{\mathbf{0}}.$$

The key point that separates this material from the previous material on Markov processes is that we now allow the total population to grow over time.

Example 103. (Fibonacci's Rabbits)
Suppose that newly born pairs of rabbits produce no offspring during the first month of their lives, but each pair produces one new pair each subsequent month. Starting with

$$\bar{\mathbf{x}}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{matrix} \text{young} \\ \text{adults,} \end{matrix}$$

pairs of rabbits, find the number of pairs $\bar{\mathbf{x}}^{(k)}$ after k months, assuming that no rabbit dies.
After k^{th} months, the total number of pairs of rabbits is

$$\begin{aligned} P_k &= (\text{number of pairs alive the previous month}) \\ &\qquad + (\text{number of pairs newly born in the } k^{\text{th}} \text{ month}) \\ &= (\text{number of pairs alive in the } (k-1)^{\text{th}} \text{ month}) \\ &\qquad + (\text{number of pairs alive in the } (k-2)^{\text{th}} \text{ month}) \\ &= P_{k-1} + P_{k-2} \end{aligned}$$

This is known as *Fibonacci's relation*: $P_k = P_{k-1} + P_{k-2}$
and generates the *Fibonacci sequence*: $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$.
We can write Fibonacci's relation in matrix form as

$$\bar{\mathbf{x}}^{(k)} = \begin{bmatrix} P_{k-1} \\ P_k \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} P_{k-2} \\ P_{k-1} \end{bmatrix} = A\bar{\mathbf{x}}^{(k-1)}$$

To see this, note that

$$A\bar{\mathbf{x}}^{(k-1)} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} P_{k-2} \\ P_{k-1} \end{bmatrix} = \begin{bmatrix} P_{k-1} \\ P_{k-1} + P_{k-2} \end{bmatrix} \begin{matrix} \text{young} \\ \text{adults} \end{matrix}$$

So that

$$\bar{\mathbf{x}}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad P_0 = 1 + 0 = 1$$

$$\bar{\mathbf{x}}^{(1)} = A\bar{\mathbf{x}}^{(0)} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad P_1 = 0 + 1 = 1$$

$$\bar{\mathbf{x}}^{(2)} = A\bar{\mathbf{x}}^{(1)} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad P_2 = 1 + 1 = 2$$

$$\bar{\mathbf{x}}^{(3)} = A\bar{\mathbf{x}}^{(2)} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad P_3 = 1 + 2 = 3$$

$$\bar{\mathbf{x}}^{(4)} = A\bar{\mathbf{x}}^{(3)} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad P_4 = 2 + 3 = 5$$

As with Markov processes, it is clear that

$$\begin{aligned}
\bar{\mathbf{x}}^{(1)} &= A\bar{\mathbf{x}}^{(0)} \\
\bar{\mathbf{x}}^{(2)} &= A\bar{\mathbf{x}}^{(1)} = A(A\bar{\mathbf{x}}^{(0)}) = A^2\bar{\mathbf{x}}^{(0)} \\
\bar{\mathbf{x}}^{(3)} &= A\bar{\mathbf{x}}^{(2)} = A^2(A\bar{\mathbf{x}}^{(0)}) = A^3\bar{\mathbf{x}}^{(0)} \\
&\vdots \\
\bar{\mathbf{x}}^{(k)} &= A^k\bar{\mathbf{x}}^{(0)}
\end{aligned}$$

In general, we can set up a population growth problem by representing the number of population members (pairs of rabbits in the previous example) at time step k with the *age distribution vector*:

$$\bar{\mathbf{x}}^{(k)} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \begin{matrix} \text{Number in first age class} \\ \text{Number in second age class} \\ \vdots \\ \text{Number in } n^{\text{th}} \text{ age class} \end{matrix}$$

and the *age transition matrix*:

$$A = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_{n-1} & b_n \\ p_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & p_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & p_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_{n-1} & (p_n) \end{bmatrix}.$$

The numbers b_1 across the first row represent the average number of offspring produced by a member of the i^{th} age class. For our rabbits example, the young produce 0 offspring, and the adults have a 100% chance of producing 1 pair of offspring.

The numbers $p_i = a_{i+1,i}$ represent the probability that a member of the i^{th} age class will survive to become a member of the $(i+1)^{\text{th}}$ age class. In our example, we are assuming the rabbits never die, so $a_{21} = 1$ indicates that each young rabbit has a 100% chance of growing into an adult rabbit, and $a_{22} = 1$ indicates that each adult rabbit has a 100% chance of sticking around as an adult rabbit.

Example 104. Suppose that a population of rabbits raised in a research lab has the following characteristics:

(a) Half of the rabbits survive their first year. Of those, half survive their second year. The maximum lifespan is 3 years.
(b) During the first year, the rabbits produce no offspring. The average number of offspring is 6 during the second year and 8 during the third year.
(c) There are currently 24 newborn rabbits, 24 one-year-old rabbits, and 20 three-year-old rabbits.

From (a), we obtain

$$A = \begin{bmatrix} b_1 & b_2 & b_3 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}.$$

From (b), we complete the age transition matrix as

$$A = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}.$$

From (c), we obtain the initial age distribution vector

$$\bar{\mathbf{x}}^{(0)} = \begin{bmatrix} 24 \\ 24 \\ 20 \end{bmatrix}.$$

How many rabbits will be in each age class after 1 year?

$$\bar{\mathbf{x}}^{(1)} = A\bar{\mathbf{x}}^{(0)} = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 24 \\ 24 \\ 20 \end{bmatrix} = \begin{bmatrix} 144+160 \\ 0.5(24) \\ 0.5(24) \end{bmatrix} = \begin{bmatrix} 304 \\ 12 \\ 12 \end{bmatrix} \begin{matrix} \text{newborn} \\ \text{1-year} \\ \text{2-year} \end{matrix}$$

And the total number of rabbits is $P_1 = 304 + 12 + 12 = 328$.

What is the stable age distribution for this population of rabbits?

This requires solving $A\bar{\mathbf{x}} = \lambda\bar{\mathbf{x}}$:

$$|\lambda I - A| = \begin{vmatrix} \lambda & -6 & -8 \\ -0.5 & \lambda & 0 \\ 0 & -0.5 & \lambda \end{vmatrix} = \lambda^3 - 2 - 3\lambda = (\lambda+1)^2(\lambda-2)$$

So we have eigenvalues $\lambda_1, \lambda_2 = -1$ and $\lambda_3 = 2$. Since it does not make sense to have a negative number as part of an age distribution, choose the positive eigenvalue $\lambda_3 = 2$:

$$\lambda_3 I - A = \begin{bmatrix} 2 & -6 & -8 \\ -0.5 & 2 & 0 \\ 0 & -0.5 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & -6 & -8 \\ -1 & 4 & 0 \\ 0 & -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & -32 \\ -1 & 0 & 16 \\ 0 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -16 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

So for $x_3 = t$, we get $x_2 = 4t$ and $x_1 = 16t$. Thus, the eigenvector is

$$\bar{\mathbf{x}} = t \begin{bmatrix} 16 \\ 4 \\ 1 \end{bmatrix} = t \begin{bmatrix} \frac{16}{21} \\ \frac{4}{21} \\ \frac{1}{21} \end{bmatrix} \approx \begin{bmatrix} 76.2\% \\ 19.0\% \\ 4.8\% \end{bmatrix} \quad \begin{matrix} \text{newborn} \\ \text{1-year} \\ \text{2-year} \end{matrix}$$

So even though the population will continue to grow over time, the population will on average (and in the long run) consist of 76% newborns, 19% 1-year-olds, and 5% 2-year-olds.

Homework Assignment:

Read: pp. 414-416
Exercises: §7.4 #2,3,6,7,9
Supplement: *none*