1. Lecture 01

1.1. **Applications and Motivation.** Since we are pressed for time in this course, and since we will be seeing the examples from §1.1 again next week (when you will actually be solving them yourself), this section has been assigned as Homework Reading.

§1.1 Homework Assignment:

Read: 1-7

Exercises: *none*

1.2. Basic Concepts.

Definition. A *differential equation* (DE) is an equation containing one or more derivatives of an unknown function.

For example,

$$y' = x^2$$
$$y'' = 2x - 2y' + y$$

In each of these examples, the unknown function is $y(x)$. In this class, y always means $y(x)$, a function of x. We say the first is a DE of order 1, since only the first derivative of y appears. The other example is a DE of order 2, since the second derivative of y also appears.

In the study of differential equations, the solution is not a number, but a *function*. Specifically,

Definition. A *solution of a differential equation* is a function that satisfies the differential equation on some open interval (a, b).

Since DE's involve derivatives, they are solved by using various techniques of integration.

Example 1.2.1. $y' = ay$. Rewriting the problem as $\frac{dy}{dx} = ay$, we solve by cross-multiplying and integrating:

$$\begin{aligned} \frac{dy}{y} &= a\,dx \\ \int \frac{dy}{y} &= \int a\,dx \\ \log|y| &= ax + C \\ y &= \pm e^{ax+C} \\ y &= \pm e^{ax}e^{C} = ce^{ax} \end{aligned}$$

If you haven't seen differential equations solved like this before (the technique just demonstrated is called "Separation of Variables") do not worry. This will be covered in the upcoming lectures.

However, if you hope to pass this course, you should know and be very comfortable with the following concepts:

$$e^{x+y} = e^x e^y \qquad \log(xy) = \log x + \log y$$

$$\frac{d}{dx}e^u = e^u \frac{du}{dx} \qquad \int e^u\,du = e^u + C$$

$$\frac{d}{dx}\log|u| = \frac{1}{u}\frac{du}{dx} \qquad \int \frac{1}{u}\,du = \log|u| + C$$

integration by u-substitution

integration by parts

and most other basic integration techniques (for trigonometric expressions, etc).

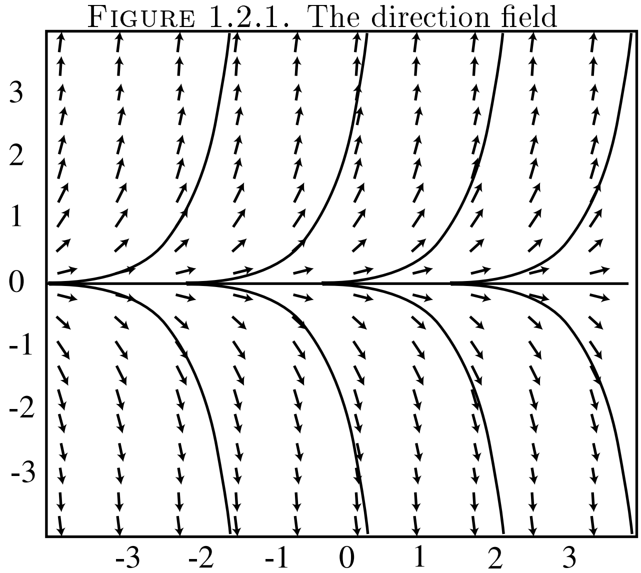
Note that the solution to the previous example

$$y = ce^{ax}$$

contains a constant c which arose during integration. This is typical of DEs: we end with an entire family of solutions. Since any value of c gives a solution to the original DE, we say that $y = ce^{ax}$ is a *(one-parameter) family of solutions* .

To see what this means, let's consider this example graphically. Recall from basic calculus that the derivative is associated with the slope. We use this interpretation to draw a direction field for $y' = 2y$:

FIGURE 1.2.1. The direction field



So the DE says that the slope is equal to the RHS of the equation, at the point (x, y).

Definition. The graph of a solution to the DE is called a *solution curve* or *integral curve* and it will be tangent to every vector in the direction field.

It is clear that there are many lines through this direction field which are everywhere tangent to the vectors; there is one solution curve for each value of c.

Definition. An *initial value problem* (IVP) is a differential equation together with an additional requirement that allows us to specify one certain solution, from a family of many solutions.

Example 1.2.2. Solve the IVP: $y' = 2y$, $y(0) = 3$.
We have already found that the general solution is given by

$$y(x) = ce^{2x},$$

so the initial condition gives

$$\begin{aligned} y(0) = ce^{2\cdot 0} &= 3 \\ ce^{0} &= 3 \\ c &= 3 \end{aligned}$$

So the solution to the IVP is $y = 3e^{2x}$. $y(0) = 3$ actually means that the y-value of the solution curve is 3 when $x = 0$. Drawing the solution curve, we see that it is exactly the one which intersects the y-axis at 3.

In this example ($y' = 2y$), x does not appear in the DE. This corresponds to the fact that the vectors on any vertical line through a given x value are the same as they are on a vertical line through any other x value. The DE does not depend on x. Let's look at a more interesting example.

Example 1.2.3. Verify that

$$y = \tan\left(\frac{x^2}{2}\right)$$

is a solution of the initial value problem

$$y' = x\left(1 + y^2\right), \quad y(0) = 0$$

Taking the derivative of the given solution,

$$y' = \tfrac{d}{dx}y = \tfrac{d}{dx}\tan\left(\frac{x^2}{2}\right)$$
$$= \sec^2\left(\frac{x^2}{2}\right)\cdot\tfrac{d}{dx}\left(\frac{x^2}{2}\right)$$
$$= x\sec^2\left(\frac{x^2}{2}\right)$$

Now check the right side of the equation:

$$x\left(1+y^2\right) = x\left(1+\tan^2\left(\frac{x^2}{2}\right)\right)$$
$$= x\left(\sec^2\left(\frac{x^2}{2}\right)\right)$$

Hence, $y' = x(1+y^2)$. Finally, note that

$$y(0) = \tan\left(\frac{0^2}{2}\right) = 0.$$

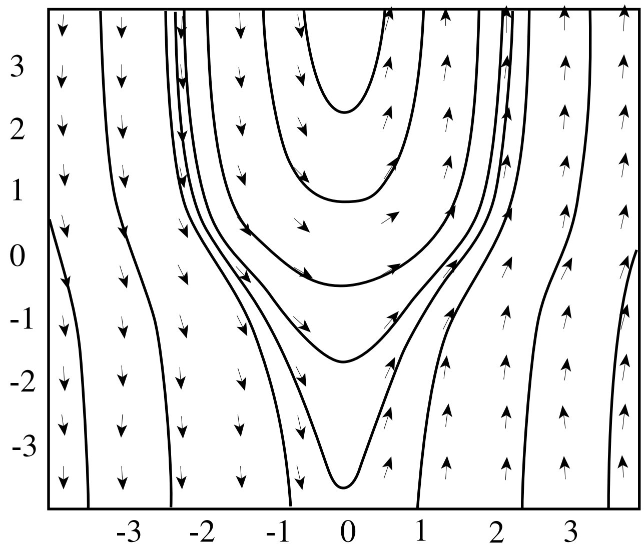
When we plot the direction field:
you can see how the vectors in this example do depend on y.

In the homework, you are asked to sketch some integral curves for various direction fields. While it is important to be able to do this by hand, there are also many computer tools that can help. A very nice one is located at

http://math.rice.edu/∼dfield/pplane.html.

Scroll down the page until you see the big DFIELD 2004.1 button. Click this button, and some windows will appear. To see the example we just completed, fill in the boxes with:

FIGURE 1.2.2. The direction field



$\boxed{y}\,' = \boxed{x(1+y\hat{\ }2)}$,

$\boxed{x}$ dependent variable

Set the Display window to $-4 < x < 4, -4 < y < 4$ and click Graph Phase Plane (you can ignore the Parameter Expressions for now).

The Direction Field will be displayed. To see the solution corresponding to the initial condition $y(0) = 0$, click $(0, 0)$ on the direction field, and the program will draw the curve through that point. Click some other points and see how different the solution curves can be!

Let's look at some examples of higher-order differential equations.

Example 1.2.4. Solve $y'' = x \cos x$.
Integrating, $y' = \int x \cos x \, dx$. Use integration by parts:

$$u = x \qquad dv = \cos x \, dx$$
$$du = dx \qquad v = \sin x$$

So

$$y' = uv - \int v \, du = x \sin x - \int \sin x \, dx = x \sin x + \cos x + c_1.$$

Now, since we need y and we only have y', we must integrate again:

$$\begin{aligned} y &= \int (x \sin x + \cos x + c_1)\, dx \\ &= \int (x \sin x)\, dx + \sin x + c_1 x + c_2 \\ &= -x \cos x + \int \cos x\, dx + \sin x + c_1 x + c_2 \\ &= -x \cos x + \sin x + \sin x + c_1 x + c_2 \\ &= -x \cos x + 2 \sin x + c_1 x + c_2, \end{aligned}$$

where the third line comes by integration by parts with

$$\begin{aligned} u &= x & dv &= \sin x\, dx \\ du &= dx & v &= -\cos x \end{aligned}$$

Notice that since this is a second-order DE, the solution has two unknown constants, c_1 and c_2. This means that in order to specify a certain solution curve, we must give two requirements. In other words, an IVP for a 2nd-order DE would look like

$$y'' = x \cos x, \qquad y'(x_0) = y_1, \qquad y(x_0) = y_2.$$

For instance, the IVP

$$y'' = x \cos x, \qquad y'(0) = 1, \qquad y(0) = 1$$

can be solved:

$$\begin{aligned} y'(0) = 1 &= 0\sin 0 + \cos 0 + c_1 \\ 1 &= 1 + c_1 \\ 0 &= c_1 \end{aligned}$$

$$\begin{aligned} y(0) = 1 &= -0\cos 0 + 2\sin 0 + c_1 \cdot 0 + c_2 \\ 1 &= c_2 \end{aligned}$$

So the solution to the IVP is

$$y(x) = -x\cos x + 2\sin x + 1.$$

Example 1.2.5. Suppose an object is falling near the surface of the earth. Formulate a differential equation that describes the motion. Since the motion takes place in a certain time interval, let t denote time. Then let $y(t)$ be the altitude of the object.
The velocity of the object is the change in altitude over time, or y'.
The acceleration of the object is the change in velocity over time, or $(y')' = y''$.
Assuming no other forces are acting on the object, its acceleration is due entirely to gravity, which we denote by $-g$. (Negative, since the direction of the force of gravity is downward.) Thus

$$y'' = -g.$$

Suppose the object is released with initial velocity v_0 from an initial height of y_0. Then y is a solution to the IVP:

$$y'' = -g, \qquad y(0) = y_0, \qquad y'(0) = v_0.$$

For instance, if you were to stand atop a building that's 100m tall, and throw a ball upwards at 3m/s, you could solve

$$y'' = -9.8, \qquad y(0) = 100, \qquad y'(0) = 3$$

to obtain a formula for the height of the ball as a function of t. Setting it equal to 0, you could solve to find how long it will take for the ball to hit the ground.

We can solve the original IVP by integrating. First,

$$\int y'' = \int -g \quad \Longrightarrow \quad y' = -gt + c_1.$$

Integrating again,

$$\int y' = \int -gt + c_1 \quad \Longrightarrow \quad y = -g\frac{t^2}{2} + c_1 t + c_2.$$

Now the initial conditions give

$$y'(0) = v_0 = -g \cdot 0 + c_1 \quad \Longrightarrow \quad v_0 = c_1,$$

and

$$y(0) = y_0 = -g\frac{0^2}{2} + c_1 \cdot 0t + c_2 \quad \Longrightarrow \quad y_0 = c_2,$$

giving

$$y = -g\frac{t^2}{2} + v_0 t + y_0.$$

§1.2 Homework Assignment:

Read: 8-14

Exercises: 2(abcf),3(abdh),4(af),5(cd)

§1.3 Homework Assignment:

Read: 16-20

Exercises: 1,2,7,9

Then go to http://math.rice.edu/ dfield/pplane.html and plot: 7,9,11 and $y' = \frac{x^2}{1-x^2-y^2}$. Draw several solution curves for each one.

2. First Order Equations

2.1. Linear First Order Equations.

Definition. A first-order differential equation (DE) is called *linear* iff it can be written as

$$y' + p(x)y = f(x).$$

So here we have all terms involving y or y' on the left, and only terms involving x remain on the right. Moreover, the coefficient of y contains only functions of x, and the coefficient of y' is 1.

For example, these are all linear 1st-order DEs:

$$y' = -2(y - x)x^2 \qquad y' + 2x^2y = 2x^3$$
$$y' + y + x = 0 \qquad y' + y = -x$$
$$xy' - 2 = \tfrac{y}{2} \qquad y' - \tfrac{1}{2x}y = \tfrac{2}{x}$$

These, however, are not:

$$yy' - \sin y = x \qquad \Longrightarrow \qquad y' - (\sin y)\tfrac{1}{y} = \tfrac{x}{y}$$
$$y' + (e^y)y = 2$$
$$y' = \sqrt{x + y}$$

Definition. When the $f(x)$ on the RHS of $y' + p(x)y = f(x)$ is 0 and we have

$$y' + p(x)y = 0,$$

then the equation is called *homogeneous* .

If we let $y(x) = 0$ be the constant function which takes the value 0 for every x, it is clear that y satisfies any homogeneous equation:

$$y' + p(x)y = 0' + p(x) \cdot 0 = 0.$$

Definition. $y \equiv 0$ is called the *trivial* solution. Any other solution is *nontrivial.*

Theorem 2.1.1. *Suppose* $p(x)$ *is continuous on* (a, b)*. Then the general solution of*

$$y' + p(x)y = 0$$

on (a, b) *is*

$$y = ce^{-\int p(x)\,dx}.$$

Proof. (Sketch) We may assume we are working on an interval where y is never 0 (there is a justification in the book on page 27, but it isn't really essential to this discussion).
Then we can solve using separation of variables:

$$\begin{aligned}
\tfrac{y'}{y} &= -p(x) \\
\tfrac{dy}{y} &= -p(x)dx \\
\int \tfrac{1}{y}dy &= -\int p(x)dx \\
\log|y| + c_1 &= -\int p(x)dx \\
e^{\log|y|+c_1} &= e^{-\int p(x)dx} \\
e^{\log|y|}e^{c_1} &= e^{-\int p(x)dx} \\
c_2(\pm y) &= e^{-\int p(x)dx} && c_2 = e^{c_1} \\
y &= ce^{-\int p(x)dx} && c = \pm 1/c_2
\end{aligned}$$

□

Example 2.1.1. Solve $xy' + y = 0$.
Rewrite the equation as $y' + \frac{y}{x} = 0$.

Then the coefficient of y is $\frac{1}{x}$, which is continuous for $x \neq 0$. Thus, we can apply the theorem when $x \neq 0$:

$$\begin{aligned} y &= ce^{-\int \frac{1}{x} dx} \\ &= ce^{-\log|x|} \\ &= ce^{\log|x|^{-1}} \\ &= c|x|^{-1} \\ &= cx^{-1} \end{aligned}$$

The last line comes about by deciding whether the solution is on $(-\infty, 0)$ or $(0, \infty)$ and setting $c = \pm c$ accordingly.
If we throw in the initial condition $y(-1) = 2$, then

$$\begin{aligned} y(-1) = 2 = c(-1)^{-1} &= \tfrac{c}{-1} \\ -2 &= c \end{aligned}$$

and the solution to the IVP is $y = -2/x$. Note that this solution is only valid on $(-\infty, 0)$ because $p(x) = \frac{1}{x}$ can only be continuous on $(0, \infty)$ or $(-\infty, 0)$, and we found the solution for $x = -1$, which is in $(-\infty, 0)$.

Example 2.1.2. Solve the IVP:

$$y' + (\tan kx)y = 0, \quad y(0) = 2.$$

Separate the variables:

$$\frac{y'}{y} = -\tan kx.$$

Then integrate (using $\int \tan u\, du = -\log|\cos u|$):

$$\begin{aligned}
\log|y| &= \tfrac{1}{k}\log|\cos kx| + c_1 \qquad\qquad u = kx, \tfrac{1}{k}du = dx \\
&= \log|\cos kx|^{1/k} + c_1 \\
|y| &= e^{\log|\cos kx|^{1/k} + c_1} \\
&= e^{\log|\cos kx|^{1/k}} e^{c_1} \\
&= c_2|\cos kx|^{1/k} \\
y &= \pm c_2|\cos kx|^{1/k} \\
y &= c|\cos kx|^{1/k}
\end{aligned}$$

Then

$$2 = y(0) = c|\cos 0|^{1/k} = c \cdot 1 = c,$$

and the solution to the IVP is

$$y(x) = 2|\cos kx|^{1/k}.$$

Definition. A solution of the form $y = y(x, c)$ is called the *general solution* of the homogeneous equation $y' + p(x)y = 0$ (where c is a parameter).

So the general solution is solved in terms of c, but is not entirely explicit. It is "general" because it gives the entire family of solutions.

Definition. A linear first-order DE

$$y' + p(x)y = f(x)$$

is *nonhomogeneous* iff $f(x) \neq 0\, \forall x$.

If we are solving a nonhomogeneous equation like this, then

$$y' + p(x)y = 0$$

is the *complementary equation / assoc'd homogeneous equation.*

Now we will see how to use a nontrivial solution of the complementary equation to obtain a solution of the nonhomogeneous equation!

This is a special case of "variation of parameters", a technique which we will see several more times. The essential idea is that the nonhomogeneous equation is too difficult to solve, so we solve the complementary equation and use this solution to find one for the original nonhomogeneous.

Suppose we have found that $y_1(x)$ is a solution of $y' + p(x)y = 0$. Guess: the solution y of the nonhomogeneous is of the form uy_1, where $u = u(x)$ is some function. Then

$$y = uy_1 \quad \Longrightarrow \quad y' = u'y_1 + uy_1',$$

by the product rule. Putting these in the original equation,

$$\begin{aligned}
y' + p(x)y &= f(x) \\
u'y_1 + uy_1' + p(x)uy_1 &= f(x) \\
u'y_1 + u(y_1' + p(x)y_1) &= f(x) \\
u'y_1 + u \cdot 0 &= f(x) \\
u'y_1 &= f(x) \\
u' &= \tfrac{f(x)}{y_1} \\
u &= \int \tfrac{f(x)}{y_1}\,dx,
\end{aligned}$$

so that

$$y = y_1(x) \int \tfrac{f(x)}{y_1(x)}\,dx$$

is the general solution.

Why is this called "variation of parameters"?
Basically, u is considered to be a parameter, but it cannot be constant. If it were,

$$\begin{aligned} f(x) &= y' + p(x)y \\ &= (u'y_1 + uy_1') + p(x)uy_1 \\ &= u'y_1 + u(y_1' + p(x)y_1) \\ &= 0 \cdot y_1 + u \cdot 0 \\ &= 0, \end{aligned}$$

which is a contradiction for $f(x) \not\equiv 0$.
So u is a parameter which varies, or at least that's how the originators of the method considered it.

Example 2.1.3. How to use variation of parameters to solve an IVP in 6 easy steps:

$$xy' + 2y = 8x^2, \quad y(1) = 3$$

(1) First, check that the equation is linear:

$$y' + \tfrac{2}{x}y = 8x. \qquad \text{(div by } x\text{)}$$

(2) Solve the complementary equation $y' + \frac{2}{x}y = 0$ by sep of vars (note $p(x) = \frac{2}{x}$ is continuous for $x \neq 0$):

$$\begin{aligned} y' &= -\tfrac{2}{x}y \\ \tfrac{1}{y}dy &= -\tfrac{2}{x}dx \\ \log|y| &= -2\int \tfrac{1}{x}\,dx \\ &= -2\log|x| + c \\ y &= ce^{\log|x|^{-2}} \\ y &= \tfrac{c}{x^2} \quad \text{on } (-\infty, 0) \text{ or } (0, \infty). \end{aligned}$$

It turns out that any choice of c will work for this part of the problem, so take $c = 1$ to make the arithmetic simpler:

$y_1(x) = \frac{1}{x^2}$ is a solution to the complementary equation.

(3) Now substitute $y = uy_1 = \frac{u}{x^2}$ into the original nonhomogeneous equation and solve for u':

$$\begin{aligned} 8x^2 &= xy' + 2y \\ &= x(u'y_1 + uy_1') + 2uy_1 \\ &= x\left(\tfrac{u'}{x^2} - \tfrac{2u}{x^3}\right) + \tfrac{2u}{x^2} \\ &= \tfrac{xu'}{x^2} - \tfrac{2u}{x^2} + \tfrac{2u}{x^2} \\ &= \tfrac{xu'}{x^2} \\ 8x^3 &= u' \end{aligned}$$

(4) Integrate to find u:

$$u = 8\tfrac{x^4}{4} + c = 2x^4 + c$$

(5) Use $y = uy_1$ to recover y:

$$uy_1 = 2x^4\tfrac{1}{x^2} + \tfrac{c}{x^2}$$
$$y = 2x^2 + \tfrac{c}{x^2}$$

(6) Use the initial condition to complete the IVP:

$$y(1) = 3 \implies 3 = 2 \cdot 1^2 + \tfrac{c}{1^2}$$
$$= 2 + c$$
$$1 = c$$

In case you're suspicious about the arbitrary choice of setting $c = 1$ in step 2, here's why it works: suppose we kept the c in there and took our solution to the complementary equation as

$$y_1 = \tfrac{c_1}{x^2}.$$

Then

$$y = uy_1 = \tfrac{c_1 u}{x^2} \qquad \text{and} \qquad y' = u'y_1 + uy_1' = u'\tfrac{c_1}{x^2} - 2u\tfrac{c_1}{x^3}.$$

So $y' + \frac{2}{x}y = 8x$ becomes

$$u'\tfrac{c_1}{x^2} - \tfrac{2c_1 u}{x^3} + \tfrac{2}{x}\tfrac{c_1 u}{x^2} = 8x$$
$$u'\tfrac{c_1}{x^2} = 8x$$
$$u' = \tfrac{1}{c_1}8x^3$$
$$u = \tfrac{2}{c_1}x^4 + c_2$$
$$y = uy_1 = \left(\tfrac{2}{c_1}x^4 + c_2\right)\tfrac{c_1}{x^2}$$
$$y = 2x^2 + \tfrac{c_1 c_2}{x^2}$$
$$y = 2x^2 + \tfrac{c}{x^2},$$

and we obtain the same answer, with a little more trouble. So next time, we will just take $c = 1$ to make life simple. We only need a nontrivial solution to the complementary equation, so we can take c to be anything other than 0 in this example.

§2.1 Homework Assignment:

Read: 26-36
Exercises: 3,5,9,10,32,36,46(ab)

2.2. Separable Equations.

Definition. A first-order differential equation is called *separable* iff it can be written as

$$h(y)y' = g(x),$$

so that all y's appear on the left, and all x's appear on the right. For example, any homogeneous linear equation is separable:

$$\begin{aligned} y' + p(x)y &= 0 \\ y' &= -p(x)y \\ \tfrac{y'}{y} &= -p(x) \end{aligned}$$

Rewriting an equation this way is called *separation of variables* .

We can use separation of variables to solve DEs. Suppose that y is a solution to the DE and seeing what conditions y must satisfy:

$$\text{If } H'(y) = h(y) \quad \text{and} \quad G'(x) = g(x),$$

then H, G are antiderivatives of h, g respectively, and by chain rule,

$$H'(y) = H'(y(x)) = H'(y(x))y'(x) = h(y)y' = g(x) = G'(x).$$

So

$$\begin{aligned} h(y)y' = g(x) \quad &\iff \quad \tfrac{d}{dx}H(y(x)) = \tfrac{d}{dx}G(x) \\ &\iff \quad H(y(x)) = G(x) + c \end{aligned}$$

and to solve the original DE, it suffices to find H and G.
In other words, to solve a separable DE, do the following:

(1) Put all the y's (and y''s) on the left and all the x's on the right.
(2) Integrate LHS with respect to y and RHS with respect to x.
(3) Combine the integration constants and you have a solution to the original DE.

Example 2.2.1. Several of the DEs we've seen solved so far have been solved using this technique. Here's yet another example:

$$y' + x\left(y^2 + y\right) = 0, \quad y(2) = 1.$$

So this is an IVP. First, make sure the equation is separable:

$$\tfrac{y'}{y^2+y} = -x.$$

Before we can integrate both sides, we need to use partial fractions to simplify the LHS:

$$\begin{aligned} \tfrac{1}{y(y+1)} &= \tfrac{A}{y} + \tfrac{B}{y+1} \\ 1 &= A(y+1) + By \\ \implies Ay + By = 0 \quad &\text{and} \quad A = 1 \end{aligned}$$

So $B = -1$ and we have

$$\left(\tfrac{1}{y} - \tfrac{1}{y+1}\right) y' = -x.$$

Next, integrate each side and get

$$\text{LHS:} \qquad \int \left(\tfrac{1}{y} - \tfrac{1}{y+1}\right) y'\, dy = \log|y| - \log|y+1| + c_1 = \log\left|\tfrac{y}{y+1}\right| + c_1$$

$$\text{RHS:} \qquad -\int x\, dx = -\tfrac{x^2}{2} + c_2.$$

Put these together and collect the integration constants:

$$\log\left|\tfrac{y}{y+1}\right| = -\tfrac{x^2}{2} + c.$$

Take the exponential of each side:

$$\tfrac{y}{y+1} = ce^{-x^2/2}.$$

Plug in the initial condition $y(2) = 1$:

$$\tfrac{1}{1+1} = \tfrac{1}{2} = ce^{-2^2/2} \implies c = \tfrac{e^2}{2}.$$

Thus

$$\begin{aligned}
\tfrac{y}{y+1} &= \tfrac{1}{2}e^{2-x^2/2} & \\
2y &= (y+1)e^{2-x^2/2} & \text{mult by } 2(y+1) \\
2y &= ye^{2-x^2/2} + e^{2-x^2/2} & \text{distribute} \\
2y - ye^{2-x^2/2} &= e^{2-x^2/2} & y\text{'s to LHS} \\
\left(2 - e^{2-x^2/2}\right) y &= e^{2-x^2/2} & \text{factor out } y \\
y &= \frac{e^{2-x^2/2}}{\left(2 - e^{2-x^2/2}\right)} & \text{solve for } y
\end{aligned}$$

Sometimes it is possible to obtain an explicit function $y(x)$ in this manner, but not always.

Example 2.2.2.

$$\left(3y^2 + 4y\right) y' + 2x + \cos x = 0, \quad y(0) = 1.$$

Separate the variables:

$$\left(3y^2 + 4y\right) y' = -2x - \cos x.$$

Integrate both sides:

$$\begin{aligned}
3\int y^2\,dy + 4\int y\,dy &= -2\int x\,dx - \int \cos x\,dx \\
y^3 + 2y^2 + c_1 &= -x^2 - \sin x + c_2 \\
y^3 + 2y^2 &= -x^2 - \sin x + c
\end{aligned}$$

While we cannot isolate (solve for) y, we can still solve the IVP. Using the initial condition, this becomes

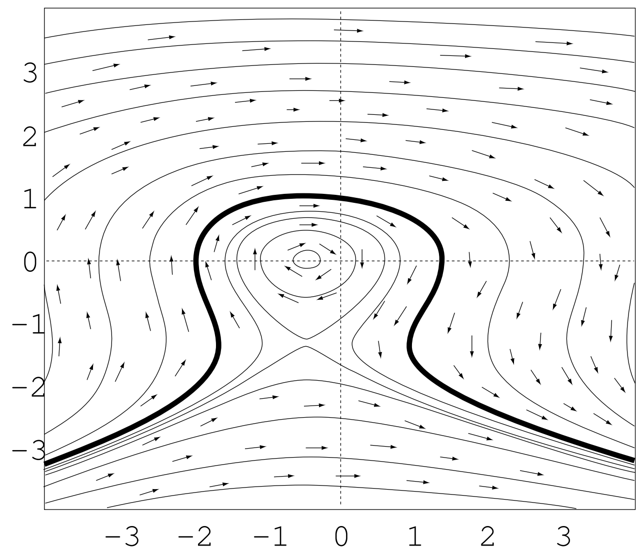
$$\begin{aligned} y^3 + 2y^2 = 1^3 + 2 \cdot 1^2 &= -0^2 - \sin 0 + c = -x^2 - \sin x + c \\ 1 + 2 &= c \\ c &= 3 \end{aligned}$$

So the solution is

$$y^3 + 2y^2 = -x^2 - \sin x + 3.$$

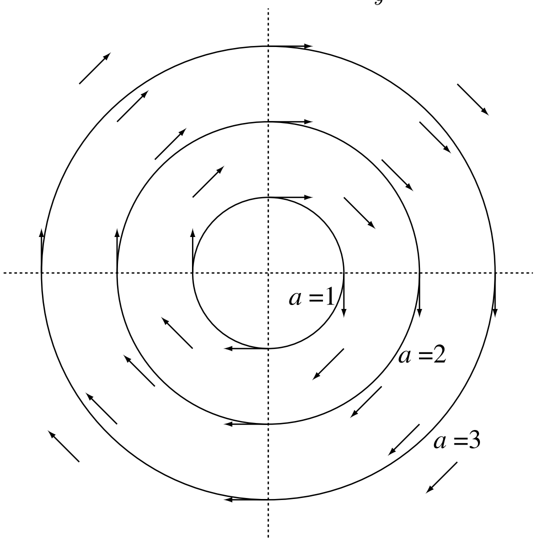
When we look at the direction field, it becomes apparent why we cannot solve explicitly for y; we have integral curves which aren't solution curves. I.e., many of the curves in the figure are not functions of x since they fail the vertical line test.

FIGURE 2.2.1. The direction field for $(3y^2 + 4y)\, y' + 2x + \cos x = 0$. The dark curve is $y(0) = 1$.



Example 2.2.3. $y' = -\frac{x}{y}$. Sketch the direction field.

FIGURE 2.2.2. A direction field for $y' = -\frac{x}{y}$ with some integral curves.



We solve this as before:

$$\begin{aligned} y\,dy &= -x\,dx \\ \tfrac{y^2}{2} &= -\tfrac{x^2}{2} + c \\ x^2 + y^2 &= a^2 \qquad\qquad a^2 = 2c, a > 0 \end{aligned}$$

Earlier, we talked about how initial conditions for different x's cause sign changes in the integration constant c, to make it appropriate for the relevant domain of definition (e.g., $(-\infty, 0)$ or $(0, \infty)$). Here, initial conditions with the same x but different y's will give different solution curves. Compare

$$y(0) = 1 \qquad \text{versus} \qquad y(0) = -1.$$

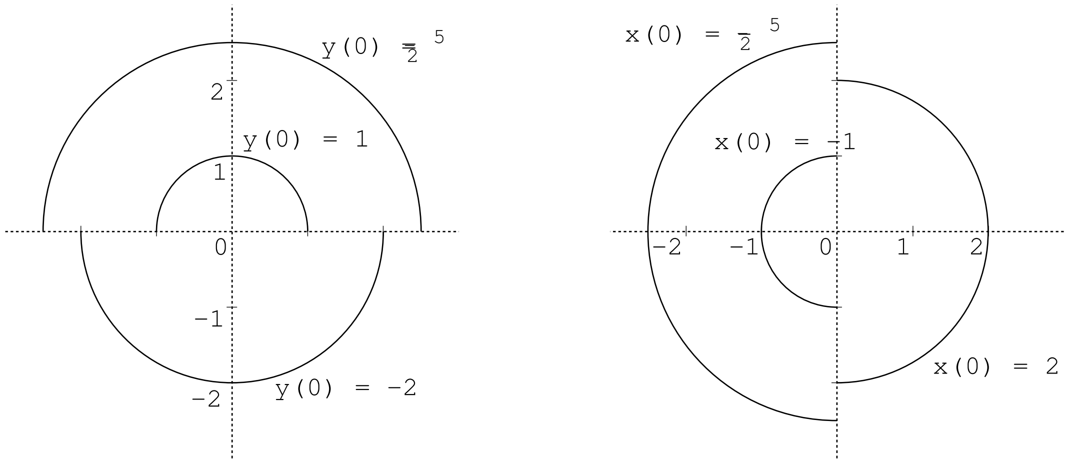
The general solution cannot be solved explicitly, but the IVP can.

$$\text{For } y(0) = 1, \qquad y = \sqrt{1 - x^2}.$$

$$\text{For } y(0) = -1, \qquad y = -\sqrt{1 - x^2}.$$

Note that both of these solutions only exist for $-1 < x < 1$.

FIGURE 2.2.3. Some solution curves for when solved as a function of x, and when solved as a function of y. The diagram at left shows the curve for $y(0) = 1$.



Example 2.2.4. Use variation of parameters followed by separation of variables:

$$y' + y = \frac{2xe^{-x}}{1+ye^{x}}$$

For variation of parameters, recall that we first solve the complementary equation by separation of variables:

$$\begin{aligned} y' + y &= 0 \\ \frac{dy}{dx} + y &= 0 \\ \frac{dy}{dx} &= -y \\ \frac{dy}{y} &= -dx \\ \log|y| &= -x + c \end{aligned}$$

So $y_1 = ce^{-x}$ is a solution to the complementary equation.
Take $c = 1$ for convenience, so $y_1 = e^{-x}$.
Then we put $y = uy_1$ into the original equation:

$$y' + y = (u'y_1 + uy_1') + uy_1 = \frac{2xe^{-x}}{1+uy_1e^{x}}$$

Solve for u':

$$\begin{aligned}
u'e^{-x} - ue^{-x} + ue^{-x} &= \tfrac{2xe^{-x}}{1+ue^{-x}e^{x}} \\
u' &= \tfrac{2xe^{-x}}{1+u}e^{x} = \tfrac{2x}{1+u} \\
(1+u)\tfrac{du}{dx} &= 2x \\
(1+u)du &= 2x\,dx \\
u + \tfrac{u^2}{2} &= x^2 + c \\
u^2 + 2u &= 2x^2 + c
\end{aligned}$$

This would be difficult to solve for u, if the constant c were not present. Since constants do not matter, however, we can complete the square on the LHS without changing the RHS:

$$\begin{aligned}
u^2 + 2u + 1 &= 2x^2 + c \\
(u+1)^2 &= 2x^2 + c \\
u + 1 &= \pm\sqrt{2x^2 + c} \\
u &= -1 \pm \sqrt{2x^2 + c}
\end{aligned}$$

The last step is to recover y: $y = uy_1 = \left(-1 \pm \sqrt{2x^2 + c}\right) e^{-x}$.

Note that instead of completing the square as we did above, we could also have integrated $(u+1)$ directly:

$$\int (u+1)\,du = \tfrac{(u+1)^2}{2} + c.$$

§2.2 Homework Assignment:

Read: 40-47

Exercises: 1-4,11,12,37

Feel free to use a computer to plot the direction fields and integral curves on the homework, to save time. However, keep in mind that you DO need to know how to do these things by hand, for example, on Midterm 1.

2.5. Exact Equations.

Definition. If we write a first-order linear DE

$$y' + p(x)y = f(x)$$

in the form

$$\frac{dy}{dx} + p(x)y = f(x)$$
$$p(x)y - f(x) + \frac{dy}{dx} = 0,$$

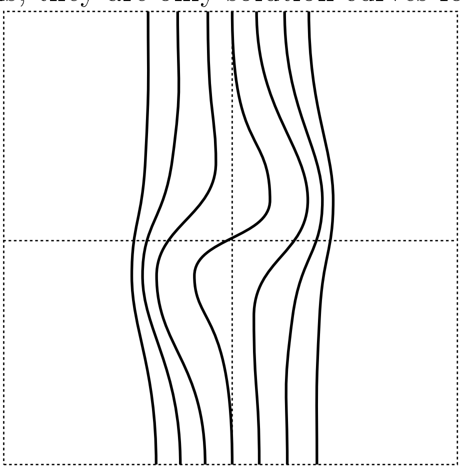
And more generally,

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$
$$M(x, y)dx + N(x, y)dy = 0$$

Then we obtain the *differential form* of the equation. This form has the advantage of begin more symmetric and thus diminishes the distinction between dependent and independent variables.

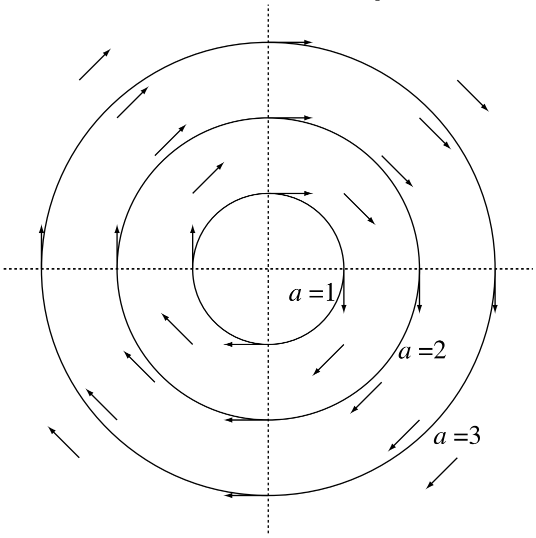
One reason for the importance of this symmetry is that some direction fields may have integral curves which may be solution curves for one variable but not the other.

FIGURE 2.5.1. These integral curves can be written as functions of y, but not as functions of x. Thus, they are only solution curves for x as a function of y.



Recall that we solved the example $y' = -\frac{x}{y}$ in the last section to obtain $y = \pm\sqrt{a^2 - x^2}$.

FIGURE 2.5.2. A direction field for $y' = -\frac{x}{y}$ with some integral curves.



If we need a solution that exists for a region containing a circle about some point on the x-axis, then we'd need to obtain this solution as a function of y. (See Figure 2.2.3.)

Now we can use some neat tricks to solve the DE implicitly, if it has the right form.

Definition. If M, N are continuous and have continuous partial derivatives M_y, N_x in an open rectangle R, then

$$M(x, y)dx + N(x, y)dy = 0$$

is called *exact* on R iff

$$M_y(x, y) = N_x(x, y) \quad \forall x, y \in R.$$

Notation. Recall the following convenient shorthand for partial derivatives:

$$M_y(x, y) := \frac{\partial}{\partial y} M(x, y), \quad \text{and} \quad N_x(x, y) := \frac{\partial}{\partial x} N(x, y).$$

Example 2.5.1. The DE $2x + y^2 + 2xyy' = 0$ is neither linear nor separable. [1]

$$\begin{aligned} 2x + y^2 + 2xy\tfrac{dy}{dx} &= 0 \\ 2xyy' + y^2 &= -2x \\ yy' + \tfrac{1}{2x}y^2 &= -1 \end{aligned}$$

However, $(2x + y^2)dx + (2xy)dy = 0$ is exact! The function

$$F(x, y) = x^2 + xy^2$$

has the property that

$$\tfrac{\partial F}{\partial x} = M(x, y) = 2x + y^2, \qquad \tfrac{\partial F}{\partial y} = N(x, y) = 2xy.$$

This implies $M_y = N_x$ because

$$M_y = \tfrac{\partial}{\partial y}M = \tfrac{\partial^2}{\partial y \partial x}F \quad \text{and} \quad \tfrac{\partial^2}{\partial x \partial y}F = \tfrac{\partial}{\partial x}N = N_y,$$

and we know from calculus that mixed partials derivatives are equal. Therefore, we can use the chain rule to rewrite this equation as

$$\tfrac{\partial F}{\partial x} + \tfrac{\partial F}{\partial y}\tfrac{dy}{dx} = (2x + y^2) + (2xy)\tfrac{dy}{dx} = 0.$$

Considering $y = y(x)$ as a function of x, this is the same as

$$\tfrac{dF}{dx} = \tfrac{d}{dx}\left(x^2 + xy^2\right) = 0.$$

Therefore, $F(x, y) = x^2 + xy^2 + c$ is an equation that defines solutions of the original equation implicitly (for any c).

To check this, note that if we implicitly differentiate the solution

$$x^2 + xy^2 = c,$$

we obtain the original equation:

$$2x + y^2 + 2xyy' = 0.$$

[1] Recall that if this DE were linear, one could write it in the form $y' + (\text{fn of } x \text{ only})y = (\text{fn of } x \text{ only})$.

So why are these called exact equations? The term comes from Calculus, where you saw in Math 10A (Marsden 2.4,2.6) that for $F(x_1, x_2, \ldots, x_n)$, the *exact differential* (sometimes also called the *total differential*) is

$$dF = \frac{\partial F}{\partial x_1}dx_1 + \frac{\partial F}{\partial x_2}dx_2 + \cdots + \frac{\partial F}{\partial x_n}dx_n.$$

Procedure for Solving an Exact Equation

1. Check that

$$M(x,y)dx + N(x,y)dy = 0$$

satisfies $M_y = N_x$. If not, stop here. When should you check to see if an equation is exact?

- If the DE is not linear or separable.
- If it looks generally nasty.

2. Integrate

$$\frac{\partial F}{\partial x} = M(x,y)$$

with respect to x to obtain

$$F(x,y) = G(x,y) + \varphi(y),$$

where G is an antiderivative of M with respect to x, and φ is some function of y only.

We include $\varphi(y)$ for the same reason we usually include c; we're integrating with respect to x in an attempt to "undo" differentiating with respect to x. Differentiating with respect to x will kill off any function $\varphi(y)$.

3. Differentiate the previous equation with respect to y to obtain

$$\frac{\partial F}{\partial y} = \frac{\partial G}{\partial y} + \varphi'(y).$$

4. Use
$$\frac{\partial F}{\partial y} = \frac{\partial G}{\partial y} + \varphi'(y) = N(x, y)$$
to solve for φ':
$$\varphi'(y) = N(x, y) - \frac{\partial G}{\partial y}.$$

5. Integrate φ' with respect to y, and take the constant of integration to be 0. This will give you
$$\varphi(y) = \int \left(N(x, y) - \frac{\partial G}{\partial y} \right) dy,$$
so you can find the solution
$$F(x, y) = G(x, y) + \varphi(y) = c.$$
In general, most terms of $N(x, y) - \frac{\partial G}{\partial y}$ will cancel out.

6. If possible, solve for y as an explicit function of x.

Example 2.5.2. Let's return to the example we just saw:
$2x + y^2 + 2xyy' = 0$. Here,
$$M(x, y) = 2x + y^2 \quad \text{and} \quad N(x, y) = 2xy.$$
We already checked that this is exact, so proceed from step 2:

2. We begin with $F(x, y) = \int \frac{\partial F}{\partial x} dx = \int M\,dx$
$$\int \left(2x + y^2\right) dx = x^2 + xy^2 + \varphi(y).$$
So we have $G(x, y) = x^2 + xy^2$.

3. Differentiating with respect to y,
$$\frac{\partial F}{\partial y} = \frac{\partial G}{\partial y} + \varphi'(y) = 2xy + \varphi'(y).$$

4. Then $\varphi'(y) = N(x,y) - \frac{\partial G}{\partial y} = 2xy - 2xy = 0.$

5. So $\varphi(y) = 0$ also, and we get

$$F(x,y) = G(x,y) + \varphi(y) = x^2 + xy^2 + 0,$$

and take the solution to be

$$x^2 + xy^2 = c.$$

Example 2.5.3. Let's see another example of this procedure.

$$(y\cos x + 2xe^y)\,dx + \big(\sin x + x^2e^y - 1\big)\,dy = 0.$$

1. $M_y(x,y) = \cos x + 2xe^y = N_x(x,y)$, so the equation is exact. Thus there is a $F(x,y)$ satisfying

$$F_x(x,y) = y\cos x + 2xe^y \quad \text{and} \quad F_y(x,y) = \sin x + x^2e^y - 1.$$

2. Integrate with respect to x to get

$$\begin{aligned}\int F_x(x,y)\,dx &= \int (y\cos x + 2xe^y)\,dx \\ &= y\int \cos x\,dx + 2e^y\int x\,dx \\ &= \underbrace{y\sin x + x^2e^y}_{G(x,y)} + \varphi(y)\end{aligned}$$

3. Differentiate with respect to y to get

$$F_y(x,y) = \sin x + x^2e^y + \varphi'(y).$$

4. Then

$$\varphi'(y) = \underbrace{\left(\sin x + x^2 e^y - 1\right)}_{N(x,y)} - \underbrace{\sin x - x^2 e^y}_{\frac{\partial G}{\partial y}} = -1.$$

5. Integrating,

$$\int \varphi'(y)\, dy = -\int dy = -y.$$

6. Thus the solutions are given implicitly by

$$F(x,y) = G(x,y) + \varphi(y) = y\sin x + x^2 e^y - y = c.$$

We can easily check from here that

$$F_x = y\cos x + 2xe^y \quad \text{and} \quad F_y = \sin x + x^2 e^y - 1.$$

Example 2.5.4. Due to the symmetry of the equation $M(x,y)dx + N(x,y)dy = 0$, we can execute the method of exact equations in the other order. This can be a considerable advantage, as sometimes it may be much easier to integrate $\int N\, dy$ than $\int M\, dx$, as in the following example:

$$\underbrace{\left(3x^2\cos xy - x^3 y\sin xy + 4x\right)}_{M(x,y)} dx + \underbrace{\left(8y - x^4\sin xy\right)}_{N(x,y)} dy = 0.$$

Following the method,

1. $M_y = -3x^3\sin xy - x^3\sin xy - x^4 y\cos xy$ and
   $N_x = -4x^3\sin xy - x^4 y\cos xy$, so exact.

2. Would you rather integrate $\int M\,dx$ or $\underline{\int N\,dy}$?

$$\int N\,dy = 8\tfrac{y^2}{2} - x^4 \int \sin xy\,dy = \underbrace{4y^2 + x^3\cos xy}_{G(x,y)} + \varphi(y).$$

3. Now apply $\frac{\partial}{\partial x}$ to obtain

$$\underbrace{3x^2\cos xy - x^3 y\sin xy}_{\frac{\partial G}{\partial x}} + \varphi'(x).$$

Note that φ is a function of x this time!

4. $\varphi'(x) = M(x,y) - \frac{\partial G}{\partial x} = 4x$.

5. $\varphi(x) = \int 4x\,dx = 2x^2$.

6. $F(x,y) = G(x,y) + \varphi(x) = 4y^2 + x^3\cos xy + 2x^2$, so the solution is

$$4y^2 + x^3\cos xy + 2x^2 = c.$$

Example 2.5.5. Now let's see an example of a differential equation which is *not* exact:

$$\left(3xy + y^2\right)dx + \left(x^2 + xy\right)dy = 0.$$

We begin the procedure:

1. Immediately,

$$M_y(x,y) = 3x + 2y \quad \text{and} \quad N_x(x,y) = 2x + y,$$

so the equation is not exact.

To see why it cannot be solved by the procedure just described, let's try to find F with

$$\frac{\partial F}{\partial x} = 3xy + y^2 \quad \text{and} \quad \frac{\partial F}{\partial y} = x^2 + xy.$$

Integrating the first equation gives

$$F(x, y) = \tfrac{3}{2}x^2 y + xy^2 + \varphi(y),$$

where $\varphi(y)$ is any function of y only (because taking the partial derivative with respect to x will kill any such function).
Now we should be able to find φ by applying $\frac{\partial}{\partial y}$:

$$\begin{aligned} F_y(x, y) &= \tfrac{\partial}{\partial y}\left(\tfrac{3}{2}x^2 y + xy^2 + \varphi(y)\right) \\ &= \tfrac{3}{2}x^2 + 2xy + \varphi'(y) \\ &= x^2 + xy && \text{(from before)} \end{aligned}$$

Isolating φ' gives

$$\varphi'(y) = -\tfrac{1}{2}x^2 - xy,$$

but the RHS of this equation depends on both x and y, so it is impossible to solve it for $\varphi(y)$. (Recall, φ is a function of y *alone*.) Thus there is no $F(x, y)$ that can satisfy both

$$\frac{\partial F}{\partial x} = 3xy + y^2 \quad \text{and} \quad \frac{\partial F}{\partial y} = x^2 + xy \tag{2.5.1}$$

on an open rectangle.

This comment about "open rectangle" is necessary because there may be some trivial solutions. For example, $F \equiv 0$ is a solution of (2.5.1) for the single point $(0, 0)$, but not for any open set.

§2.5 Homework Assignment:

Read: 66-73

Exercises: 1-4,12,13

2.6. **Integrating Factors.**

We just saw that

$$\left(3xy + y^2\right) dx + \left(x^2 + xy\right) dy = 0.$$

is not exact, but we can actually *correct* this situation, in a manner of speaking.
If we multiply across by x,

$$\left(3x^2y + xy^2\right) dx + \left(x^3 + x^2y\right) dy = 0.$$

Now,

$$\frac{\partial}{\partial y}\left(3x^2y + xy^2\right) = 3x^2 + 2xy = \frac{\partial}{\partial x}\left(x^3 + x^2y\right),$$

so the criterion for exactness is satisfied for this latter equation. Also, the method previously discussed easily allows one to find that its solutions are implicitly given by $x^3y + \frac{1}{2}x^2y^2 = c$.

$$\underbrace{\left(3x^2y + xy^2\right)}_{M(x,y)} dx + \underbrace{\left(x^3 + x^2y\right)}_{N(x,y)} dy = 0.$$

We just checked exactness, so carry on with the method:

2. Pick which term to integrate: $F(x, y) = \int M\, dx$. Then

$$\int \left(3x^2y + xy^2\right) dx = \underbrace{x^3y + \tfrac{1}{2}x^2y^2}_{G(x,y)} + \varphi(y).$$

3. $F_y = x^3 + x^2y + \varphi(y)$.

4. $\varphi'(y) = 0$.

5. $\varphi(y) = 0$.

6. $F(x,y) = G(x,y) + \varphi(y) = x^3y + \frac{1}{2}x^2y^2$, so
$$x^3y + \tfrac{1}{2}x^2y^2 = c.$$

Definition. An *integrating factor* is a function $\mu = \mu(x,y)$ which, when multiplied against a DE, yields an exact equation. In other words, μ is an integrating factor for
$$M(x,y)dx + N(x,y)dy = 0$$
iff
$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$
is exact.

How does this work? If the previous equation is exact, then
$$(\mu M)_y = (\mu N)_x$$
$$\mu_y M + \mu M_y = \mu_x N + \mu N_x,$$
so μ must satisfy the partial differential equation
$$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0.$$
↑ partials! ↑

This equation may have more than one solution. For example, you can verify that
$$\mu(x,y) = \frac{1}{xy(2x+y)}$$
is also an integrating factor for the previous example.

Unfortunately, the partial differential equation
$$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0.$$
is often at least as difficult to solve as the original equation. Therefore, while integrating factors are in principle a very powerful tool for

solving differential equations, they can often only be found in special cases. The most important situation in which simple integrating factors can be found is when μ is a function of only one of the variables x, y.

In this case, let's determine necessary conditions on M, N so that we have an integrating factor $\mu = \mu(x)$ which depends on x alone. In this case,

$$(\mu M)_y = \mu M_y \quad \text{and} \quad (\mu N)_x = \mu N_x + N\tfrac{d\mu}{dx},$$

so $(\mu M)_y = (\mu N)_x$ implies that

$$\tfrac{d\mu}{dx} = \tfrac{M_y - N_x}{N}\mu.$$

Thus, if $\frac{M_y - N_x}{N}$ is a function of x alone, then there is an integrating factor μ that depends only on x. Further, μ can be found by solving the last equation (it is linear & separable):

$$\begin{aligned} \tfrac{d\mu}{dx} &= \tfrac{M_y - N_x}{N}\mu \\ \tfrac{d\mu}{\mu} &= \tfrac{M_y - N_x}{N}\, dx \\ \log|\mu| &= \int \left(\tfrac{M_y - N_x}{N}\right) dx \\ \mu &= \pm e^{\int \left(\frac{M_y - N_x}{N}\right) dx} = \pm e^{\int p(x) dx}, \end{aligned}$$

where $p(x) = \frac{M_y - N_x}{N}$, as in the book.

The book does the next best thing: try to find a μ which is a product of a function of x and a function of y:

$$\mu(x, y) = P(x)Q(y).$$

However, this approach leads to exactly the same result.

Example 2.6.1.

$$\text{Solve } \left(27xy^2 + 8y^3\right) dx + \left(18x^2y + 12xy^2\right) dy = 0.$$

First off, note that this equation is not exact, since

$$M_y = 54xy + 24y^2$$
$$N_x = 36xy + 12y^2.$$

So we look for an integrating factor.

$$M_y - N_x = (54xy + 24y^2) - (36xy + 12y^2) = 18xy + 12y^2$$
$$p(x) = \tfrac{M_y - N_x}{N} = \tfrac{18xy+12y^2}{18x^2y+12xy^2} = \tfrac{1}{x}$$
$$\int p(x)\,dx = \int \tfrac{1}{x}\,dx = \log|x| \qquad (\text{take } c = 0)$$
$$\mu = \pm e^{\int p(x)dx} = x$$

Then

$$\underbrace{\left(27x^2y^2 + 8xy^3\right)}_{M(x,y)} dx + \underbrace{\left(18x^3y + 12x^2y^2\right)}_{N(x,y)} dy = 0$$

is exact and we use the method of the previous section.

2. Choose $F(x,y) = \int F_x\,dx = \int M\,dx$.

$$27y^2 \int x^2\,dx + 8y^3 \int x\,dx = \underbrace{9x^3y^2 + 4x^2y^3}_{G(x,y)} + \varphi(y).$$

3. Differentiating, $F_y = \underbrace{18x^3y + 12x^2y^2}_{\frac{\partial G}{\partial y}} + \varphi'(y).$

4. $\varphi'(y) = N(x,y) - \frac{\partial G}{\partial y} = 0.$

5. Integrating with respect to y, $\varphi(y) = 0$.

6. $F(x, y) = G(x, y) + \varphi(y) = 9x^3y^2 + 4x^2y^3 + 0$, so

$$x^2y^2(9x + 4y) = c.$$

Notice the shortcut: if $G(x, y) = N(x, y)$, you're done in step 2:

$$G(x, y) = c.$$

Example 2.6.2.

$$\text{Solve } y^2dx + \left(xy^2 + 3xy + \tfrac{1}{y}\right) dy = 0.$$

$$M_y - N_x = 2y - (y^2 + 3y) = -y - y^2$$
$$p(x) = -\tfrac{y+y^2}{xy^2+3xy+1/y} = -\tfrac{1+y}{xy+3x+y^{-2}}$$

This looks tough to integrate, so let's try something else. There is no particular reason why we always need to find $\mu(x)$; just as in the previous section, we can exploit symmetry and make do with $\mu(y)$, interchanging the roles of x and y all the way through.

$$q(y) = \tfrac{N_x - M_y}{M} = \tfrac{y+y^2}{y^2} = \tfrac{1}{y} + 1$$
$$\int q(y)\, dy = \int \tfrac{1}{y}\, dy + \int dy = \log|y| + y$$
$$\mu(y) = \pm e^{\int q(y) dy} = \pm e^{\log|y|}e^y = ye^y$$

Then

$$y^3e^ydx + \left(xy^3e^y + 3xy^2e^y + e^y\right) dy = 0$$

is exact and we use the method of the previous section.

2. Choose $F(x,y) = \int M\,dx$.

$$y^3e^y \int dx = \underbrace{xy^3e^y}_{G(x,y)} + \varphi(y).$$

3. $F_y = \underbrace{3xy^2e^y + xy^3e^y}_{\frac{\partial G}{\partial y}} + \varphi'(y)$.

4. $\varphi'(y) = \underbrace{\left(xy^3e^y + 3xy^2e^y + e^y\right)}_{N(x,y)} - \underbrace{\left(3xy^2e^y + xy^3e^y\right)}_{\frac{\partial G}{\partial y}} = e^y$.

5. $\varphi(y) = e^y$.

6. $F(x,y) = xy^3e^y + e^y$ so the solution is

$$xy^3e^y + e^y = c.$$

§2.6 Homework Assignment:

Read: 76-84

Exercises: 4,6-10

4. Applications of First Order Equations

4.1. Growth and Decay.

Definition. An *exponential model* for a process is one in which the rate of change of a given quantity Q is proportional to its value at a that time. This model occurs naturally in the study of bacterial propagation and radioactive decay and leads to methods of solving problems involving compound interest and the mixing of solutions.

An exponential model is based on an equation of the form

$$Q' = aQ.$$

Note: a is referred to as the *constant of proportionality*. Also, since most of these processes are describing the state of a system as time varies, we will use t for the dependent variable. We solve this equation:

$$\begin{aligned} \tfrac{dQ}{dt} &= aQ \\ \tfrac{dQ}{Q} &= a\,dt \\ \log|Q| &= at + c \\ Q(t) &= ce^{at} \end{aligned}$$

If we're also given an initial condition $Q(t_0) = Q_0$, then

$$\begin{aligned} Q(t_0) = ce^{at_0} &= Q_0 \\ c &= Q_0 e^{-at_0} \\ Q(t) &= Q_0 e^{a(t-t_0)} \end{aligned}$$

Example 4.1.1. For example, how would you find the constant of proportionality k for a radioactive substance undergoing decay, given that the mass of the substance is Q_1 at time t_1 and Q_2 at time t_2? Radioactive decay is known to occur at a rate proportional to the amount of the radioactive substance, so we can use what we just found:

$$Q_1 = Q(t_1) = Q_0 e^{a(t_1 - t_0)} \quad \text{and} \quad Q_2 = Q(t_2) = Q_0 e^{a(t_2 - t_0)}.$$

Then we need to solve for the decay constant $k = -a$. The constant of proportionality is called a *growth* constant when it is positive, and a *decay* constant when it is negative. When it is negative, we replace it by $-k$ where $k > 0$.
To get an answer that is no longer in terms of Q_0, we need to cancel it out.[2] Hence:

$$\begin{aligned}
\frac{Q_2}{Q_1} &= \frac{Q_0 e^{-k(t_2 - t_0)}}{Q_0 e^{-k(t_1 - t_0)}} = \frac{e^{-k(t_2 - t_0)}}{e^{-k(t_1 - t_0)}} = e^{-k(t_2 - t_1)} \\
\log\left(\frac{Q_2}{Q_1}\right) &= -k(t_2 - t_1) \\
k &= -\frac{\log\left(Q_2/Q_1\right)}{(t_2 - t_1)} \\
k &= \frac{\log\left(Q_1/Q_2\right)}{(t_2 - t_1)}
\end{aligned}$$

Definition. The *half-life* of a radioactive substance is defined to be the amount of time it takes for half of its mass to decay.

Thus, suppose we start with an amount of substance $Q(t_0) = Q_0$ which has decay constant k, and it is reduced by half after time τ

[2]For example, $\frac{Q_1}{Q_0} = e^{a(t_1 - t_0)} \implies \frac{\log(Q_1/Q_0)}{t_1 - t_0} = a$, but we don't know Q_0, so this doesn't help much.

passes. Then

$$\begin{aligned}
\tfrac{Q_0}{2} &= Q(\tau + t_0) = Q_0 e^{-k(\tau+t_0-t_0)} = Q_0 e^{-k\tau} \\
\tfrac{1}{2} &= e^{-k\tau} \\
\log\left(\tfrac{1}{2}\right) &= -k\tau \\
\tau &= -\tfrac{1}{k}\log\left(\tfrac{1}{2}\right) \\
\tau &= \tfrac{1}{k}\log 2
\end{aligned}$$

Example 4.1.2. Suppose a substance decays at a yearly rate equal to half the square of the mass of the substance present. If we start with 50g of the substance, how long will it be until on 25g remains? So we need to calculate the half-life.

$$Q' = -\tfrac{Q^2}{2}, Q(0) = 50.$$

First, solve the DE using separation of variables:

$$\begin{aligned}
\tfrac{dQ}{dt} &= -\tfrac{Q^2}{2} \\
\tfrac{dQ}{Q^2} &= -\tfrac{dt}{2} \\
-\tfrac{1}{Q} &= -\tfrac{t}{2} + c \\
\tfrac{1}{Q} &= \tfrac{t}{2} - c \\
Q &= \frac{1}{\frac{t}{2} - c}
\end{aligned}$$

Then add the initial condition

$$50 = \frac{1}{0 - c} \quad \Longrightarrow \quad c = -\tfrac{1}{50}$$

so that

$$Q(t) = \frac{1}{\frac{t}{2} + \frac{1}{50}} = \frac{50}{25t+1}$$

Now let τ be the amount of time it takes the substance to decrease by half. Then

$$\begin{aligned} 25 = Q(\tau) &= \tfrac{50}{25\tau+1} \\ 25\tau + 1 &= 2 \\ 25\tau &= 1 \\ \tau &= \tfrac{1}{25} \text{ of a year,} \end{aligned}$$

or about 14.6 days.

Another common example of exponential growth arises with compound interest. An account with an annual interest rate r is *compounded* every time the growth due to interest is assessed. Every time this occurs, the balance is multiplied by $(1+pr)$ where r is the interest rate, and p is the portion of the year being compounded. For example:

times / year	factor	value after t years
1	$(1+r)$	$Q(t) = Q_0\,(1+r)^t$
2	$\left(1+\frac{r}{2}\right)$	$Q(t) = Q_0\left(1+\frac{r}{2}\right)^{2t}$
3	$\left(1+\frac{r}{3}\right)$	$Q(t) = Q_0\left(1+\frac{r}{3}\right)^{3t}$
365	$\left(1+\frac{r}{365}\right)$	$Q(t) = Q_0\left(1+\frac{r}{365}\right)^{365t}$
n	$\left(1+\frac{r}{n}\right)$	$Q(t) = Q_0\left(1+\frac{r}{n}\right)^{nt}$

In fact, we can even find out what would happen if we were to compound continuously, by taking the limit

$$Q(t) = \lim_{n\to\infty} Q_0\left(1+\tfrac{r}{n}\right)^{nt} = Q_0 e^{rt}.$$

Example 4.1.3. Suppose you open a savings account with an initial deposit of \$1000 and subsequently deposit \$50 per week. Find the value $Q(t)$ of the account at time $t > 0$, assuming that the bank pays 6% interest, compounded continuously.
To formulate the question as a differential equation, we assume that the weekly payments are deposited continuously at a rate of \$50/week, rather than in a lump sum at the end of each week. This corresponds to a continuous rate of deposit equal to \$2600/year. Now the balance increases at the rate

$$Q' = 2600 + 0.06Q,$$

which gives the separable equation

$$Q' - 0.06Q = 2600.$$

Now we use variation of parameters. Solve the complementary equation:

$$\begin{aligned}
\tfrac{dQ}{dt} - 0.06Q &= 0\\
\tfrac{dQ}{dt} &= 0.06Q\\
\tfrac{dQ}{Q} &= 0.06dt\\
\log|Q| &= \pm 0.06t + c\\
Q &= ce^{0.06t}
\end{aligned}$$

Now $Q_1 = e^{0.06t}$, so $Q = uQ_1 = ue^{0.06t}$:

$$\begin{aligned}
u'Q_1 + uQ_1' - 0.06uQ_1 &= 2600\\
u'e^{0.06t} + 0.06ue^{0.06t} - 0.06uQ_1 &= 2600\\
u'e^{0.06t} &= 2600\\
u' &= 2600e^{-0.06t}\\
u &= -\tfrac{2600}{0.06}e^{-0.06t} + c
\end{aligned}$$

Then $Q = \left(-\frac{2600}{0.06}e^{-0.06t} + c\right)e^{0.06t} = -\frac{2600}{0.06} + ce^{0.06t}$, so

$$\begin{aligned} Q(0) = \$1000 &= -\tfrac{2600}{0.06} + ce^{0.06t} \\ 1000 + \tfrac{2600}{0.06} &= c \end{aligned}$$

Thus,

$$\begin{aligned} Q(t) &= -\tfrac{2600}{0.06} + \left(1000 + \tfrac{2600}{0.06}\right)e^{0.06t} \\ Q(t) &= 1000e^{0.06t} + \tfrac{2600}{0.06}\left(e^{0.06t} - 1\right) \end{aligned}$$

Example 4.1.4. Suppose a homebuyer borrows P_0 dollars at an annual interest rate r, agreeing to repay the loan with equal monthly payments of M dollars per month over N years.

(a) Derive a differential equation for the loan principal $P(t)$ at time $t > 0$.
Again, we make the assumption that the loan is being repaid continuously. Then the principal is growing at a rate of rP per year, and shrinking at a rate of $12M$ per year.

$$P' = rP - 12M.$$

(b) Solve this equation. Using variation of parameters, start with

$$\begin{aligned} \tfrac{dP}{dt} &= rP \\ \tfrac{dP}{P} &= r\,dt \\ \log|P| &= \pm rt + c \\ P &= ce^{rt} \end{aligned}$$

Then $P_1 = e^{rt} \implies P = uP_1 = ue^{rt}$, so

$$\begin{aligned} u'P_1 + ruP_1 - ruP_1 &= -12M \\ u'P_1 &= -12M \\ u' &= -12Me^{-rt} \\ u &= \tfrac{12M}{r}e^{-rt} + c \\ P = uP_1 &= \left(\tfrac{12M}{r}e^{-rt} + c\right)e^{rt} \\ P &= \tfrac{12M}{r} + ce^{rt} \end{aligned}$$

Then the initial condition gives

$$\begin{aligned} P(0) = P_0 &= \tfrac{12M}{r} + ce^{r\cdot 0} = \tfrac{12M}{r} + c \\ P_0 - \tfrac{12M}{r} &= c \end{aligned}$$

Finally,

$$\begin{aligned} P(t) &= \tfrac{12M}{r} + \left(P_0 - \tfrac{12M}{r}\right)e^{rt} \\ P(t) &= P_0e^{rt} + \tfrac{12M}{r}(1 - e^{rt}) \end{aligned}$$

(c) Use this result to determine an approximate value for M (assume all months have equal length).

After N years, the principal remaining should be zero, i.e.,

$$P(N) = 0.$$

Plugging this into the previous result,

$$\begin{aligned} P_0e^{rN} + \tfrac{12M}{r}(1 - e^{rN}) &= 0 \\ \tfrac{12M}{r}(1 - e^{rN}) &= -P_0e^{rN} \end{aligned}$$

$$\frac{12M}{r} = \frac{-P_0 e^{rN}}{1 - e^{rN}}$$
$$M = \frac{rP_0}{12} \cdot \frac{e^{rN}}{e^{rN} - 1}$$
$$M = \frac{rP_0}{12} \cdot \frac{1}{1 - e^{-rN}}$$

(d) The exact value of M is

$$M_2 = \frac{rP_0}{12} \cdot \frac{1}{1 - \left(1 + \frac{r}{12}\right)^{-12N}}$$

P_0	r	N	M_1	M_2
\$50,000	0.075	20	402.255	402.797
\$150,000	0.09	30	1206.05	1206.93

§4.1 Homework Assignment:

Read: 122-130

Exercises: 1,6-8,10,12,13,17,21-23

4.2. **Cooling and Mixing.**

4.2.1. *Newton's Law of Cooling.* Newton's law of cooling states that the rate at which an object cools is proportional to the difference between its own temperature and the ambient temperature.

$$T' = -k(T - T_m).$$

Thus, as the object approaches the same temperature as its environment, this difference becomes small, and the rate of cooling slows down. We solve

$$T' + kT = kT_m$$

using variation of parameters:

$$\begin{aligned} \tfrac{dT}{dt} &= -kT \\ \tfrac{dT}{T} &= -k\,dt \\ \log|T| &= -kt + c \\ T &= ce^{-kt} \end{aligned}$$

Now $T_1 = e^{-kt}$ and $T = uT_1 = ue^{-kt}$. $T_1' = -ke^{-kt} = -kT_1$, so

$$\begin{aligned} u'T_1 - kuT_1 + kuT_1 &= kT_m \\ u'T_1 &= kT_m \\ u' &= kT_m e^{kt} \\ u &= T_m e^{kt} + c \end{aligned}$$

Then

$$T = uT_1 = \left(T_m e^{kt} + c\right) e^{-kt} = T_m + ce^{-kt}.$$

If we also have an initial condition $T(0) = T_0$, then

$$\begin{aligned} T(0) = T_0 &= T_m + ce^{-k\cdot 0} \\ T_0 - T_m &= c \end{aligned}$$

So

$$T(t) = T_m + (T_0 - T_m)e^{-kt}.$$

Example 4.2.1. An object with initial temperature 150°C is placed outside where the temperature is 35°C. Its temperatures at 12:15 and 12:20 are 120°C and 90°C, respectively.

(a) At what time was the object placed outside?
We would like to solve

$$T(t) = 150 = 35 + 115e^{-kt}$$

for t. Unfortunately, this is not possible, as we have the unknown k. Thus, we need to determine the cooling constant first. We use the given info:

$$T_m = 35 \quad \text{and} \quad T_0 = 120.$$

This time, take $t = 0$ at 12:15. Then

$$T(t) = 35 + 85e^{-kt}.$$

Since $t = 5$ at 12:20, the other recorded temperature gives

$$\begin{aligned} 35 + 85e^{-5k} &= 90 \\ 85e^{-5k} &= 55 \\ e^{-5k} &= \tfrac{11}{17} \\ k &= -\tfrac{1}{5}\log\left(\tfrac{11}{17}\right) \\ k &= \tfrac{1}{5}\log\left(\tfrac{17}{11}\right) \end{aligned}$$

So we can use the formula

$$\begin{aligned} T(t) = 35 + 85e^{-\frac{t}{5}\log(17/11)} &= 150 \\ 85e^{-\frac{t}{5}\log(17/11)} &= 115 \\ e^{-\frac{t}{5}\log(17/11)} &= \tfrac{23}{17} \\ \tfrac{t}{5}\log(11/17) &= \log\left(\tfrac{23}{17}\right) \\ t &= 5\,\tfrac{\log 23/17}{\log 11/17} \approx -3.47195 \end{aligned}$$

So the object was placed outside approximately 3.5 minutes before 12:15. (12:11:32pm)

(b) When will its temperature be 40°C?
We can use the formula

$$\begin{aligned} T(t) = 35 + 85e^{-\frac{t}{5}\log(17/11)} &= 40 \\ 85e^{-\frac{t}{5}\log(17/11)} &= 5 \\ e^{-\frac{t}{5}\log(17/11)} &= \tfrac{1}{17} \\ \tfrac{t}{5}\log(11/17) &= \log\left(\tfrac{1}{17}\right) \\ t &= 5\,\tfrac{\log(1/17)}{\log(11/17)} \approx 32.5419 \end{aligned}$$

So the object will be 40°C at approximately 32.5 minutes after 12:15. (12:47:54pm)
In general, the object will be T°C at $t = 5\,\frac{\log(T-35/85)}{\log(11/17)}$.

4.2.2. *Mixing Problems.* The next applications consider a situation in which a salt and water are added to a tank. The solution mixes in the tank (instantly, for ease of calculation), and perhaps some is drawn from a spigot on the tank. In order to determine information about the concentration of salt in the water (or the total mass of the salt in the tank) at any given time, we consider differential equations like

$$Q' = \text{rate in} - \text{rate out}.$$

Example 4.2.2. A 200-gallon tank initially contains 100 gallons of water with 20 pounds of salt. A salt solution with 1/4 lb of salt per gallon is added to the tank at 4 gal/min, and the resulting mixture is drained out at 2 gal/min. Find the quantity of salt in the tank when it overflows.

Start by determining

$$\text{rate in} = \left| \begin{array}{c|c} 1/4 \text{ lb} & 4 \text{ gal} \\ \hline \text{gal} & \text{min} \end{array} \right| = 1 \text{ lb/min}$$

and since there are $100 + 4t - 2t = 100 + 2t$ gallons of mixture in the tank at time t,

$$\text{rate out} = \left| \begin{array}{c|c} Q(t) \text{ lb} & 2 \text{ gal} \\ \hline 100 + 2t \text{ gal} & \text{min} \end{array} \right| = \tfrac{Q(t)}{t+50} \text{ lb/min}.$$

Then

$$\begin{aligned} Q' &= \text{rate in} - \text{rate out} \\ Q' &= 1 - \tfrac{Q(t)}{t+50} \end{aligned}$$

We solve this by variation of parameters:

$$\begin{aligned} \tfrac{dQ}{dt} &= -\tfrac{Q(t)}{t+50} \\ \tfrac{dQ}{Q} &= -\tfrac{dt}{t+50} \\ \log|Q| &= -\log(t+50) + c \\ Q &= c\tfrac{1}{t+50} \end{aligned}$$

Then $Q_1 = \frac{1}{t+50}$ and $Q = uQ_1 = \frac{u}{t+50}$. So $Q_1' = -\frac{1}{(t+50)^2}$ and

$$\begin{aligned} u'Q_1 - \tfrac{u}{t+50}Q_1 &= 1 - \tfrac{u}{t+50}Q_1 \\ u'\tfrac{1}{t+50} &= 1 \\ u' &= t + 50 \\ u &= \tfrac{(t+50)^2}{2} + c \end{aligned}$$

So

$$Q(t) = (\tfrac{(t+50)^2}{2} + c)\tfrac{1}{t+50} = \tfrac{t+50}{2} + \tfrac{c}{t+50}.$$

Applying the initial condition,

$$\begin{aligned} Q(0) &= \tfrac{0+50}{2} + \tfrac{c}{0+50} = 20 \\ \tfrac{c}{50} &= -5 \\ c &= -250 \end{aligned}$$

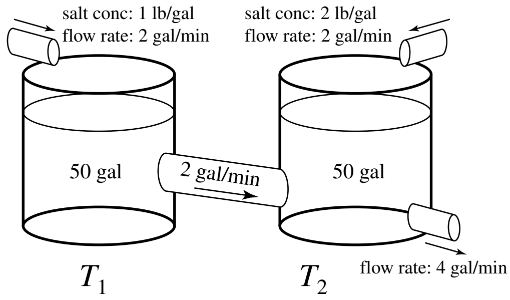
Then overflow occurs when $t = 50$, so at that time

$$Q(50) = \tfrac{50+50}{2} - \tfrac{250}{50+50} = 50 - 2.5 = 47.5.$$

Example 4.2.3. Suppose a tank contains 50 gallons of pure water. Starting at $t_0 = 0$, water containing 1 pound of salt per gallon is poured into T_1 at the rate of 2 gal/min. The mixture is drained from T_1 at the same rate into a second tank T_2, which initially contains 50 gallons of pure water. Also starting at t_0, a mixture from another source containing 2 pounds of salt per gallon is poured into T_2 at a rate of 2 gal/min. The mixture is drained from T_2 at the rate of 4 gal/min. Start by determining

$$Q_1^{in} = \left|\begin{array}{c|c} 1 \text{ lb} & 2 \text{ gal} \\ \hline \text{gal} & \text{min} \end{array}\right| = 2 \text{ lb/min},$$

FIGURE 4.2.1. The two tanks



and since there are always 50 gallons of mixture in T_1,

$$Q_1^{out} = \left| \begin{array}{c|c} Q_1(t) \text{ lb} & 2 \text{ gal} \\ \hline 50 \text{ gal} & \text{min} \end{array} \right| = \frac{Q_1(t)}{25} \text{ lb/min}.$$

Then

$$Q_2^{in} = \left| \begin{array}{c|c} 2 \text{ lb} & 2 \text{ gal} \\ \hline \text{gal} & \text{min} \end{array} \right| + Q_1^{out} = 4 + \frac{Q_1(t)}{25} \text{ lb/min},$$

and since there are always 50 gallons of mixture in T_2,

$$Q_2^{out} = \left| \begin{array}{c|c} Q_2(t) \text{ lb} & 4 \text{ gal} \\ \hline 50 \text{ gal} & \text{min} \end{array} \right| = \frac{2Q_2(t)}{25} \text{ lb/min}.$$

(a) Find a differential equation for the quantity $Q_2(t)$ of salt in tank T_2 at time $t > 0$.
First, we figure out what's going on with the first tank.

$$\begin{aligned} Q_1' = Q_1^{in} - Q_1^{out} &= 2 - \tfrac{1}{25} Q_1 \\ \frac{dQ_1}{Q_1} &= -\tfrac{1}{25} dt \\ \log |Q_1| &= -\tfrac{1}{25} t + c \\ Q_1 &= c e^{-\frac{1}{25} t} \end{aligned}$$

So take $Q_1 = uq = ue^{-\frac{1}{25}t}$ with $q' = -\frac{1}{25} e^{-\frac{1}{25}t} = -\frac{1}{25} q$.

$$\begin{aligned} u'q - \tfrac{1}{25}uq &= 2 - \tfrac{1}{25}uq \\ u'q &= 2 \\ u' &= 2e^{t/25} \\ u &= 50e^{t/25} + c \end{aligned}$$

Then $Q_1(t) = 50 + ce^{-t/25}$. The initial condition $Q_1(0) = 0$ gives $c = -50$, so

$$Q_1(t) = 50\left(1 - e^{-t/25}\right).$$

Now we can find out what's going on with the second tank.

$$\begin{aligned} Q_2' = Q_2^{in} - Q_2^{out} &= 4 + \tfrac{Q_1}{25} - \tfrac{2Q_2}{25} \\ Q_2' &= 4 + 2\left(1 - e^{-t/25}\right) - \tfrac{2Q_2}{25} \end{aligned}$$

Thus, the quantity of salt in T_2 is the solution to

$$Q_2'(t) + \tfrac{2}{25}Q_2 = 6 - 2e^{-t/25}, \quad Q_2(0) = 0.$$

(b) Solve the equation derived in (a) to determine $Q_2(t)$. Taking the complementary equation of the above,

$$\begin{aligned} \tfrac{dQ_2}{dt} &= -\tfrac{2Q_2}{25} \\ \tfrac{dQ_2}{Q_2} &= -\tfrac{2}{25}dt \\ \log|Q_2| &= -\tfrac{2}{25}t + c \\ Q_2(t) &= ce^{-2t/25} \end{aligned}$$

So take $Q_2 = uq = ue^{-2t/25}$ with $q' = -\frac{2}{25}e^{-2t/25} = -\frac{2}{25}q$.

$$
\begin{aligned}
u'q - \tfrac{2}{25}uq &= 6 - 2e^{-t/25} - \tfrac{2}{25}uq \\
u'e^{-2t/25} &= 6 - 2e^{-t/25} \\
u' &= 6e^{2t/25} - 2e^{t/25} \\
u &= \tfrac{25}{2}6e^{2t/25} - 25 \cdot 2e^{t/25} + c \\
u &= 75e^{2t/25} - 50e^{t/25} + c
\end{aligned}
$$

Then $Q_2 = \left(75e^{2t/25} - 50e^{t/25} + c\right) e^{-2t/25}$, so

$$Q_2(t) = 75 - 50e^{-t/25} + ce^{-2t/25}.$$

Finally, $Q_2(0) = 0$ gives $c = 50 - 75 = -25$ and we have

$$Q_2(t) = 75 - 50e^{-t/25} - 25e^{-2t/25}.$$

(c) Find the equilibrium state of T_2.

So we must consider what happens when $t \to \infty$:

$$
\begin{aligned}
\lim_{t\to\infty} Q_2(t) &= \lim_{t\to\infty} \left(75 - 50e^{-t/25} - 25e^{-2t/25}\right) \\
&= 75 - 50 \lim_{t\to\infty} e^{-t/25} - 25 \lim_{t\to\infty} e^{-2t/25} \\
&= 75 - 50 \cdot 0 - 25 \cdot 0 \\
&= 75
\end{aligned}
$$

§4.2 Homework Assignment:

Read: 133-139

Exercises: 2,5,7,11-13,18

4.3. **Elementary Mechanics.**

4.3.1. *Newton's Second Law of Motion.* Newton's Second Law of Motion asserts that the force F on an object and its acceleration a are related by

$$F = ma.$$

For our discussion, we will assume that Earth is a perfect sphere so that its gravitational field is the same as a point mass (of the same magnitude) located at the center of the Earth.

Acceleration due to gravity will always be taken to be $-9.8\,\frac{m}{s^2}$.
If $y(t)$ gives the height of an object above the Earth's surface, then $y'(t) = v(t)$ gives its vertical velocity, and $y''(t) = v'(t) = a(t)$ gives its vertical acceleration. Thus for an object solely under the influence of gravity, acceleration is constant with $y''(t) = a(t) = -9.8$.

In general, force may depend on t, y, y' and we have

$$F(t, y, y') = my''.$$

This is a second-order equation, which we cannot solve.
For the time being, we consider only problems in which F does not depend on y. In this case, we can rewrite as

$$F(t, y') = my''$$
$$F(t, v) = mv'$$

and obtain a first-order equation. Equations of this form occur in problems involving motion through a resisting medium: the medium produces a drag force proportional to the speed of the moving object, and in the direction opposite to its motion.

In such a situation, the total force acting on an object is

$$F = -mg - kv.$$

4.3.2. *Motion Through a Resisting Medium.*

Consider an object in freefall near the Earth's surface. From Newton's second law,

$$F = ma = mv'$$

gives

$$\begin{aligned} mv' &= -mg - kv \\ v' &= -g - \tfrac{k}{m}v \\ v' + \tfrac{k}{m}v &= -g \end{aligned}$$

We can solve this using variation of parameters.

$$\begin{aligned} \tfrac{dv}{dt} &= -\tfrac{k}{m}v \\ \tfrac{dv}{v} &= -\tfrac{k}{m}\,dt \\ \log|v| &= -\tfrac{k}{m}t + c \\ v &= ce^{-kt/m} \end{aligned}$$

So take $v_1 = e^{-kt/m}$, that $v_1' = -\frac{k}{m}e^{-kt/m} = -\frac{k}{m}v_1$ and

$$\begin{aligned} u'v_1 - \tfrac{k}{m}uv_1 + \tfrac{k}{m}v &= -g \\ u'e^{-kt/m} &= -g \\ u' &= -ge^{kt/m} \\ u &= -\tfrac{mg}{k}e^{kt/m} + c \\ v = uv_1 &= \left(-\tfrac{mg}{k}e^{kt/m} + c\right)e^{-kt/m} \\ v(t) &= -\tfrac{mg}{k} + ce^{-kt/m} \\ v(0) &= -\tfrac{mg}{k} + c = v_0 \\ c &= v_0 + \tfrac{mg}{k} \\ v(t) &= -\tfrac{mg}{k} + \left(v_0 + \tfrac{mg}{k}\right)e^{-kt/m} \end{aligned}$$

The terminal velocity of the object is the limiting velocity; what it tends towards as $t \to \infty$:

$$\begin{aligned}\lim_{t\to\infty} v(t) &= \lim_{t\to\infty}\left(-\tfrac{mg}{k} + \left(v_0 + \tfrac{mg}{k}\right) e^{-kt/m}\right)\\ &= -\tfrac{mg}{k} + \left(v_0 + \tfrac{mg}{k}\right)\left(\lim_{t\to\infty} e^{-kt/m}\right)\\ &= -\tfrac{mg}{k} + \left(v_0 + \tfrac{mg}{k}\right)\cdot 0\\ &= -\tfrac{mg}{k}\end{aligned}$$

Example 4.3.1. The book gives several examples of an object being accelerated back towards earth by gravity, so I will give an example where the acceleration is given by a different kind of force.

Suppose a constant horizontal force of 10N pushes a 20kg mass through a medium that resists its motion with 0.5N for ever m/s of speed. The initial velocity of the mass is 7m/s in the direction opposite to the direction of the applied force. Find the velocity of the mass for $t > 0$.

Suppose we take the initial direction of motion to be negative, so that the direction of acceleration is positive. We have two equivalent expressions for the force acting on the object. One is Newton's law:

$$F = ma = mv' = 20v'.$$

The other is the force being applied to the object, minus the drag force:

$$\text{constant force} - \text{drag force} = 10 - 0.5v.$$

Putting these together, we obtain the IVP

$$20v' = 10 - 0.5v, \quad v(0) = -7.$$

Using variation of parameters,

$$
\begin{aligned}
v' &= \tfrac{1}{2} - \tfrac{v}{40} \\
\tfrac{dv}{dt} &= -\tfrac{v}{40} \\
\tfrac{dv}{v} &= -\tfrac{dt}{40} \\
\log|v| &= -\tfrac{t}{40} + c \\
v &= ce^{-t/40}
\end{aligned}
$$

So take $v_1 = e^{-t/40}$ that $v_1' = -\frac{1}{40}e^{-t/40} = -\frac{1}{40}v_1$.

$$
\begin{aligned}
u'v_1 - \tfrac{uv_1}{40} &= \tfrac{1}{2} - \tfrac{uv_1}{40} \\
u'e^{-t/40} &= \tfrac{1}{2} \\
u' &= \tfrac{1}{2}e^{t/40} \\
u &= 20e^{t/40} + c \\
v = uv_1 &= \left(20e^{t/40} + c\right) e^{-t/40} \\
v(t) &= 20 + ce^{-t/40} \\
v(0) &= 20 + c = -7 \\
c &= -27 \\
v(t) &= 20 - 27e^{-t/40}
\end{aligned}
$$

The terminal velocity will be given by

$$
\begin{aligned}
\lim_{t\to\infty} v(t) &= \lim_{t\to\infty} \left(20 - 27e^{-t/40}\right) \\
&= 20 - 27 \lim_{t\to\infty} e^{-t/40} \\
&= 20
\end{aligned}
$$

4.3.3. *Escape Velocity.*

Suppose we attempt to launch a space vehicle to an altitude h. The motion of the vehicle is that of a mass m under the influence of Earth's gravity. Remember that we are assuming Earth to be a perfect sphere: this allows us to think of its mass as being entirely concentrated at the center. For points on the surface of the Earth, the distance to this mass is given by R, the radius of the Earth. For objects at an altitude y above the surface, the distance is given by $y + R$. Since Newton's law of gravity says that the force due to gravity is inversely proportional to the square of the distance between the objects, the gravitational force on our space vehicle is

$$F = -\tfrac{K}{(y+R)^2}.$$

Since $F = -mg$ near the Earth's surface, put $y = 0$ to get

$$-mg = -\tfrac{K}{R^2} \quad \Longrightarrow \quad K = mgR^2,$$

so we obtain

$$F = -\tfrac{mgR^2}{(y+R)^2}.$$

But since $F = m\frac{d^2y}{dt^2}$, this becomes

$$\tfrac{d^2y}{dt^2} = -\tfrac{gR^2}{(y+R)^2}. \tag{4.3.1}$$

Again, this is a second-order equation, so let's convert it into a first-order equation regarding v:

$$\frac{d^2y}{dt^2} = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy},$$

so we get a first-order equation with v as dependent variable and y as independent variable:

$$v\tfrac{dv}{dy} = -\tfrac{gR^2}{(y+R)^2}.$$

To remove the dependence on t from the initial condition, note that when $t = 0$, the altitude is h. Thus, the velocity at altitude h is $v(h) = v_0$. Now it is easy to solve this IVP by separation of variables:

$$\begin{aligned} v\frac{dv}{dy} &= -\frac{gR^2}{(y+R)^2} \\ v\,dv &= -\frac{gR^2}{(y+R)^2}\,dy \\ \frac{v^2}{2} &= \frac{gR^2}{y+R} + c \end{aligned}$$

Then applying $v(h) = v_0$,

$$\begin{aligned} \frac{v_0^2}{2} &= \frac{gR^2}{h+R} + c \\ \frac{v_0^2}{2} - \frac{gR^2}{h+R} &= c \end{aligned}$$

So that

$$\frac{v^2}{2} = \frac{gR^2}{y+R} + \left(\frac{v_0^2}{2} - \frac{gR^2}{h+R}\right)$$

Now if $\left(\frac{v_0^2}{2} - \frac{gR^2}{h+R}\right)$ is nonnegative, then the entire RHS will be positive, and $v(y) > 0$ for $y > h$. In this case, the rocket will continue to fly away from Earth, even if it stops thrusting at this point. Define the escape velocity

$$v_e = \left(\frac{2gR^2}{h+R}\right)^{1/2}.$$

Then

$$\frac{v_e^2}{2} - \frac{gR^2}{h+R} = \frac{2gR^2}{2(h+R)} - \frac{gR^2}{h+R} = 0,$$

so if the rocket is going at least v_e when it reaches altitude h, its velocity will be at least

$$v = \sqrt{\frac{2gR^2}{y+R}} > 0,$$

and it will never stop moving away from Earth.[3] v_e also has the special property that if $v_0 < v_e$, the rocket will fall back to Earth. If

$$v_0 < \left(\frac{2gR^2}{h+R}\right)^{1/2},$$

then let us define

$$-\beta := \frac{v_0^2}{2} - \frac{gR^2}{h+R} < \frac{2gR^2}{2(h+R)} - \frac{gR^2}{h+R} < 0.$$

so β is a small positive number. Since

$$\frac{gR^2}{y+R} \xrightarrow{y\to\infty} 0,$$

there will be some large value of y for which

$$\frac{gR^2}{y+R} = \beta.$$

At this point,

$$\frac{v^2(y)}{2} = \frac{gR^2}{y+R} - \beta = \beta - \beta = 0$$

so the rocket will stop moving away from Earth. Then, under the influence of gravity, it will begin to fall back to Earth.

Example 4.3.2. A space probe is launched from a space station 200 miles above Earth. Determine its escape velocity in miles/s. Take

[3] To see this last fact, suppose that the rocket never gets further than y_{max} from Earth. Then $y(t) \leq y_{max}$ $\forall t \geq 0$. If we take $\alpha = \frac{gR^2}{(y_{max}+R)^2}$, then

$$\begin{aligned}
\frac{d^2y}{dt^2} &= -\frac{gR^2}{(y+R)^2} \leq -\frac{gR^2}{(y_{max}+R)^2} = -\alpha, \text{ so} \\
\frac{dv}{dt} &= \frac{d^2y}{dt^2} \leq -\alpha < 0, \quad \forall t \geq 0 \\
dv &\leq -\alpha\, dt \\
\int_0^T dv &\leq -\alpha \int_0^T dt \\
v(T) - v(0) &\leq -\alpha\, [t]_{t=0}^{t=T} = -\alpha T \\
v(T) &\leq v(0) - \alpha T
\end{aligned}$$

Then for large enough time T (specifically, for $T > \frac{v_0}{\alpha}$), $v(T)$ will be negative, indicating that the rocket is falling back to Earth.

$\lightning$ $v = \sqrt{\frac{2gR^2}{y+R}} > 0.$

the Earth's radius to be 3960 miles.

We just found

$$\begin{aligned} v_e &= \left(\tfrac{2gR^2}{h+R}\right)^{1/2} \\ &= R\left(\tfrac{2g}{h+R}\right)^{1/2}, \end{aligned}$$

so convert into stupid units:

$$\begin{aligned} R &= 3960 \\ g &= \left| \frac{32 \text{ feet}}{\text{seconds}^2} \,\middle|\, \frac{\text{miles}}{5280 \text{ feet}} \right| = \tfrac{32}{5280} \ {}^{\text{mi}}\!/\!_{\text{s}^2} \end{aligned}$$

Then

$$\begin{aligned} v_e &= 3960\left(\frac{2\cdot\frac{32}{5280}}{200+3960}\right)^{1/2} \\ &\approx 6.75961 \text{ miles} \end{aligned}$$

§4.3 Homework Assignment:

Read: 142-151

Exercises: 3,5,6,17,18,20

4.4. Autonomous Second Order Equations.

Definition. A differential equation that can be written as

$$y' = F(y) \qquad \text{or} \qquad y'' = F(y, y'),$$

where F is independent of t, is said to be *autonomous*. Thus, an autonomous system is one in which the external forces acting on the system do not change over time.

An autonomous second-order equation can be rewritten as a first-order equation by putting $v = y'$. Then $v' = y''$ and

$$v' = F(y, v).$$

Since

$$v' = \tfrac{dv}{dt} = \tfrac{dv}{dy}\tfrac{dy}{dt} = v\tfrac{dv}{dy},$$

this can also be written as

$$v\tfrac{dv}{dy} = F(y, v).$$

Now if we have a solution y of this equation,

$$\begin{aligned} y &= y(t) \\ v &= y'(t) \end{aligned}$$

is a parametric equation for an integral curve of $v\frac{dv}{dy} = F(y, v)$.

Definition. A *trajectory* or *orbit* of the DE $y'' = F(y, y')$ is the integral curve on the (y, v)-plane given by $y = y(t), v = y'(t)$.

We can study the trajectories to obtain qualitative information about the system. For example:

- Does the system tend toward a rest state?
- Is the system periodic? Is there some $T > 0$ such that $y(t) = y(t + T)$, so that the system returns to the same state every T seconds (or other time units).

- Is the system stable? Does it remain constrained to a certain collection of states, or do certain variables grow without bound?

Definition. The (y, v)-plane is called the *phase plane* and a representative set of trajectories is referred to as a *phase portrait* of the system.

Example 4.4.1. Let's look at the phase portrait of an oscillating pendulum to get an idea of what the phase plane is all about. To make life simple, assume for the moment that there is no air resistance or friction acting on the pendulum. For this model, we'll denote the angular displacement of the pendulum by $\theta(t)$ and the angular velocity by $v(t) = \theta'(t)$.

So we pull the pendulum up to its starting position and release. What happens?

FIGURE 4.4.1. The pendulum starts with some positive initial position $\theta(0) = \theta_0$ and a velocity of $0 = v'(t)$.

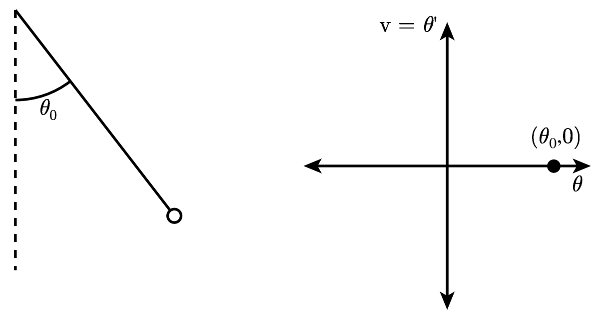


Figure 4.4.2. As the pendulum swings into the vertical position, we see the velocity go negative, with speed hitting its maximum as the displacement $\theta = 0$.

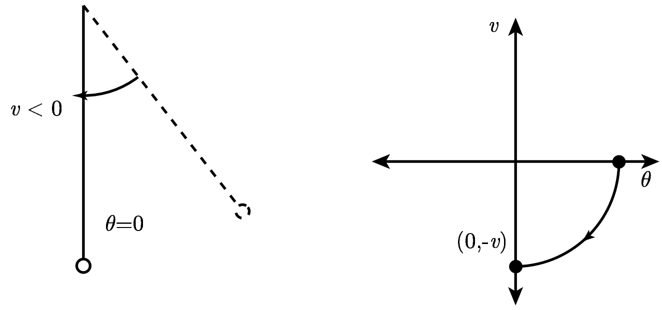


Figure 4.4.3. Then the pendulum begins to slow down again as its displacement approaches $-\theta_0$, where it stops.

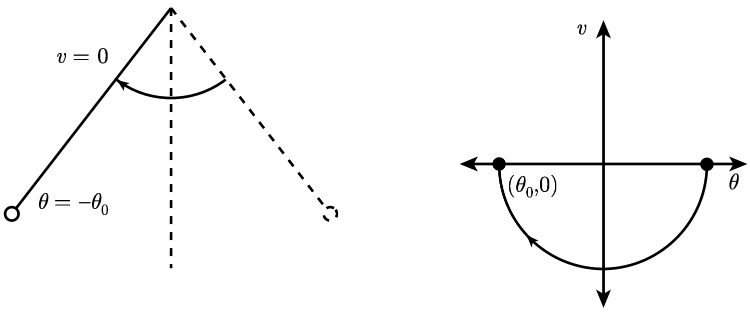
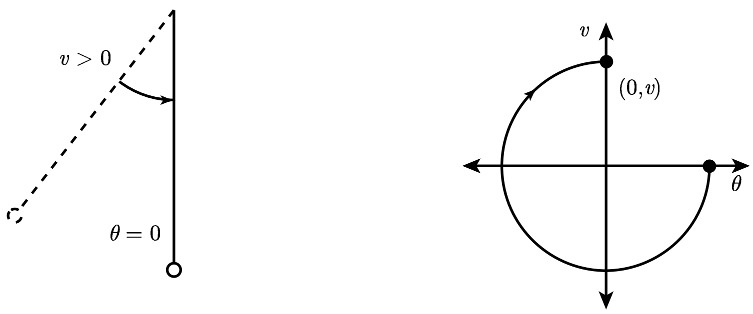


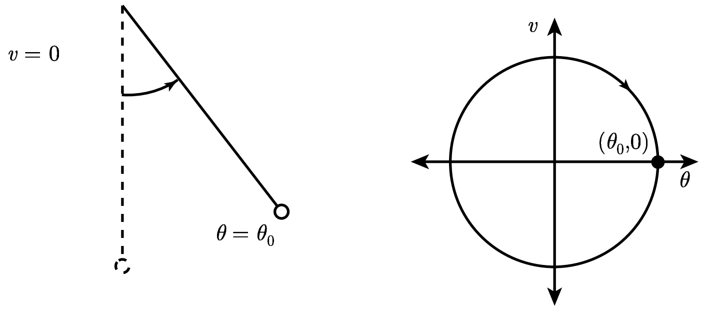
Figure 4.4.4. The pendulum begins its return swing, and the orbit indicates positive velocity this time, as displacement moves back towards 0.



Several characteristics of the system are apparent from the phase portrait:

- The system does not tend toward a rest state.

FIGURE 4.4.5. Finally, the pendulum slows down as it returns to the original starting position.



- The system has at least one periodic orbit; for initial displacement θ_0, the system returns to the same state every T seconds, for some $T > 0$.
- The system is stable; it remains constrained to a certain collection of states for all time.

Probably the most important thing about phase space is that a single point in phase space gives the entire state of the system at a particular point in time.

Also, notice that time does not appear directly in the phase portrait. Neither axis has time on it. The phase portrait indicates how the variables of the system change with respect to each other, not with respect to time.
Time is indirectly present in the phase portrait; as time passes, the state of the system moves along the trajectory in the direction indicated by the arrow. However, there is nothing to indicate the speed at which the orbit follows this flow.

Consider autonomous equations of the form

$$y'' + q(y, y')y' + p(y) = 0,$$

or equivalently,

$$v\frac{dv}{dy} + q(y, v)v + p(y).$$

Such equations are very common in the study of mechanics.
If y is the displacement of a mass m, then gravity would appear as $-mp(y)$ in this equation. Forces which depend on velocity are called *damping forces* and would appear as $-mq(y,v)v$.

Some facts about the phase space of $v\frac{dv}{dy} + q(y,v)v + p(y) = 0$:

(1) For $y(0) = y_0, v(0) = v_0$, this DE has a unique solution that exists for all time $t \in (-\infty, \infty)$.
(2) If $y_1 = y(t)$ is a solution, then $y_2 = y(t-\tau)$ is also a solution. y_1, y_2 share the same trajectory.
(3) Trajectories in phase space either coincide or do not intersect at all.
(4) If the trajectory of a solution is a closed curve, then the solution is periodic and traverses the entire trajectory in finite time.

So any point in phase space lies on some trajectory, and no two trajectories intersect. It is, however, possible to have a trajectory which consists of a single point – this is called a critical point.

Definition. A *critical point* is a point in phase space corresponding to a constant solution or equilibrium solution; if the system starts at a critical point, it will never leave.

To find critical points, we need to find the constant solutions. Since the derivative of a constant is always 0, we need to set the derivatives equal to 0. In this case,

$$v\frac{dv}{dy} + q(y,v)v + p(y) = 0$$

becomes

$$p(y) = 0.$$

Suppose we solve $p(y) = 0$ and find the solution $y(t) = k$, where k is a fixed constant. Then we say k is an *equilibrium* of $y'' + q(y,y')y' + p(y) = 0$, or that $(k, 0)$ is a critical point for $v\frac{dv}{dy} + q(y,v)v + p(y) = 0$.

4.4.1. *Behavior of trajectories near a critical point.*

Example 4.4.2. Let's look at the pendulum again. Let's take the pendulum to consist of a rigid but weightless rod of length L, with a mass m on the tip. The gravitational force mg always acts downward, and the damping force $c|\frac{d\theta}{dt}|$ always acts in the direction opposite to the motion.

We use Newton's law for angular momentum: the time rate of change of angular momentum about any point is equal to the moment of the resultant force about that point. The angular momentum about the origin is $mL^2\frac{d^2\theta}{dt^2}$, so the governing equation is

$$mL^2\frac{d^2\theta}{dt^2} = cL\frac{d\theta}{dt} - mgL\sin\theta.$$

We rewrite this as

$$\frac{d^2\theta}{dt^2} + \frac{c}{mL}\frac{d\theta}{dt} + \frac{g}{L}\sin\theta = 0.$$

To find the critical points, we look for solutions where y is a constant function; in this case, the first two terms drop out and we're left with

$$\frac{g}{L}\sin\theta = 0.$$

Thus the solutions are $(\theta, v) = (n\pi, 0)$, where n is an integer. These correspond to the points where the pendulum bob is directly below or above the point of rotation.

The first is stable and the second is unstable. Intuitively, this should be clear: if you start a pendulum with no initial velocity it displacement, it ain't going anywhere. Also, if you were in a physically perfect universe, you could possibly balance the pendulum so that it remained perfectly still above the point of rotation.

Definition. A critical point is said to be *stable* iff $\forall \varepsilon > 0, \exists \delta > 0$ such that

$$|(y_0, v_0) - (k, 0)| < \delta \implies |(y(t), v(t)) - (k, 0)| < \varepsilon, \quad \forall t > 0.$$

In plain English, this just means that if the system starts close enough to (within δ of) $(k, 0)$, it will always remain close to (within ε of) $(k, 0)$.

So here is the formalization of the previous example of instability: if you start the pendulum very near (but not exactly) balanced straight up, it will always swing away from the critical point. More rigourously, take $\varepsilon = 0.01L$. Then there is no distance $\delta > 0$ for which any initial position within δ of the vertical will guarantee that the rod remains within ε of the vertical.

In fact, if we have a damping $c > 0$, then $\delta > 0$ guarantees that the rod will swing downwards and, losing energy, eventually come to rest in the other equilibrium state. This indicates another kind of stability.

Definition. A critical point is *asymptotically stable* iff it is stable and nearby trajectories eventually approach it. Rigourously, $(k, 0)$ is stable if there exists $\delta > 0$ such that

$$|(y(t_0), v(t_0)) - (k, 0)| < \delta \quad \implies \quad \lim_{t\to\infty} y(t) = (k, 0).$$

In other words, any trajectory passing near the critical point will not only remain close to the critical point forever, it will actually approach it as $t \to \infty$.

Let's solve the pendulum equation without damping:

$$v\tfrac{dv}{dy} = -\tfrac{g}{L}\sin\theta, \quad \theta(0) = 0, v(0) = v_0.$$

Integrating,

$$\frac{v^2}{2} = \frac{g}{L}\cos\theta + c.$$

Applying the initial condition,

$$\frac{v_0^2}{2} = \frac{g}{L} + c$$
$$\frac{v_0^2}{2} - \frac{g}{L} = c$$

so we get

$$\begin{aligned} \frac{v^2}{2} &= \frac{v_0^2}{2} - \frac{g}{L}(1-\cos\theta) \\ &= \frac{v_0^2}{2} - \frac{2g}{L}\sin^2\frac{\theta}{2} \\ v^2 &= v_0^2 - v_c^2\sin^2\frac{\theta}{2} \end{aligned}$$

where $v_c = 2\sqrt{\frac{g}{L}}$ is the critical velocity.

Why is this called the critical velocity? Note that if $|v_0| = v_c$, then

$$v^2 = v_0^2 - v_c^2\sin^2\frac{\theta}{2} = 0 \quad \text{when } \theta = \pi,$$

and the pendulum stops standing straight up, but if $|v_0| > v_c$, then the pendulum whirls in the same direction forever (though the speed varies, it never reaches 0).

Similarly, for $0 < |v_0| < v_c$ we get $v = 0$ whenever $\theta = \pm\theta_{max}$, where

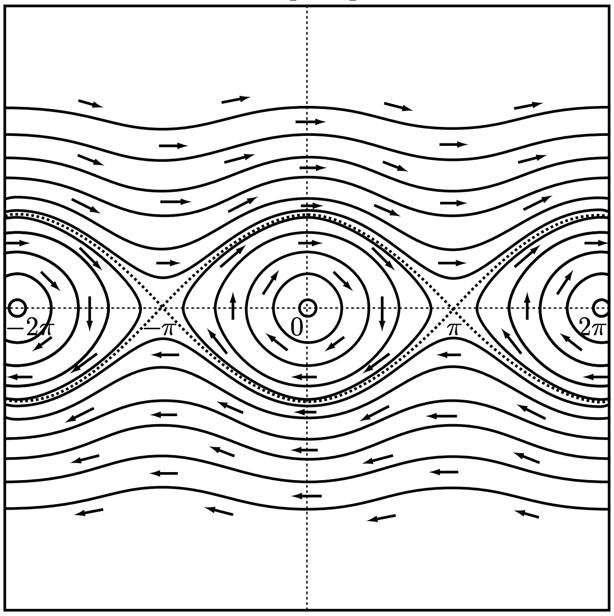
$$\theta_{max} = 2\sin^{-1}\left(\frac{|v_0|}{v_c}\right),$$

and the pendulum oscillates between $-\theta_{max}$ and θ_{max}.

§4.4 Homework Assignment:

Read: 153-160

FIGURE 4.4.6. Undamped pendulum oscillations



Exercises:

1. Plot a nice phase portrait for the undamped pendulum
$$y'' = -\sin y$$
for $-2\pi < y < 2\pi$.
If you use the Rice web site, use PPLANE instead of DFIELD and enter
$$y' = x$$
$$x' = -\sin(y).$$
Include some characteristic trajectories and indicate the critical value v_c.

2. Plot a nice phase portrait for the damped pendulum
$$y'' = -\sin y - 0.2x.$$

If you use the Rice web site, enter

$$y' = x$$
$$x' = -\sin(y) - 0.2x.$$

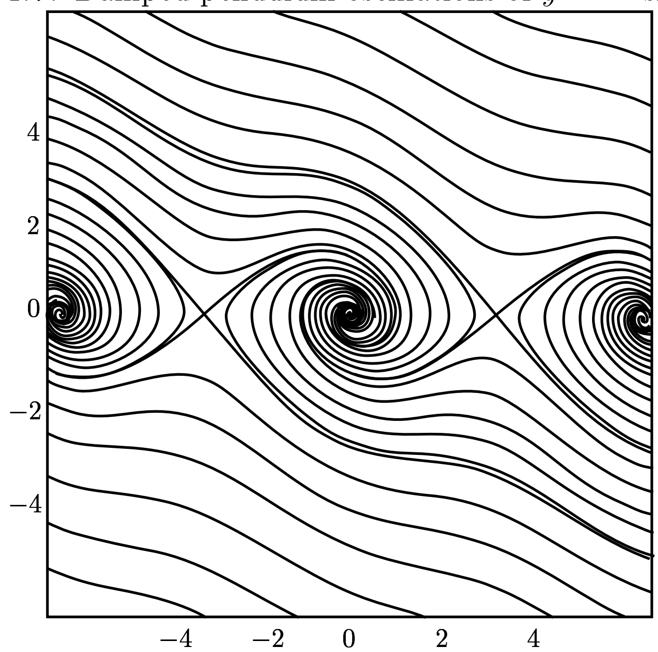
Plot the trajectories for $v_0 \approx 2$ and $v_0 \approx 2.5$ and take a guess as to the value of v_c.

3. Compare the previous plot to a plot of

$$y'' = -\sin y - 0.6x.$$

In your own words describe the qualitative difference between the two systems; what sets the two phase portraits apart, and what does this indicate about the underlying system?

FIGURE 4.4.7. Damped pendulum oscillations of $y'' = -\sin y - 0.4x$.



5. Linear Second Order Equations

5.1. Homogeneous Linear Equations.

Definition. A second-order differential equation is said to be *linear* iff it can be written as

$$y'' + p(x)y' + q(x)y = f(x).$$

The function $f(x)$ is called a *forcing function* since it generally corresponds to an external force on the system, when modelling physical applications.

Definition. As earlier, we say $y'' + p(x)y' + q(x)y = f(x)$ is homogeneous iff $f \equiv 0$ or nonhomogeneous if $f \not\equiv 0$.

We saw earlier that the homogeneous linear equation

$$y' + p(x)y = 0$$

can be solved using

$$y = ce^{-\int p(x)\,dx},$$

and that we could combine variation of parameters with this technique to solve nonhomogeneous linear equations by finding a solution of the form

$$y = uy_1.$$

Unfortunately, nice formulae like these do not exist for second-order equations and more work is generally required. One thing that does remain similar, however, is that we still need to solve the complementary equation before solving the nonhomogeneous equation.

Recall that a given differential equation will have many solutions, but if we specify the initial conditions, this will determine a unique solution. When we are solving a DE, we want to find all possible solutions. Just as first-order systems usually have a one-parameter family of solutions (this parameter is the ubiquitous c in the general solution), second-order families usually have two-parameter families of solutions. General solutions will be *linear combinations*

$$y = c_1 y_1 + c_2 y_2,$$

and initial value problems will have two initial conditions

$$y(x_0) = k_0, \quad y'(x_0) = k_1,$$

because we will need two equations to help determine the two unknowns c_1, c_2.

This leads to the important principle that combinations of solutions are again solutions.

Theorem 5.1.1. *If y_1 and y_2 are both solutions of*

$$y'' + p(x)y' + q(x)y = 0$$

on (a, b), then any linear combination

$$y = c_1 y_1 + c_2 y_2$$

is also a solution.

Proof. We differentiate y once:

$$y' = c_1 y_1' + c_2 y_2'$$

and then again:

$$y'' = c_1 y_1'' + c_2 y_2''$$

and substitute back into the original equation, to check.

$$\begin{aligned}
&y'' + p(x)y' + q(x)y \\
&\quad = (c_1y_1'' + c_2y_2'') + p(x)(c_1y_1' + c_2y_2') + q(x)(c_1y_1 + c_2y_2) \\
&\quad = (c_1y_1'' + p(x)c_1y_1' + q(x)c_1y_1) + (c_2y_2'' + p(x)c_2y_2') + q(x)c_2y_2) \\
&\quad = c_1\left(y_1'' + p(x)y_1' + q(x)y_1\right) + c_2\left(y_2'' + p(x)y_2') + q(x)y_2\right) \\
&\quad = c_1 \cdot 0 + c_2 \cdot 0
\end{aligned}$$

□

Example 5.1.1. Consider the differential equation

$$y'' + y = 0.$$

From what you know of basic calculus, you might notice[4] that both $y_1 = \sin x$ and $y_2 = \cos x$ are solutions:

$$(\sin x)'' = -\sin x \quad \text{and} \quad (\cos x)'' = \cos x.$$

Take $y = 3\sin x + 4\cos x$. Then

$$\begin{aligned}
y' &= 3\cos x + 4\sin x \\
y'' &= -3\sin x - 4\cos x
\end{aligned}$$

So this solution y clearly satisfies the DE. It is also apparent that linear combinations of solutions are again solutions because taking derivatives is a linear operation:

$$\tfrac{d}{dx}\left(af(x) + bg(x)\right) = a\tfrac{df}{dx}(x) + b\tfrac{dg}{dx}(x).$$

Since we need to find *all* solutions of a given differential equation, we define the following concept:

[4]If not, don't worry - we'll shortly see better methods for how to find solutions that don't involve such lucky insights.

Definition. $\{y_1, y_2\}$ is a *fundamental set of solutions* of $y'' + p(x)y' + q(x)y = 0$ on (a, b) if every solution on (a, b) can be written as a linear combination of y_1 and y_2. In this way, y_1 and y_2 form a basis for the set of all solutions; all other solutions can be built out of these ones.

Definition. Just analogous to the first-order definition, if $\{y_1, y_2\}$ is a fundamental set of solutions, then

$$y = c_1 y_1 + c_2 y_2$$

is the *general solution* of the DE.

So how do we determine whether or not $\{y_1, y_2\}$ is a fundamental set of solutions? We need one more concept first.

Definition. y_1, y_2 are linearly independent iff neither is a constant multiple of the other. In general, a collection of functions $\{y_1, y_2, \ldots, y_n\}$ is linearly independent if none of them can be written as a linear combination of the others. The idea is that if we had, say,

$$y_1 = ay_2 + by_3 - cy_4,$$

then we could "build" y_1 out of the other functions. Thus, the set of all linear combinations of $\{y_1, y_2, \ldots, y_n\}$ would be the same as the set of all linear combinations of $\{y_2, \ldots, y_n\}$; y_1 is redundant. In a linearly independent set, we want no redundancies (this is where the "independent" comes in).
For two functions, however, this amounts to simply meaning that neither function is a constant multiple of the other:

$$y_1 \neq ay_2, \qquad \text{for any } a.$$

Theorem 5.1.2. *If p, q are continuous on (a, b), then $\{y_1, y_2\}$ is a fundamental set of solutions iff it is linearly independent.*

$\{y_1, y_2\}$ will be a fundamental set of solutions iff the solution of any IVP

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

can be written as $y = c_1y_1 + c_2y_2$, or, equivalently, that

$$\begin{aligned} c_1y_1(x_0) + c_2y_2(x_0) &= k_0 \\ c_1y_1'(x_0) + c_2y_2'(x_0) &= k_1 \end{aligned}$$

has a solution (i.e., we can solve for c_1, c_2) for every choice of k_0, k_1. We can solve this system of linear equations using Cramer's Rule, a result from linear algebra.

$$c_1 = \frac{\begin{vmatrix} k_0 & y_2(x_0) \\ k_1 & y_2'(x_0) \end{vmatrix}}{\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix}}, \qquad c_2 = \frac{\begin{vmatrix} y_1(x_0) & k_0 \\ y_1'(x_0) & k_1 \end{vmatrix}}{\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix}}$$

This is well-defined whenever the denominator is not zero.

Definition. The *Wronskian* of $\{y_1, y_2\}$ is

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1y_2' - y_1'y_2.$$

It can be shown that the Wronskian satisfies Abel's formula

$$W(x) = W(x_0)e^{\left(-\int_{x_0}^{x} p(t)\,dt\right)}.$$

Since $e^{f(x)}$ is never 0, $W(x) = 0$ iff $W(x_0) = 0$. But then if $W(x_0) = 0$, we will have $W(x) \equiv 0$! This shows that the Wronskian is either always 0, or never 0.

Theorem 5.1.3. *$y = c_1y_1 + c_2y_2$ is the general solution on (a, b) iff $W(x) \neq 0$ for $a < x < b$.*

Proof. (Sketch)

$$\begin{aligned}
y = c_1y_1+c_2y_2 \text{ is the general solution}\\
&\iff \{y_1, y_2\} \text{ is a fundamental set of solutions}\\
&\iff \{y_1, y_2\} \text{ is linearly independent}\\
&\iff W(x_0) \neq 0 \ \text{ for some } x \in (a, b)\\
&\iff W(x) \neq 0 \ \text{ for all } a < x < b.
\end{aligned}$$

□

Example 5.1.2. Find the Wronskian of the set $\{y_1, y_2\}$ of solutions of Legendre's equation

$$\left(1 - x^2\right) y'' - 2xy' + \alpha(\alpha + 1)y = 0.$$

We will use Abel's formula, so first rewrite this as

$$y'' - \tfrac{2x}{1-x^2}y' + \tfrac{\alpha(\alpha+1)}{1-x^2}y = 0.$$

Then $p(x) = \frac{2x}{x^2-1}$, which we integrate

$$\begin{aligned}
\int_0^x p(t)\,dt &= \int_0^x \frac{2t}{t^2-1}\,dt\\
&= \int_{-1}^{x^2-1} \frac{du}{u} && u = t^2 - 1, du = 2t\,dt\\
&= \left[\log|t^2 - 1|\right]_0^x\\
&= \log|x^2 - 1| - \log 1\\
&= \log(x^2 - 1)
\end{aligned}$$

Then we use Abel's formula

$$\begin{aligned} W(x) &= W(0)e^{\left(-\int_0^x p(t)\,dt\right)} \\ &= 1 \cdot e^{-\log(x^2-1)} \\ &= \tfrac{1}{x^2-1} \end{aligned}$$

Example 5.1.3. Let's use Abel's formula to construct one solution from another. First, suppose p, q are continuous and we already have one solution y_1 of

$$y'' + p(x)y' + q(x)y = 0.$$

Additionally, assume that $y_1(x) \neq 0$ for $a < x < b$.

(a) Show that if $K \neq 0$ is any constant and y_2 satisfies

$$y_1y_2' - y_1'y_2 = Ke^{-\int p(x)\,dx}$$

on (a, b), then y_2 also satisfies the equation on (a, b), and $\{y_1, y_2\}$ is a fundamental set of solutions on (a, b).

Differentiating both side of this equation yields

$$\begin{aligned} y_1y_2'' + y_1'y_2' - y_1'y_2' - y_1''y_2 &= -p(x)Ke^{-\int p(x)\,dx} \\ y_1y_2'' - y_1''y_2 &= -p(x)Ke^{-\int p(x)\,dx} \\ y_1y_2'' + (p(x)y_1' + q(x)y_1)y_2 &= -p(x)(y_1y_2' - y_1'y_2) \\ y_1(y_2'' + p(x)y_2') + (p(x)y_1' + q(x)y_1)y_2 &= p(x)y_1'y_2 \\ y_1(y_2'' + p(x)y_2' + q(x)y_2) &= 0 \\ y_2'' + p(x)y_2' + q(x)y_2 &= 0 \qquad (y_1 \neq 0) \end{aligned}$$

So y_2 also satisfies the differential equation. Then we can look at the Wronskian of $\{y_1, y_2\}$

$$W = y_1y_2' - y_1'y_2 = Ke^{-\int p(x)\,dx}.$$

Since $K = W(x_0) \neq 0$, we have a fundamental set of solutions. Note that the differentiation of y_2' is justified by rewriting the Wronskian as

$$y_2' = \tfrac{y_1'}{y_1} y_2 + \tfrac{K}{y_1} e^{-\int p(x)\,dx}$$

and noting that everything on the right is differentiable.

(b) Now we show that if $y_2 = uy_1$, where $u' = \frac{K}{y_1^2} e^{-\int p(x)\,dx}$, then $\{y_1, y_2\}$ is a fundamental set of solutions on (a, b).

From the Wronskian, we have the linear nonhomogeneous equation

$$y' - \tfrac{y_1'}{y_1} y = \tfrac{K}{y_1} e^{-\int p(x)\,dx},$$

Which we can solve by variation of parameters. Since y_1 is a solution of the complementary equation

$$y' - \tfrac{y_1'}{y_1} y = 0,$$

we get $y_2 = uy_1$ where

$$\begin{aligned} u'y_1 + uy_1' - \tfrac{y_1'}{y_1} uy_1 &= \tfrac{K}{y_1} e^{-\int p(x)\,dx} \\ u'y_1 &= \tfrac{K}{y_1} e^{-\int p(x)\,dx} \\ u' &= \tfrac{K}{y_1^2} e^{-\int p(x)\,dx} \end{aligned}$$

Thus, if we are given that $u' = \frac{K}{y_1^2} e^{-\int p(x)\,dx}$, the equation

$$y_1 y_2' - y_1' y_2 = K e^{-\int p(x)\,dx}$$

will be satisfied, and by part (a) we will have a fundamental set of solutions.

Example 5.1.4. Apply this method to

$$(3x-1)y'' - (3x+2)y' - (6x-8)y = 0, \qquad y_1 = e^{2x}.$$

We are looking for $y_2 = uy_1$, where $u' = \frac{K}{y_1^2}e^{-\int p(x)\,dx}$. We need to find u', then integrate it. Start by rewriting the original equation:

$$y'' - \tfrac{3x+2}{3x-1}y' - \tfrac{6x-8}{3x-1}y = 0.$$

Now construct u from $p(x)$:

$$p(x) = -\tfrac{3x+2}{3x-1} = -\tfrac{3x-1}{3x-1} - \tfrac{3}{3x-1} = -1 - \tfrac{3}{3x-1}$$

$$\int p(x)\,dx = -\int dx - \int \tfrac{3}{3x-1}\,dx = -x - \log|3x-1|$$

$$u' = \tfrac{K}{y_1^2}e^{-\int p(x)\,dx} = Ke^{-4x}e^{x+\log(3x-1)} = Ke^{-3x}(3x-1)$$

$$u = K\int e^{-3x}(3x-1)\,dx$$

$$= 3K\int xe^{-3x}\,dx - K\int e^{-3x}\,dx$$

Put $u = x$, $du = dx$, $dv = e^{-3x}dx$, $v = -\frac{1}{3}e^{-3x}$.

$$= 3K\left(-\tfrac{1}{3}xe^{-3x} + \tfrac{1}{3}\int e^{-3x}dx\right) + \tfrac{K}{3}e^{-3x}$$

$$= -Kxe^{-3x} - \tfrac{K}{3}e^{-3x} + \tfrac{K}{3}e^{-3x}$$

$$= -Kxe^{-3x}$$

Then $y_2 = uy_1 = -Kxe^{-3x} \cdot e^{2x} = xe^{-x}$, if we take $K = -1$.

§5.1 Homework Assignment:

Read: 184-193, skip proofs

Exercises: 1,2,5,7-10,12,19,20,30

5.2. Homogeneous Equations with Constant Coefficients.

Definition. A *constant coefficient equation* is a second-order linear differential equation

$$ay'' + by' + cy = f(x),$$

where $a, b, c \in \mathbb{R}$.

For now, we consider only the homogeneous case

$$ay'' + by' + cy = 0.$$

Note that $y = e^{rx}$ is a solution to this equation iff

$$\begin{aligned} ay'' + by' + cy &= 0 \\ ar^2e^{rx} + bre^{rx} + ce^{rx} &= 0 \\ \left(ar^2 + br + c\right) e^{rx} &= 0 \\ \left(ar^2 + br + c\right) &= 0 \qquad \text{(since } e^{rx} \neq 0\text{)} \end{aligned}$$

Definition.

$$p(r) = ar^2 + br + c$$

is the *characteristic polynomial* of the equation above, and

$$p(r) = 0$$

is the *characteristic equation* .

There are three cases for the roots of the characteristic equation, depending on the discriminant of

$$r = \frac{-b \pm \sqrt{b^2 - 4ab}}{2a}.$$

case (i) $b^2 - 4ac > 0$. Two distinct real roots.

Consider the IVP

$$y'' + 5y' + 6y = 0, \quad y(0) = 2, \quad y'(0) = 3.$$

Assuming that $y = e^{tx}$, r must be a root of the characteristic equation, so

$$r^2 + 5r + 6 = (r+2)(r+3) = 0 \quad \Longrightarrow \quad r_1 = -2, r_2 = -3$$

and the general solution is

$$y = c_1 e^{-2x} + c_2 e^{-3x}.$$

To solve the IVP, we differentiate the solution an get

$$y' = -2c_1 e^{-2x} - 3c_2 e^{-3x}.$$

Then the initial conditions give

$$y(0) = c_1 + c_2 = 2$$
$$y'(0) = -2c_1 - 3c_2 = 3$$

Solve this system of linear equations to get

$$c_2 = -7, c_1 = 9,$$

so the solution is

$$y = 9e^{-2x} - 7e^{-3x}.$$

case (ii) $b^2 - 4ac = 0$. A repeated real root.

Solve the differential equation

$$y'' + 4y' + 4 = 0.$$

The characteristic equation is

$$r^2 + 4r + r = (r + 2)^2 = 0,$$

so the roots are $r_1 = r_2 = -2$. Therefore one solution is $y_1 = e^{-2x}$. To find a fundamental set of solutions, we need a second solution which is not a multiple of y_1. The second solution can be found in several ways. We use the following idea:

Assume that the other solution is of the form $v(x)y_1(x)$. Then

$$\begin{aligned} y &= v(x)y_1(x) = v(x)e^{-2x} \\ y' &= v'(x)e^{-2x} - 2v(x)e^{-2x} \\ y'' &= v''(x)e^{-2x} - 4v'(x)e^{-2x} + 4v(x)e^{-2x} \end{aligned}$$

Substituting these back into the original DE gives

$$\begin{aligned} \left(v'' - 4v' + 4v + 4v' - 8v + 4v\right) e^{-2x} &= 0 \\ v'' &= 0 \\ v' &= c_1 \\ v(x) &= c_1 x + c_2 \end{aligned}$$

Now plugging this back into the earlier equation,

$$\begin{aligned} y &= v(x)e^{-2x} \\ &= (c_1 x + c_2)e^{-2x} \\ &= c_1 x e^{-2x} + c_2 e^{-2x} \end{aligned}$$

The second term is the solution we had previously, but the first term is something new. To verify that they are linearly independent, we check the Wronskian

$$W(x) = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & (1-2x)e^{-2x} \end{vmatrix}$$
$$= e^{-4x} - 2xe^{-4x} + 2xe^{-4x}$$
$$= e^{-4x} \neq 0$$

Therefore, $\{e^{-2x}, xe^{-2x}\}$ is a fundamental set of solutions.

How does this work?
Since y_1 is a solution, we know cy_1 will also be a solution, for any constant c.
We generalize this by replacing c with a function $v(x)$, then try to determine $v(x)$ so that $v(x)y_1(x)$ is a solution to the differential equation.

This method is called *Reduction of Order*.
Suppose we have a nontrivial solution y_1 of the equation

$$y'' + p(x)y' + q(x)y = 0.$$

To find a second solution, let $y = vy_1$. Then

$$y' = v'y_1 + vy_1'$$
$$y'' = v''y_1 + 2v'y_1' + vy_1''$$

Substituting back and collecting terms gives

$$y_1v'' + (2y_1' + py_1)v' + (y_1'' + py_1 + qy_1)v = 0.$$

Since y_1 is a solution of the original equation, the coefficient of v here is just 0, and it reduces to

$$y_1v'' + (2y_1' + py_1)v' = 0.$$

This is actually a first order equation for the function v'!
So we have reduced a second-order equation to a first-order

equation.
Thus, it can be solved as a first order linear equation or a separable equation. Once you have v', you can get v by integrating, then multiply by y_1 to get your new, linearly independent solution.

case (iii) $b^2 - 4ac < 0$. Two distinct complex roots.

Find the solution of the DE

$$y'' + 14y' + 50, \quad y(0) = 2, \quad y'(0) = -17.$$

The characteristic equation is

$$r^2 + 14r + 50 = 0,$$

which has roots

$$r = \frac{-14 \pm \sqrt{196 - 200}}{2} = -7 \pm i.$$

Thus the general solution of the DE is

$$y = c_1 e^{(-7+i)x} + c_2 e^{(-7-i)x}.$$

So the first part of this is

$$y_1(x) = c_1 e^{(-7+i)x} = c_1 e^{-7x} e^{ix},$$

but what in the heck is the exponential of an imaginary number? Actually, it turns out that

$$y_1(x) = c_1 e^{-7x}(\cos x + i \sin x).$$

We can verify this by taking the derivatives and substituting, but the more interesting question is: why does it work?!? The book uses a long and drawn-out discussion to explain this because it wants to avoid scary-looking formulae. However, we have all had calculus 9C or an equivalent, so we have no fear. The tool we're using is The[5] Euler Formula:

$$e^{aix} = \cos ax + i \sin ax,$$

and it is not too difficult to prove if we recall the power series definition of the exponential function:

$$e^x := \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \tfrac{x^2}{2} + \tfrac{x^3}{3!} + \tfrac{x^4}{4!} + \dots$$

Notice that if we separate the even and odd terms, we get

$$= \Big(1 + \tfrac{x^2}{2} + \tfrac{x^4}{4!} + \dots\Big) + \Big(x + \tfrac{x^3}{3!} + \tfrac{x^5}{5!} + \dots\Big)$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

These series are very close to

$$\cos x := \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad \sin x := \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

Since $i^2 = -1$, the even and odd powers of i are

$$i^{2n} = (i^2)^n = (-1)^n, \qquad \text{and} \qquad i^{2n+1} = i \cdot i^{2n} = i(-1)^n.$$

[5] Actually, just one of a great many formulae developed by Euler ...

So if we plug an imaginary number into the series for e^x, we get

$$\begin{aligned} e^{ix} &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(ix)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(ix)^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{i^{2n}x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{i^{2n+1}x^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + i\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\ &= \cos x + i \sin x \end{aligned}$$

More generally, for $z \in \mathbb{C}$ we have

$$e^z = e^{a+ib} = e^a e^{ib} = e^a(\cos bx + i \sin bx).$$

Note that if we plug in $i\pi$, we get

$$e^{i\pi} = \cos \pi + i \sin \pi = -1 + 0 = -1,$$

which gives Euler's Identity:

$$e^{i\pi} + 1 = 0.$$

This tiny and elegant formula ties together all five of the fundamental constants of mathematics. This should blow your mind if you haven't seen it before.

There is one final upshot in the case of complex conjugate roots: by combining the real and imaginary parts of the answer, we can obtain a real-valued solution. This is invaluable for studying the solution, graphing it, etc.

Solve the IVP

$$y'' + 14y' + 50, \quad y(0) = 2, \quad y'(0) = -17.$$

We saw previously that the general solution is

$$y = c_1 e^{(-7+i)x} + c_2 e^{(-7-i)x},$$

so we apply Euler's formula and regroup the terms:

$$\begin{aligned} y &= c_1 e^{-7x}(\cos x + i \sin x) + c_2 e^{-7x}(\cos(-x) + i\sin(-x)) \\ &= c_1 e^{-7x} \cos x + i c_1 e^{-7x} \sin x) + c_2 e^{-7x} \cos x - i c_2 e^{-7x} \sin x \\ &= c_1 e^{-7x} \cos x + c_2 e^{-7x} \cos x + i c_1 e^{-7x} \sin x) - i c_2 e^{-7x} \sin x \\ &= (c_1 + c_2) e^{-7x} \cos x + i(c_1 - c_2) e^{-7x} \sin x \\ y &= e^{-7x} \left(k_1 \cos x + k_2 \sin x\right) \end{aligned}$$

So we collected all the imaginary terms together and simplified the constants by setting

$$k_1 := c_1 + c_2 \qquad \text{and} \qquad k_2 := i(c_1 - c_2),$$

effectively removing the i from view. Is this allowed? Well, i is just a constant, so YES. If you differentiate this solution, you will see that it works. Now the first initial condition gives

$$\begin{aligned} 2 &= k_1 \cos 0 + k_2 \sin 0 \\ 2 &= k_1 \end{aligned}$$

$$\text{and since} \quad \begin{aligned} y' = &-7e^{-7x} \left(k_1 \cos x + k_2 \sin x\right) \\ &+ e^{-7x} \left(-k_1 \sin x + k_2 \cos x\right), \end{aligned}$$

the second initial condition gives

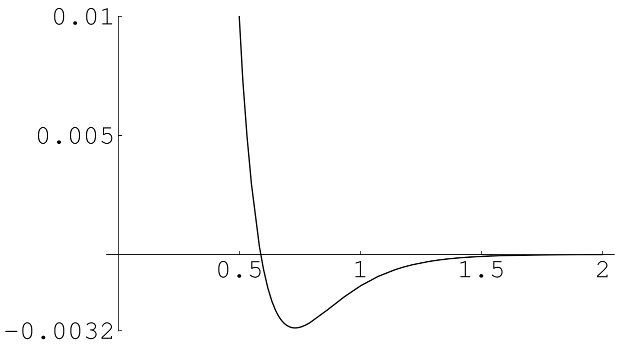
$$\begin{aligned}-17 &= -7\left(k_1\cos 0 + k_2\sin 0\right)\\ &\qquad + \left(-k_1\sin 0 + k_2\cos 0\right),\\ &= -7k_1 + k_2,\\ &= -14 + k_2,\\ -3 &= k_2,\end{aligned}$$

So the solution to the IVP is

$$y = e^{-7x}\left(2\cos x - 3\sin x\right).$$

Despite the initial complex numbers, we have ended with a real-valued function! This is invaluable for understanding what is going on with the system, graphing it, etc. For example, as you might expect for a solution involving sines and cosines, there is some oscillation going on.

FIGURE 5.2.1. The solution curve is an extremely damped oscillation.



§5.2 Homework Assignment:

Read: 200-206

Exercises: 1-4,13,14,17,22-24,34

5.3. Nonhomogeneous Equations.

Definition. Any solution of a nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = f(x)$$

is called a *particular solution* .

Theorem 5.3.1. *Suppose we have a particular solution y_p of*

$$y'' + p(x)y' + q(x)y = f(x)$$

and a fundamental set of solutions $\{y_1, y_2\}$ of the complementary equation

$$y'' + p(x)y' + q(x)y = 0.$$

Then the general solution of the nonhomogeneous equation is

$$y = y_p + c_1y_1 + c_2y_2.$$

This theorem allows us to break the task of finding a general solution into the two smaller tasks of finding a general solution for the complementary equation and finding a particular solution. The first of these we already know how to do. The next few chapters will be all about finding a particular solution.

Theorem 5.3.2. *(Principle of Superposition)*
If y_{p_1} is a particular solution of

$$y'' + p(x)y' + q(x)y = f_1(x),$$

and y_{p_2} is a particular solution of

$$y'' + p(x)y' + q(x)y = f_2(x),$$

Then

$$y = y_{p_1} + y_{p_2}$$

is a particular solution of

$$y'' + p(x)y' + q(x)y = f_1(x) + f_2(x).$$

This theorem allows us to break down the task of finding a solution for $f(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$ into the n smaller tasks of finding particular solutions for each $f_i(x)$.

We first saw this idea (for first-order systems) in the homework §2.1, #46. In the reading for this chapter, concentrate on the theorems. The examples are rather unenlightening, as you don't yet have any methods for finding particular solutions.

§5.3 Homework Assignment:

Read: 209-210,213-214 (Theorems)
Exercises: 1-4,7,10-14

5.4. **Method of Undetermined Coefficients I.**

Definition. The *Method of Undetermined Coefficients* is the first method we'll see for how to obtain a particular solution y_p, and it works as follows:

(1) Make an initial assumption about the form of the particular solution y_p, but with the coefficients unspecified.

(2) Then take derivatives and substitute into the nonhomogeneous equation and attempt to determine the coefficients so as to satisfy the equation.

(3) If successful, we have a particular solution. If not, there is no solution of the assumed form, but we can modify the assumption and try again.

The main advantage of this method is that it is straightforward to execute once you have picked a y_p to try. Its disadvantage is that it is primarily limited to equations for which we can easily write down the correct form of the particular solution in advance: exponentials, polynomials, sines, and cosines.

We begin by considering polynomial forcing functions:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

Example 5.4.1. Find the solution of

$$y'' - 4y' + 4y = 2 + 8x - 4x^2.$$

The roots of the characteristic polynomial $r^2 - 4r + 4 = (r-2)^2$ are $r_1, r_2 = 2$, so $\{e^{2x}, xe^{2x}\}$ is a fundamental set of solutions.

1. The forcing function is a second-degree polynomial. If we substitute a second-degree polynomial into the LHS, we will

obtain another second-degree polynomial, so let

$$y_p = A + Bx + Cx^2,$$

where A, B, C are some unknown coefficients, yet to be determined.

2. To find these coefficients, we compute

$$y_p'(x) = B + 2Cx, \qquad y_p''(x) = 2C,$$

and substitute into the original problem.

$$\begin{aligned} 2 + 8x - 4x^2 &= y_p'' - 4y_p' + 4y_p \\ &= 2C - 4(B + 2Cx) + 4(A + Bx + Cx^2) \\ &= (4A - 4B + 2C) + (4B - 8C)x + 4Cx^2 \end{aligned}$$

Now equating the coefficients of like powers of x, we obtain

$$\begin{aligned} -4x^2 &= 4Cx^2 && \Longrightarrow \quad C = -1 \\ 8x &= (4B - 8C)x && \Longrightarrow \quad 0 = 4Bx \quad \Longrightarrow \quad B = 0 \\ 2 &= (4A - 4B + 2C) && \Longrightarrow \quad 4 = 4A \quad \Longrightarrow \quad A = 1 \end{aligned}$$

Thus, $y_p = 1 - x^2$ is a particular solution and the general solution is

$$y = 1 - x^2 + c_1e^{2x} + c_2xe^{2x}.$$

3. Important Note: if one of the terms in $(1-x^2)$ were a solution to the complementary equation, we would need to take

$$y_p = x(1 - x^2),$$

unless $x(1-x^2)$ also contained a solution to the complementary equation, in which case take

$$y_p = x^2(1 - x^2), \text{etc.}$$

Now we consider exponential forcing functions:

$$f(x) = ke^{\alpha x}.$$

Example 5.4.2. Find the general solution of

$$y'' - 3y' - 4y = 3e^{2x}.$$

The roots of the characteristic polynomial $r^2 - 3r - 4 = (r-4)(r+1)$ are $r_1 = -1, r_2 = 4$, so $\{e^{-x}, e^{4x}\}$ is a fundamental set of solutions.

1. Since the exponential function propagates itself when differentiated, the most plausible way to obtain a particular solution is to assume y_p is some multiple of e^{2x}, that is, take

$$y_p = Ae^{2x},$$

where A is some unknown coefficient, yet to be determined.

2. To find A, we compute

$$y_p'(x) = 2Ae^{2x}, \qquad y_p''(x) = 4Ae^{2x},$$

and substitute into the original problem.

$$\begin{aligned} 3e^{2x} &= y_p'' - 3y_p' - 4y \\ &= 4Ae^{2x} - 6Ae^{2x} - 4Ae^{2x} \\ &= -6Ae^{2x} \\ -\tfrac{1}{2}e^{2x} &= Ae^{2x} \\ -\tfrac{1}{2} &= A \end{aligned}$$

Thus, $y_p = -\frac{1}{2}e^{2x}$ is a particular solution and the general solution is

$$y = -\tfrac{1}{2}e^{2x} + c_1e^{-x} + c_2e^{4x}.$$

Now consider forcing functions with exponentials and polynomials:

$$f(x) = e^{\alpha x}(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n).$$

Example 5.4.3. Find the general solution of

$$y'' - y' - 2y = e^x(9 + 2x - 4x^2).$$

The roots of the characteristic polynomial $r^2 - r - 2 = (r-2)(r+1)$ are $r_1 = -1, r_2 = 2$, so $\{e^{-x}, e^{2x}\}$ is a fundamental set of solutions.

1. Based on the two previous examples, try

$$y_p = e^x(A + Bx + Cx^2),$$

where A, B, C are some unknown coefficients.

2. To find these coefficients, we compute

$$\begin{aligned} y_p'(x) &= e^x(A + Bx + Cx^2) + e^x(B + 2Cx) \\ &= e^x\left((A+B) + (B+2C)x + Cx^2\right) \\ y_p''(x) &= e^x\left((A+B) + (B+2C)x + Cx^2\right) \\ &\qquad + e^x\left((B+2C) + 2Cx\right) \\ &= e^x\left((A+2B+2C) + (B+4C)x + Cx^2\right) \end{aligned}$$

and substitute into the original problem.

$$\begin{aligned} &e^x(9+2x - 4x^2) \\ &\quad = y_p'' - y_p' - 2y_p \\ &\quad = e^x\left((A+2B+2C) + (B+4C)x + Cx^2\right) \\ &\qquad\qquad - e^x\left((A+B) + (B+2C)x + Cx^2\right) \\ &\qquad\qquad - 2e^x(A + Bx + Cx^2) \\ &\quad = e^x\left((-2A+B+2C) + (-2B+2C)x - 2Cx^2\right) \end{aligned}$$

Now equating the coefficients of like powers of x, we obtain

$$\begin{array}{lllll} -2Cx^2 = -4x^2 & \Longrightarrow & C = 2 & & \\ (-2B + 2C)x = 2x & \Longrightarrow & -2B = -2 & \Longrightarrow & B = 1 \\ -2A + B + 2C = 9 & \Longrightarrow & -2A = 4 & \Longrightarrow & A = -2 \end{array}$$

Thus, $y_p = e^x(-2 + x + 2x^2)$ is a particular solution and the general solution is

$$y = e^x(-2 + x + 2x^2) + c_1 e^{-x} + c_2 e^{2x}.$$

The book introduces another way to solve DEs of this type. I will include it here in case you find it easier to apply or remember.

When the forcing function is

$$f(x) = G(x)e^{\alpha x},$$

(where G is any function) look for a particular solution of the form

$$y_p = ue^{\alpha x},$$

where $u = u(x)$ is some function that needs to be determined. Then, after you compute y_p' and y_p'' and substitute back into the original DE, you will be able to cancel all the $e^{\alpha x}$s.

Example 5.4.4. Find the general solution of

$$y'' + 2y' + y = e^{2x}(-7 - 15x + 9x^2).$$

The roots of the characteristic polynomial $r^2 + 2r + 1 = (r + 1)^2$ are $r_1, r_2 = -1$, so $\{e^{-x}, xe^{-x}\}$ is a fundamental set of solutions.

We look for a solution of the form $y = ue^{2x}$.

$$y'(x) = u'e^{2x} + 2ue^{2x} = (u' + 2u)e^{2x}$$

$$y''(x) = (u'' + 2u')e^{2x} + 2(u' + 2u)e^{2x} = (u'' + 4u' + 4u)e^{2x}$$

and substitute into the original problem.

$$\begin{aligned} e^{2x}(-7 - 15x + 9x^2) \\ &= y'' + 2y' + y \\ &= e^{2x}\left((u'' + 4u' + 4u) + 2(u' + 2u) + u\right) \\ -7 - 15x + 9x^2 &= u'' + 6u' + 9u \end{aligned}$$

We suppose

$$\begin{aligned} u_p &= A + Bx + Cx^2 \\ u_p' &= B + 2Cx \\ u_p'' &= 2C \end{aligned}$$

This gives

$$\begin{aligned} -7 - 15x + 9x^2 &= u_p'' + 6u_p' + 9u_p \\ &= 2C + 6B + 12Cx + 9A + 9Bx + 9Cx^2 \\ &= (9A + 6B + 2C) + (9B + 12C)x + 9Cx^2, \end{aligned}$$

so we collect coefficients:

$$\begin{aligned} 9x^2 &= 9Cx^2 && \implies C = 1 \\ -15x &= 9Bx + 12Cx && \implies -27 = 9B && \implies B = -3 \\ -7 &= 9A + 6B + 2C && \implies 9 = 9A && \implies A = 1 \end{aligned}$$

giving $u_p = 1 - 3x + x^2$ and $y_p = (1 - 3x + x^2)e^{2x}$. The general solution is

$$y = (1 - 3x + x^2)e^{2x} + c_1 e^{-x} + c_2 x e^{-x}.$$

Definition. *Resonance* is a phenomenon that occurs when the frequency of the forcing function is the same as the natural frequency of the system. For example, the characteristic polynomial of

$$y'' - 5y' + 6y = e^{2x}$$

is $p(r) = r^2 - 5r + 6 = (r-2)(r-3)$, which has roots $r_1 = 2, r_2 = 3$. Thus the fundamental set of solutions is $\{e^{2x}, e^{3x}\}$. One of these functions, e^{2x}, occurs in the forcing function, so this system will have resonance. Problems with resonance require special attention.

In terms of solving, you need to multiply your guessed y_p by x until no term in y_p is a solution of the complementary equation.

In terms of applications, resonance can either be good or bad, but must be considered. It can produce instabilities leading to the catastrophic failure of a structure, like a building or a bridge; suspension bridges are susceptible to the resonance from wind. Also, soldiers traditionally break step when crossing a bridge to eliminate the *periodic* force of their marching that could resonate with the natural frequency of the bridge. Another example occurred in the design of the high-pressure fuel turbopump for the space shuttle engine. The turbopump was unstable and could not be operated over 20,000 RPM as compared to the design speed of 39,000 RPM. This difficulty led to shutdown of the space program for 6 months at an estimated $500,000 per day.

On the other hand, resonance can be put to good use in the design of instruments, like seismographs, intended to detect weak incoming signals.

Example 5.4.5. Find a particular solution of

$$y'' + y' - 12y = e^{3x}(-6 + 7x).$$

The roots of the characteristic polynomial $r^2+r-12 = (r+4)(r-3)$ are $r_1 = -4, r_2 = 3$, so $\{e^{-4x}, e^{3x}\}$ is a fundamental set of solutions. Since e^{3x} appears in the forcing function and in the solution to the complementary equation, we will encounter resonance.

We look for a solution of the form $y = ue^{3x}$.

$$\begin{aligned} y'(x) &= u'e^{3x} + 3ue^{3x} = (u' + 3u)e^{3x} \\ y''(x) &= (u'' + 3u')e^{3x} + 3(u' + 3u)e^{3x} = (u'' + 6u' + 9u)e^{3x} \end{aligned}$$

and substitute into the original problem.

$$\begin{aligned} &e^{3x}(-6 + 7x) \\ &\qquad = y'' + y' - 12y \\ &\qquad = e^{3x}\left((u'' + 6u' + 9u) + (u' + 3u) - 12u\right) \\ -6 + 7x &= u'' + 7u' \end{aligned}$$

The lack of a u term is evidence of resonance. Consequently, this equation does not have a particular solution of the form $u_p = A+Bx$:

$$u_p' = B, \qquad u_p'' = 0 \qquad \Longrightarrow \qquad 7B = -6 + 7x. \ \lightning$$

This difficulty occurs because the first term of

$$y_p = u_p e^{3x} = Ae^{3x} + Bxe^{3x}$$

is a solution to the complementary equation. Thus, we need to multiply u_p by x until no term of y_p is a solution of the complementary equation. Once will suffice:

$$\begin{aligned} u_p &= Ax + Bx^2 \\ u_p' &= A + 2Bx \\ u_p'' &= 2B \\ \implies \qquad 2B + 7A + 14Bx &= -6 + 7x. \end{aligned}$$

Now equating the coefficients of like powers of x, we obtain

$$\begin{aligned} 14Bx &= 7x & &\implies B = \tfrac{1}{2} \\ 2B + 7A &= -6 & &\implies 7A = -7 & &\implies A = -1 \end{aligned}$$

Thus, $u_p = -x + \frac{1}{2}x^2$ and $y_p = (-x + \frac{1}{2}x^2)e^{3x}$ is a particular solution, so the general solution is

$$y = \left(-x + \frac{x^2}{2} + c_1\right) e^{3x} + c_2 e^{-4x}.$$

Example 5.4.6. Find the general solution of

$$y'' - 4y' + 4y = e^{2x}\left(1 - 3x + 6x^2\right).$$

The roots of the characteristic polynomial $r^2 - 4r + 4 = (r-2)^2$ are $r_1, r_2 = 2$, so $\{e^{2x}, xe^{2x}\}$ is a fundamental set of solutions. Since e^{2x} appears in the forcing function and in the solution to the complementary equation, we will encounter resonance. Note that xe^{2x} also appears in both the forcing function and in the solution to the complementary equation! Major resonance!

We look for a solution of the form $y = ue^{2x}$.

$$\begin{aligned} y'(x) &= u'e^{2x} + 2ue^{2x} = (u' + 2u)e^{2x} \\ y''(x) &= (u'' + 2u')e^{2x} + 2(u' + 2u)e^{2x} = (u'' + 4u' + 4u)e^{2x} \end{aligned}$$

and substitute into the original problem.

$$\begin{aligned} e^{2x}\left(1-3x+6x^2\right) & \\ &= y'' - 4y' + 4y \\ &= e^{2x}\left((u''+4u'+4u) - 4(u'+2u) + 4u\right) \\ 1-3x+6x^2 &= u'' \end{aligned}$$

The lack of a u and a u' term is evidence of resonance. This equation does not have a particular solution of the form $u_p = A + Bx + Cx^2$. Thus, we need to multiply u_p by x until no term of y_p is a solution of the complementary equation. This time, we need to do it twice:

$$\begin{aligned} u_p &= Ax^2 + Bx^3 + Cx^4 \\ u_p' &= 2Ax + 3Bx^2 + 4Cx^3 \\ u_p'' &= 2A + 6Bx + 12Cx^2 \end{aligned}$$

$$\Longrightarrow \quad u_p'' = 2A + 6Bx + 12Cx^2 = 1 - 3x + 6x^2.$$

Now equating the coefficients of like powers of x, we obtain

$$\begin{aligned} A &= \tfrac{1}{2} \\ 6Bx &= -3x & \Longrightarrow\ B = -\tfrac{1}{2} \\ 12Cx^2 &= 6x^2 & \Longrightarrow\ C = \tfrac{1}{2} \end{aligned}$$

Thus, $u_p = \frac{x^2}{2} - \frac{x^3}{2} + \frac{x^4}{2}$ and $y_p = \frac{x^2}{2}(1 - x + x^2)e^{2x}$ is a particular solution, so the general solution is

$$y = \left(c_1 + c_2 x + \tfrac{x^2}{2} - \tfrac{x^3}{2} + \tfrac{x^4}{2}\right) e^{2x}.$$

§5.4 Homework Assignment:

Read: 225-232

Exercises: 1-13 odd,24,27,38,39(ab)

5.5. **Method of Undetermined Coefficients II.** We continue the study of the Method of Undetermined Coefficients, examining solution techniques for

$$y'' + p(x)y' + q(x)y = f_1(x),$$

when the forcing function contains trigonometric terms.

Now we consider forcing functions of the form:

$$f(x) = k \sin \omega x.$$

Example 5.5.1. Find a particular solution of

$$y'' - 3y' - 4y = 2 \sin x.$$

1. We suppose the solution to have the form

$$y_p = A \sin x.$$

where A is some unknown coefficient, yet to be determined.

2. To find A, we compute

$$y_p'(x) = A \cos x, \qquad y_p''(x) = -A \sin x,$$

and substitute into the original problem.

$$\begin{aligned} 2 \sin x &= y_p'' - 3y_p' - 4y_p \\ &= -A \sin x - 3A \cos x - 4A \sin x \\ &= -5A \sin x - 3A \cos x \end{aligned}$$

$$(2 + 5A) \sin x + 3A \cos x = 0$$

The functions sine and cosine are linearly independent, so this can only hold on an interval if

$$2 + 5A = 3A = 0$$

$$-1 = A \qquad A = 0,$$

which is a contradiction. This indicates that our choice of y_p was inadequate.

1. We suppose the solution to have the form

$$y_p = A\sin x + B\cos x.$$

where A, B are unknown coefficients.

2. To find A, B, we compute

$$y_p'(x) = A\cos x - B\sin x, \qquad y_p''(x) = -A\sin x - B\cos x,$$

and substitute into the original problem.

$$\begin{aligned} 2\sin x &= y_p'' - 3y_p' - 4y_p \\ &= (-A\sin x - B\cos x) - 3(A\cos x - B\sin x) \\ &\qquad - 4(A\sin x + B\cos x) \\ &= -A\sin x - B\cos x - 3A\cos x + 3B\sin x \\ &\qquad - 4A\sin x - 4B\cos x \\ &= (-5A + 3B)\sin x - (3A + 5B)\cos x \end{aligned}$$

Now equating the coefficients of like powers of x, we obtain

$$-5A + 3B = 2 \quad \text{and} \quad 3A + 5B = 0$$

which resolves to $A = -5/17$ and $B = 3/17$, so a particular solution is

$$y_p = -\tfrac{5}{17}\sin x + \tfrac{3}{17}\cos x.$$

In conclusion, if a forcing functions includes at least one sine or cosine, the guess at the solution needs to contain both. For any of

$$f(x) = a\sin\omega x, \quad g(x) = b\cos\omega x, \quad h(x) = a\sin\omega x + b\cos\omega x,$$

start with the guess

$$y_p = A\sin x + B\cos x.$$

Now we consider forcing functions of the form:

$$f(x) = k\sin\omega x.$$

Example 5.5.2. Find a particular solution of

$$y'' + 3y' + 2y = (12 + 20x + 10x^2)\cos x + 8x\sin x.$$

1. We suppose the solution to have the form

$$y_p = (A_0 + A_1 x + A_2 x^2)\sin x + (B_0 + B_1 x + B_2 x^2)\cos x.$$

where all A_is and B_is are unknown coefficients.

2. To find the A_is and B_is, we compute

$$\begin{aligned}
y_p'(x) &= (A_1 + 2A_2 x)\sin x + (A_0 + A_1 x + A_2 x^2)\cos x \\
&\qquad + (B_1 + 2B_2 x)\cos x - (B_0 + B_1 x + B_2 x^2)\sin x \\
&= \big((A_1 - B_0) + (2A_2 - B_1)x - B_2 x^2\big)\sin x \\
&\qquad + \big((A_0 + B_1) + (A_1 + 2B_2)x + A_2 x^2\big)\cos x \\
y_p''(x) &= (2A_2 - B_1 - 2B_2 x)\sin x \\
&\qquad + (A_1 + 2A_2 x - B_0 - B_1 x - B_2 x^2)\cos x \\
&\qquad + (A_1 + 2A_2 x + 2B_2)\cos x \\
&\qquad - (A_0 + A_1 x + A_2 x^2 + B_1 + 2B_2 x)\sin x \\
&= \big((-A_0 + 2A_2 - 2B_1) - (A_1 + 4B_2)x - A_2 x^2\big)\sin x \\
&\qquad + \big((2A_1 - B_0 + 2B_2) + (4A_2 - B_1)x - B_2 x^2\big)\cos x
\end{aligned}$$

and substitute into the original problem.

$$
\begin{aligned}
(12 + 20x + 10x^2)\cos x + 8x\sin x \\
&= y_p'' + 3y_p' + 2y_p \\
&= \big((-A_0 + 2A_2 - 2B_1) - (A_1 + 4B_2)x - A_2x^2\big)\sin x \\
&\qquad + \big((2A_1 - B_0 + 2B_2) + (4A_2 - B_1)x - B_2x^2\big)\cos x \\
&\qquad + 3\big((A_1 - B_0) + (2A_2 - B_1)x - B_2x^2\big)\sin x \\
&\qquad + 3\big((A_0 + B_1) + (A_1 + 2B_2)x + A_2x^2\big)\cos x \\
&\qquad + 2(A_0 + A_1x + A_2x^2)\sin x \\
&\qquad + 2(B_0 + B_1x + B_2x^2)\cos x \\
&= \big((A_0 + 3A_1 + 2A_2 - 3B_0 - 2B_1) \\
&\qquad\qquad + (A_1 + 6A_2 - 3B_1 - 4B_2)x \\
&\qquad\qquad + (A_2 - 3B_2)x^2\big)\sin x \\
&\qquad + \big((3A_0 + 2A_1 + B_0 + 3B_1 + 2B_2) \\
&\qquad\qquad + (3A_1 + 4A_2 + B_1 + 6B_2)x \\
&\qquad\qquad + (3A_2 + B_2)x^2\big)\cos x
\end{aligned}
$$

Now equating the coefficients of like terms, we obtain a system of six equations and six unknowns

$$
\begin{aligned}
A_0 + 3A_1 + 2A_2 - 3B_0 - 2B_1 &= 0 && c\sin x \\
3A_0 + 2A_1 + B_0 + 3B_1 + 2B_2 &= 12 && c\cos x \\
A_1 + 6A_2 - 3B_1 - 4B_2 &= 8 && x\sin x \\
3A_1 + 4A_2 + B_1 + 6B_2 &= 20 && x\cos x \\
A_2 - 3B_2 &= 0 && x^2\sin x \\
3A_2 + B_2 &= 10 && x^2\cos x
\end{aligned}
$$

The x^2 terms give $A_2 = 3, B_2 = 1$.
Substituting this into the x terms yields

$$A_1 - 3B_1 = -6$$
$$3A_1 + B_1 = 2,$$

so $A_1 = 0$ and $B_1 = 2$.
Substituting into the constant terms yields

$$A_0 - 3B_0 = -2$$
$$3A_0 + B_0 = 4,$$

so $A_0 = 1$ and $B_0 = 1$. Thus, a particular solution is

$$y_p = (1 + 3x^2)\sin x + (1 + 2x + x^2)\cos x.$$

§5.5 Homework Assignment:

Read: 225-232
Exercises: 1-4,13-16,31-32

5.6. **Reduction of Order.** We already discussed Reduction of Order briefly in §5.2 when we did equations with constant coefficients, where the characteristic polynomial has repeated real roots. Recall that this is a method for reducing a second-order equation to a first-order equation, hence the name; and recall that it involves using a solution y_1 of the complementary equation to find a solution of the nonhomogeneous equation.

Earlier, we used Reduction of Order to solve homogeneous equations. Now we use it to solve the more general equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x).$$

Suppose we have a nontrivial solution y_1 of the equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0.$$

To find a second solution, look for solutions of the form $y = uy_1$. Then

$$\begin{aligned} y' &= u'y_1 + uy_1' \\ y'' &= u''y_1 + 2u'y_1' + uy_1'' \end{aligned}$$

Substituting back and collecting terms gives

$$P_0y_1u'' + (2P_0y_1' + P_1y_1)u' + (P_0y_1'' + P_1y_1 + P_2y_1)u = F.$$

Since y_1 is a solution of the complementary equation, the coefficient of u here is just 0, and it reduces to

$$P_0y_1u'' + (2P_0y_1' + P_1y_1)u' = F.$$

This is actually a first order equation for the function u'!
So we have reduced a second-order equation to a first-order equation. Thus, it can be solved as a first order linear equation or a separable equation, using the methods of Chapter 2. Once you have u', you can get u by integrating, then multiply by y_1 to get your particular solution.

Handy tip: note that y_1'' only appears in the coefficient of u, upon substitution into the original nonhomogeneous equation. Since this coefficient drops out in the next step, it is not actually necessary to compute y_1''. This can save a substantial amount of time.

Example 5.6.1. Solve the differential equation, given the solution y_1 of the complementary equation.

$$y'' + 4xy' + (4x^2 + 2)y = 8e^{-x(x+2)}, \quad y_1 = e^{-x^2}.$$

Put

$$\begin{aligned} y &= uy_1 && = ue^{-x^2} \\ y' &= u'y_1 + uy_1' && = u'e^{-x^2} - 2xue^{-x^2} \\ y'' &= u''y_1 + 2u'y_1' + uy_1'' && = u''e^{-x^2} - 4xu'e^{-x^2} + uy_1'' \end{aligned}$$

into the original equation and obtain

$$\begin{aligned} 8e^{-x(x+2)} &= (u''e^{-x^2} - 4xu'e^{-x^2} + uy_1'') \\ &\qquad + 4x(u'e^{-x^2} - 2xue^{-x^2}) + (4x^2+2)ue^{-x^2} \\ &= u''e^{-x^2} + (-4xe^{-x^2} + 4xe^{-x^2})u' + 0 \end{aligned}$$

(All u terms vanish because y_1 is a solution, hence the 0.)

$$\begin{aligned} &= u''e^{-x^2} \\ 8e^{-x^2-2x} &= u''e^{-x^2} \\ 8e^{-2x} &= u'' \end{aligned}$$

Now integrate:

$$\begin{aligned} u' &= \int 8e^{-2x}dx = -4e^{-2x} + c_1 \\ u &= \int (-4e^{-2x} + c_1)dx = 2e^{-2x} + c_1x + c_2 \end{aligned}$$

Thus,

$$y = uy_1 = (2e^{-2x} + c_1 x + c_2)e^{-x^2}.$$

Example 5.6.2. Solve the differential equation, given the solution y_1 of the complementary equation.

$$4xy'' + 2y' + y = 0, \quad y_1 = \sin\sqrt{x}.$$

Put

$$\begin{aligned} y &= uy_1 & &= u\sin\sqrt{x} \\ y' &= u'y_1 + uy_1' & &= u'\sin\sqrt{x} + \tfrac{u}{2\sqrt{x}}\cos\sqrt{x} \\ y'' &= u''y_1 + 2u'y_1' + uy_1'' & &= u''\sin\sqrt{x} + \tfrac{u'}{\sqrt{x}}\cos\sqrt{x} + uy_1'' \end{aligned}$$

into the original equation and obtain

$$\begin{aligned} 0 &= 4x(u''\sin\sqrt{x} + \tfrac{u'}{\sqrt{x}}\cos\sqrt{x} + uy_1'') \\ &\qquad + 2(u'\sin\sqrt{x} + \tfrac{u}{2\sqrt{x}}\cos\sqrt{x}) + u\sin\sqrt{x} \\ &= u''4x\sin\sqrt{x} + (4\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x})u' + 0 \end{aligned}$$

Now separate the variables:

$$\begin{aligned} u''4x\sin\sqrt{x} &= -(4\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x})u' \\ &= \left(-\tfrac{1}{\sqrt{x}}\cot\sqrt{x} + \tfrac{1}{2x}\right)u' \\ \tfrac{d(u')}{u'} &= \left(-\tfrac{1}{\sqrt{x}}\cot\sqrt{x} + \tfrac{1}{2x}\right)dx \end{aligned}$$

Now integrate:

$$\log|u'| = -\int \tfrac{1}{\sqrt{x}}\cot\sqrt{x}dx + \tfrac{1}{2}\int \tfrac{1}{x}dx$$

$$= -2\log\left|\sin\sqrt{x}\right| - \tfrac{1}{2}\log|x| + c_1 \qquad (\textstyle\int \cot u = \log|\sin u|,\ u = \sqrt{x}, 2du = x^{-1/2}dx$$

$$u' = e^{-2\log|\sin\sqrt{x}| - \frac{1}{2}\log|x| + c_1}$$

$$= c_1 e^{-2\log|\sin\sqrt{x}|}e^{-\frac{1}{2}\log|x|}$$

$$= c_1(\sin\sqrt{x})^{-2}x^{-\frac{1}{2}}$$

$$= \tfrac{c_1}{\sqrt{x}\sin^2\sqrt{x}}$$

$$u = c_1\int \tfrac{dx}{\sqrt{x}\sin^2\sqrt{x}}$$

$$= -2c_1\cot\sqrt{x} + c_2 \qquad (\textstyle\int \csc^2 u = -\cot u,\ u = \sqrt{x}, 2du = x^{-1/2}dx$$

Thus,

$$y = uy_1 = (-2c_1\cot\sqrt{x} + c_2)\sin\sqrt{x}$$
$$y = -2c_1\cos\sqrt{x} + c_2\sin\sqrt{x}.$$

§5.6 Homework Assignment:

Read: 235-239

Exercises: 2,4,6,8,18,21,23,24,36

5.7. **Variation of Parameters.** This is the third, and most powerful method for obtaining a particular solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x).$$

Variation of parameters requires that we begin with 2 linearly independent solutions of the complementary equation. If you have a fundamental set of solutions, then variation of parameters will likely be easier than reduction of order. If not, you may find it quicker to do reduction of order.

Example 5.7.1. Given that two solutions of the complementary equation are $y_1 = x, y_2 = x^3$, solve the DE

$$x^2y'' - 3xy' + 3y = 2x^4 \sin x. \tag{5.7.1}$$

We know that the general solution of the DE is

$$y = c_1x + c_2x^3.$$

The basic idea of variation of parameters is to replace these constants c_1, c_2 with functions u_1, u_2 and then to determine these functions so that the resulting expression

$$y = u_1(x)x + u_2(x)x^3$$

is a solution to the nonhomogeneous equation.

To determine u_1 and u_2, we need to substitute this last equation into the original nonhomogeneous equation. This will give a single equation involving some combination of u_1, u_2 and their first two derivatives. Since there will be only one equation, and two unknown coefficients, there will always be many possible choices of u_1, u_2 which meet our needs.

Alternatively, we may impose another condition of our choosing to determine u_1, u_2. There is a way to choose the second condition which will make the computations much simpler.

First, we compute the derivative of $y = u_1 x + u_2 x^3$ as

$$y' = u_1 + 3u_2 x^2 + u_1' x + u_2' x^3.$$

Now we impose the condition that the last two terms be 0 so that

$$y' = u_1 + 3u_2 x^2.$$

This is the condition that will simplify the computation, although we do not prove this fact in this course. Clearly, it at least has the effect of simplifying y'.

Then we differentiate again and get

$$y'' = u_1' + 3u_2' x^2 + 6u_2 x.$$

Now substitute this back into the original nonhomogeneous equation:

$$\begin{aligned} 2x^4 \sin x &= x^2 y'' - 3xy' + 3y \\ &= u_1' x^2 + 3u_2' x^4 + 6u_2 x^3 - 3xu_1 - 9u_2 x^3 + 3u_1 x + 3u_2 x^3 \\ &= u_1' x^2 + 3u_2' x^4 \\ 2x^2 \sin x &= u_1' + 3u_2' x^2 \end{aligned}$$

So to recover u_1, u_2, it turns out that we first need to solve the linear system

$$\begin{aligned} u_1' x + u_2' x^3 &= 0 \\ u_1' + 3u_2' x^2 &= 2x^2 \sin x \end{aligned}$$

for the variables u_1', u_2'. Note that the second equation is actually

$$u_1' y_1' + u_2' y_2' = \frac{2x^4 \sin x}{x^2} = \frac{F}{P_0}.$$

To solve the system, subtract $\frac{1}{x}$ times the first from the second:

$$\begin{aligned} 2u_2' x^2 &= 2x^2 \sin x \\ u_2' &= \sin x \end{aligned}$$

substituting back in,

$$u_1'x + x^3 \sin x = 0$$
$$u_1' = -x^2 \sin x$$

Now we integrate, taking the constants of integration to be 0:

$$u_1 = -\int x^2 \sin x \, dx = x^2 \cos x - 2 \sin x - 2 \cos x$$
$$u_2 = \int \sin x \, dx = -\cos x$$

Thus the particular solution is

$$\begin{aligned} y_p &= u_1 x + u_2 x^3 \\ &= \left(x^2 \cos x - 2x \sin x - 2 \cos x\right) x + (-\cos x)\, x^3 \\ &= x^3 \cos x - 2x^2 \sin x - 2x \cos x - x^3 \cos x \\ y_p &= -2x^2 \sin x - 2x \cos x. \end{aligned}$$

Let's collect the steps we have just taken.

Method of Variation of Parameters:

1. Given a fundamental set of solutions $\{y_1, y_2\}$, take the particular solution to be
$$y_p = u_1 y_1 + u_2 y_2,$$
where u_1, u_2 are some functions yet to be discovered.

2. Write down the system
$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = \tfrac{F}{P_0}.$$

3. Solve the linear system for u_1', u_2' by any means available.

4. Integrate to obtain u_1, u_2, taking the constants of integration to be 0.

5. Substitute the resulting u_1, u_2 into $y_p = u_1y_1 + u_2y_2$ to obtain the particular solution y_p.

Example 5.7.2. Solve the DE, given the two solutions of the complementary equation:

$$x^2y'' + xy' - y = 2x^2 + 2, \quad y_1 = x, \quad y_2 = \tfrac{1}{x}.$$

1. $y_p = u_1x + \frac{u_2}{x}$.

2. Then

$$\begin{aligned} u_1'x + \tfrac{u_2'}{x} &= 0 \\ u_1' - \tfrac{u_2'}{x^2} &= \tfrac{2x^2+2}{x^2} = 2 + \tfrac{2}{x^2} \end{aligned}$$

So solve the second for $u_1' = \frac{u_2'}{x^2} + 2 + \frac{2}{x^2}$ and substitute:

$$\begin{aligned} \left(\tfrac{u_2'}{x^2} + 2 + \tfrac{2}{x^2}\right) x + \tfrac{u_2'}{x} &= 0 \\ 2\tfrac{u_2'}{x} &= -2x - \tfrac{2}{x} \\ u_2' &= -x^2 - 1 \end{aligned}$$

Use this to solve for u_1:

$$\begin{aligned} u_1'x - \tfrac{1}{x}(x^2+1) &= 0 \\ u_1'x &= x + \tfrac{1}{x} \end{aligned}$$

and obtain

$$\begin{aligned} u_1' &= 1 + \tfrac{1}{x^2} \\ u_2' &= -x^2 - 1 \end{aligned}$$

3. Now integrate:

$$u_1 = \int dx + \int \frac{dx}{x^2} = x - \frac{1}{x}$$

$$u_2 = -\int x^2 dx - \int dx = -\frac{x^3}{3} - x$$

4. So a particular solution is

$$y_p = \left(x - \frac{1}{x}\right) x + \left(-\frac{x^3}{3} - x\right) \frac{1}{x}$$

$$y_p = x^2 - 1 - \frac{x^2}{3} - 1$$

$$y_p = \frac{2x^2}{3} - 2$$

5.7.1. *Comparison of methods.*

The method of undetermined coefficients is good for constant coefficient equations with forcing functions which are combinations of the familiar function families: polynomials, exponentials, sines & cosines.

For equations with coefficients that are functions of x, or forcing functions other than those listed, the method of Undetermined Coefficients doesn't apply. If you have only one complementary solution, use Reduction of Order, or else form another solution (e.g., by using Abel's Formula) and use Variation of Parameters. If you have two solutions, you probably want to use Variation of Parameters.

§5.7 Homework Assignment:

Read: 242-249

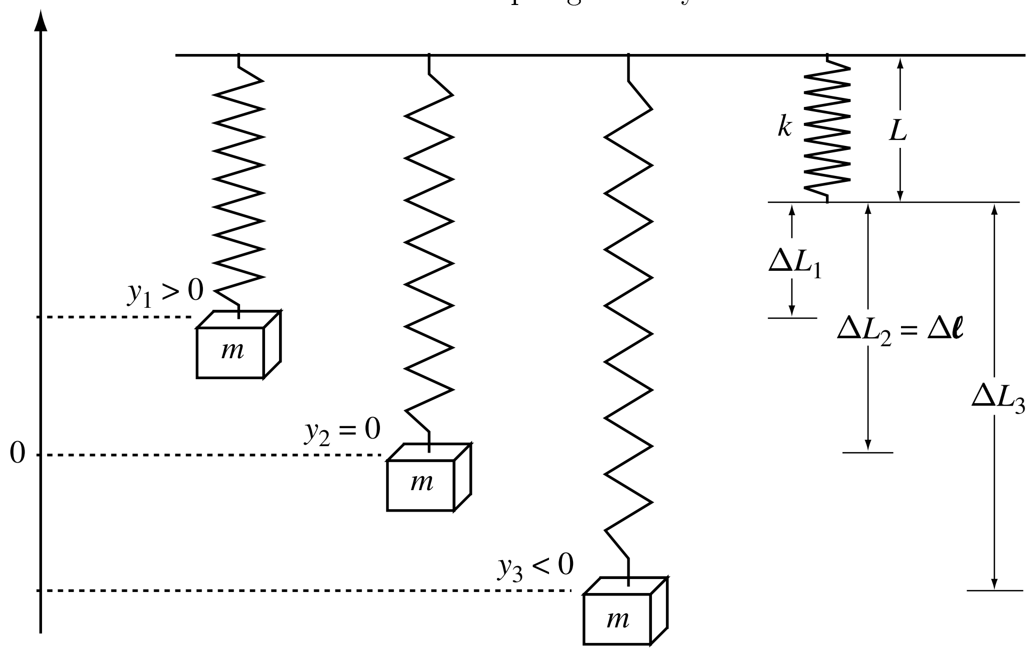
Exercises: 1,4,7,10-12,27,30-31

6. Applications of Second Order Equations

6.1. Spring Problems 1.

Our model for spring problems requires a bit of set-up and some definitions. We will be considering the motion of a mass m suspended from a spring with spring constant k like so:

Figure 6.1.1. Spring-mass system



The equilibrium position of the mass is where it rests when the forces on it sum to zero. Then $y(t)$ will be the displacement of the mass from the equilibrium (measured positive upwards), so that equilibrium corresponds to when $y(t) = y'(t) = y''(t) = 0$.

We consider the following forces on the system:

1. Gravity: $F_g = -mg$, a constant force.

2. The spring: $F_s = k\Delta L$, a force which depends on the distance ΔL that the spring is compressed or extended. The *natural length* L of a spring is its length with no mass attached. We

assume that all springs obey Hooke's Law: if the length of a spring is changed ΔL from its natural length, then it exerts a force $F_s = k\Delta L$ where k is the spring constant.

3. Damping: $F_d = -cy'$, a force which depends on the velocity of the object (e.g., friction, air resistance).

4. External forces: $F(t)$, independent of displacement or velocity (e.g., hitting the spring with your hand). We say the motion is *free* iff $F \equiv 0$ or *forced* iff $F \not\equiv 0$.

We have two equivalent expressions for the forces on the system:
(1) $F = ma = my''$.
(2) $F = F_g + F_d + F_s + F = -mg - cy' + F_s + F$

Equating these gives a second-order DE:

$$my'' = -mg - cy' + F_s + F.$$

Now what is F_s? We need to relate F_s to y. When we first attach the mass to the spring, it will stretch by an amount $\Delta\ell$ to assume its equilibrium position. At this point (ignoring external forces) the object will be at rest, so the forces must be balanced, i.e., $mg = k\Delta\ell$. If the mass is displaced, then the total change in the length of the spring is $\Delta L = \Delta\ell - y$, so Hooke's law gives

$$F_s = k\Delta L = k\Delta\ell - ky,$$

which we substitute back into the previous equation and obtain

$$my'' = -mg - cy' + k\Delta\ell - ky + F.$$

Since $mg = k\Delta\ell$ in equilibrium, this first and third term cancel, and the equation can be rewritten as

$$my'' + cy' + ky = F.$$

This is the *equation of motion* for the spring system. From now on, this is really all we need to consider.

6.1.1. *Undamped free oscillation: simple harmonic motion.*

In the absence of a damping term, the equation of motion is

$$my'' + ky = 0.$$

If we divide by m and set $\omega_0 = \sqrt{k/m}$, this becomes

$$y'' + \omega_0^2 y = 0.$$

We can solve this as a constant coefficient equation— the characteristic equation is

$$r^2 + \omega_0^2 = 0,$$

which has solutions

$$r = \frac{\pm\sqrt{-4\omega_0^2}}{2} = \frac{2\omega_0 i}{2} = \omega_0 i,$$

so that the general solution is (using Euler's formula)

$$y = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t.$$

So this is why we chose that funny definition

$$\omega_0 = \sqrt{k/m};$$

it is easy to go from

$$y'' + \omega_0^2 y = 0 \qquad \text{to} \qquad y = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t.$$

However, we can rewrite this in a more useful form converting to polar coordinates; let

$$R = \sqrt{c_1^2 + c_2^2}, \quad c_1 = R\cos\varphi, \text{ and } \quad c_2 = R\sin\varphi.$$

Now (R, φ) is (c_1, c_2) in polar coordinates.

Next, use the trig identity

$$\cos\omega_0 t\cos\varphi + \sin\omega_0 t\sin\varphi = \cos(\omega_0 t - \varphi)$$

to get

$$\begin{aligned} y &= c_1\cos\omega_0 t + c_2\sin\omega_0 t \\ &= R\cos\varphi\cos\omega_0 t + R\sin\varphi\sin\omega_0 t \\ &= R\cos(\omega_0 t - \varphi) \end{aligned}$$

Definition.
$R = \sqrt{c_1^2 + c_2^2}$ is the *amplitude* of the oscillation: this last equation shows that y attains its maximum and minimum values at $\pm R$.

$\omega_0 = \sqrt{\frac{k}{m}}$ is the (*natural*) *frequency* of the motion: it tells how many cycles the oscillation goes through every second.

φ is called the *phase angle*: it gives the initial offset from which the oscillation begins.

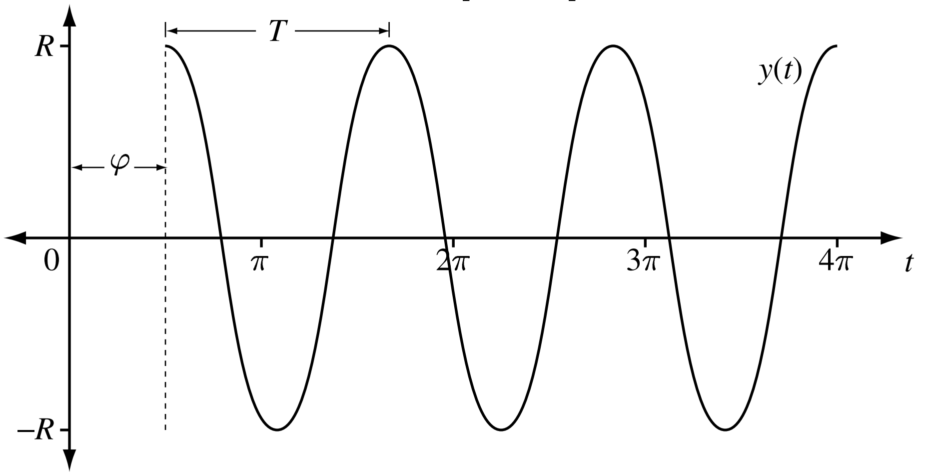
Thus, the equation

$$y = R\cos(\omega_0 t - \varphi)$$

is called the *amplitude-phase form* of the displacement. We may always take $-\pi \leq \varphi \leq \pi$.

$T = 2\pi/\omega_0$ is the *period*, and gives the time it takes for the mass to complete one full cycle of motion (e.g., from $y(t) = R$ to $y(t) = -R$ and back to $y(t) = R$).

Figure 6.1.2. Amplitude-phase form.



Example 6.1.1. A 10-kg mass stretches a spring 70 cm in equilibrium. Suppose that a 2-kg mass is attached to the spring, initially displaced 25 cm below equilibrium, and given an upward velocity of 2 m/s. Find its displacement for $t > 0$. Find the frequency, period, amplitude, and phase angle of the motion.

Since we have already derived the general solution, this will simply be a matter of plugging in to formulae, once we have the problem set up. Let's do this in mks units so $\Delta\ell = .7m$ for $m = 10$.

To find the spring constant, use the relation $mg = k\Delta\ell$ to get

$$k = \frac{mg}{\Delta\ell} = \frac{9.8 \cdot 10}{.7} = 140.$$

Put this into the equation of motion:

$$\begin{aligned} my'' + ky &= 0 \\ 2y'' + 140y &= 0 \\ y'' + 70y &= 0 \end{aligned}$$

Giving the general solution

$$y = c_1 \cos \sqrt{70}t + c_2 \sin \sqrt{70}t.$$

To solve the IVP, plug in the initial displacement

$$y(0) = -0.25 = c_1 \cos 0 + c_2 \sin 0 \quad \Longrightarrow \quad c_1 = -0.25.$$

and the initial velocity

$$\begin{aligned} y' &= -c_1\sqrt{70} \sin \sqrt{70}t + c_2\sqrt{70} \cos \sqrt{70}t \\ y'(0) = 2 &= -c_1\sqrt{70} \sin 0 + c_2\sqrt{70} \cos 0 = c_2\sqrt{70} \\ c_2 &= \tfrac{2}{\sqrt{70}} \end{aligned}$$

to get the solution

$$y = -\tfrac{1}{4} \cos \sqrt{70}t + \tfrac{2}{\sqrt{70}} \sin \sqrt{70}t.$$

Then we have

$$\begin{aligned} \text{frequency: } & \omega_0 = \sqrt{70} \\ \text{period: } & T = \tfrac{2\pi}{\omega_0} \\ \text{amplitude: } & R = \sqrt{c_1^2 + c_2^2} = \sqrt{\tfrac{1}{16} + \tfrac{4}{70}} = \sqrt{\tfrac{67}{560}} = \tfrac{1}{4}\sqrt{\tfrac{67}{35}} \\ \text{phase angle: } & \cos\varphi = \tfrac{c_1}{R} = -\tfrac{1}{4} \cdot 4\sqrt{\tfrac{35}{67}} = -\sqrt{\tfrac{35}{67}} \\ & \sin\varphi = \tfrac{c_2}{R} = \tfrac{2}{\sqrt{70}} \cdot 4\sqrt{\tfrac{35}{67}} = \tfrac{4}{\sqrt{67}} \\ & \varphi = \arccos\left(-\sqrt{\tfrac{35}{67}}\right) \qquad \text{(the positive one)} \end{aligned}$$

6.1.2. *Undamped forced oscillation.*

We didn't even have to worry about solving a differential equation for that last example! We only had to plug the numbers into the formula we derived. This simplicity is due to the fact that we were only working with a homogeneous equation. Things become more complicated when there is an external force on the system; when the forcing function is not identically 0, we have to solve a nonhomogeneous equation using the techniques of chapter 5.

Consider a system which is subject to an external force

$$F(t) = F_0 \cos \omega t,$$

where F_0 is some constant. In this case, the equation of motion is

$$my'' + ky = F_0 \cos \omega t,$$

which we rewrite as

$$y'' + \omega_0^2 y = \frac{F_0}{m} \cos \omega t,$$

where $w_0 = \sqrt{\frac{k}{m}}$ as before. Let's derive a solution to this differential equation, taking initial conditions

$$y(0) = 0, \quad y'(0) = 0.$$

First, notice that $\cos \omega_0 t$ is a solution of the complementary equation

$$y'' + \omega_0^2 y = 0.$$

If $\omega = \omega_0$, interesting things will occur, like we talked about in §5.4. First, however, let's consider the case when $\omega \neq \omega_0$. In this case, $\cos \omega t$ is NOT a complementary solution, and we know from §5.4 that there will be a particular solution of the form

$$y_p = A \cos \omega t + B \sin \omega t.$$

Since

$$y_p'' = -\omega^2(A\cos\omega t + B\sin\omega t),$$

y_p will be a solution to the nonhomogeneous equation iff

$$(\omega_0^2 - \omega^2)(A\cos\omega t + B\sin\omega t) = \tfrac{F_0}{m}\cos\omega t.$$

Since there's no sin term on the right, $B = 0$, and

$$\begin{aligned}(\omega_0^2 - \omega^2)(A\cos\omega t) &= \tfrac{F_0}{m}\cos\omega t\\ A\cos\omega t &= \tfrac{F_0}{m(\omega_0^2-\omega^2)}\cos\omega t\\ A &= \tfrac{F_0}{m(\omega_0^2-\omega^2)}\end{aligned}$$

So we get $y_p = \frac{F_0}{m(\omega_0^2-\omega^2)}\cos\omega t$. Then the general solution is

$$y = \tfrac{F_0}{m(\omega_0^2-\omega^2)}\cos\omega t + c_1\cos\omega_0 t + c_2\sin\omega_0 t.$$

and the initial conditions give

$$y(0) = 0 = \tfrac{F_0}{m(\omega_0^2-\omega^2)} + c_1 \quad\Longrightarrow\quad c_1 = -\tfrac{F_0}{m(\omega_0^2-\omega^2)},$$

and

$$\begin{aligned}y' &= \tfrac{-\omega F_0}{m(\omega_0^2-\omega^2)}\sin\omega t - c_1\omega_0\sin\omega_0 t + c_2\omega_0\cos\omega_0 t\\ y'(0) = 0 &= \tfrac{-\omega F_0}{m(\omega_0^2-\omega^2)}\cdot 0 - c_1\omega_0\cdot 0 + c_2\omega_0 \quad\Longrightarrow\quad c_2 = 0\end{aligned}$$

So the solution to the IVP is

$$\begin{aligned}y &= \tfrac{F_0}{m(\omega_0^2-\omega^2)}\cos\omega t - \tfrac{F_0}{m(\omega_0^2-\omega^2)}\cos\omega_0 t\\ y &= \tfrac{F_0}{m(\omega_0^2-\omega^2)}(\cos\omega t - \cos\omega_0 t)\end{aligned}$$

Again, we will be able to gain better understanding if we rewrite this equation in another form. Using trig identities,

$$\cos\omega t - \cos\omega_0 t = 2\sin\tfrac{(\omega_0-\omega)t}{2}\sin\tfrac{(\omega_0+\omega)t}{2},$$

and we can rewrite y as

$$\begin{aligned} y &= \tfrac{2F_0}{m(\omega_0^2-\omega^2)}\sin\tfrac{(\omega_0-\omega)t}{2}\sin\tfrac{(\omega_0+\omega)t}{2} \\ y &= R(t)\sin\tfrac{(\omega_0+\omega)t}{2} \end{aligned}$$

where

$$R(t) = \tfrac{2F_0}{m(\omega_0^2-\omega^2)}\sin\tfrac{(\omega_0-\omega)t}{2}$$

is the variable amplitude (as it varies sinusoidally with time). The oscillations of y between successive zeroes of $R(t)$ is called a *beat*, and a period of $R(t)$ is called the *beat frequency*. Note that the solution is bounded for all time:

$$\begin{aligned} |y(t)| &= \left|\tfrac{2F_0}{m(\omega_0^2-\omega^2)}\sin\tfrac{(\omega_0-\omega)t}{2}\sin\tfrac{(\omega_0+\omega)t}{2}\right| \\ &\le \tfrac{|2F_0|}{|m(\omega_0^2-\omega^2)|}\cdot 1\cdot 1 \\ &= \tfrac{2|F_0|}{m|\omega_0^2-\omega^2|} \end{aligned}$$

If $\omega+\omega_0$ is quite large in comparison to $\omega-\omega_0$, there will be many oscillations in each beat, and y will get very close to this maximum.

Now consider the case when $\omega=\omega_0$. Since $\cos\omega t$ is a complementary solution, the form for y_p is

$$y_p = t(A\cos\omega t + B\sin\omega t).$$

This time, we need to use the product rule to compute the derivative and it is not so simple:

$$y_p' = (A\cos\omega t + B\sin\omega t) + t\omega(-A\sin\omega t + B\cos\omega t)$$
$$y_p'' = 2\omega(-A\sin\omega t + B\cos\omega t) - \omega^2 t(A\cos\omega t + B\sin\omega t)$$

So y_p will be a particular solution iff

$$2\omega(-A\sin\omega t + B\cos\omega t) = \tfrac{F_0}{m}\cos\omega t,$$

since the ω^2 terms of y_p and y_p'' cancel. This gives

$$A = 0 \qquad \text{and} \qquad B = \tfrac{F_0}{2m\omega}.$$

Thus, $y_p = \frac{F_0 t}{2m\omega}\sin\omega t$ and we get the general solution

$$y = \tfrac{F_0 t}{2m\omega}\sin\omega t + c_1\cos\omega t + c_2\sin\omega t.$$

Suppose we were to consider this system for the initial conditions $y(0) = 0, y'(0) = 0$.

$$y(0) = \tfrac{F_0\cdot 0}{2m\omega}\cdot 0 + c_1 + c_2\cdot 0 \quad \Longrightarrow \quad c_1 = 0$$

and

$$y' = \tfrac{F_0}{2m\omega}\sin\omega t + \tfrac{F_0 t}{2m}\cos\omega t - c_1\omega\sin\omega t + c_2\omega\cos\omega t$$
$$y'(0) = 0 = \tfrac{F_0}{2m\omega}\cdot 0 + \tfrac{F_0\cdot 0}{2m} - c_1\omega\cdot 0 + c_2 \quad \Longrightarrow \quad c_2 = 0$$

This gives the solution as

$$y(t) = \tfrac{F_0 t}{2m\omega}\sin\omega t,$$

a function which oscillates between asymptotes that grow linearly with t; y varies between $-\frac{F_0 t}{2m\omega} < y < \frac{F_0 t}{2m\omega}$. In fact, since y_p is unbounded, $y(t)$ will be unbounded regardless of the initial conditions. Thus, resonance guarantees that the spring must eventually break or fail to obey Hooke's Law (when it reaches the point of plasticity).

Example 6.1.2. A mass of 1 kg is attached to a spring with a constant $k = 4N/m$. An external force $F(t) = -\cos\omega t - 2\sin\omega t$N is applied to the mass. Find the displacement y for $t > 0$ if ω equals the natural frequency of the system. Assume that the mass is initially displaced 3m above equilibrium and given an upward initial velocity of 450 cm/s.

Since the natural frequency of the system is

$$\omega_0 = \sqrt{\tfrac{k}{m}} = \sqrt{4/1} = 2,$$

we have $\omega = 2$. Thus the equation we need to solve is

$$y'' + 4y = -\cos 2t - 2\sin 2t.$$

The characteristic equation

$$r^2 + 4 = 0$$

has roots

$$r = \pm\sqrt{-16}/2 = \pm 2i,$$

so the complementary solution is

$$y = c_1 \cos 2t + c_2 \sin 2t.$$

We use the method of undet'd coeffs:

$$\begin{aligned}
y_p &= t(A\cos 2t + B\sin 2t) \\
y_p' &= A\cos 2t + B\sin 2t + t(-2A\sin 2t + 2B\cos 2t) \\
&= (A + 2Bt)\cos 2t + (B - 2At)\sin 2t \\
y_p'' &= 2B\cos 2t - 2(A + 2Bt)\sin 2t - 2A\sin 2t + 2(B - 2At)\cos 2t \\
&= 4(B - At)\cos 2t - 4(A + Bt)\sin 2t
\end{aligned}$$

So that

$$
\begin{aligned}
-\cos 2t - 2\sin 2t &= y_p'' + 4y_p \\
&= 4(B - At)\cos 2t - 4(A + Bt)\sin 2t \\
&\qquad + 4t(A\cos 2t + B\sin 2t) \\
&= 4B\cos 2t - 4At\cos 2t - 4A\sin 2t - 4Bt\sin 2t \\
&\qquad + 4At\cos 2t + 4Bt\sin 2t \\
&= 4B\cos 2t - 4A\sin 2t \\
4B &= -1 \qquad \text{and} \qquad -4A = -2 \\
B &= -\tfrac{1}{4} \qquad \text{and} \qquad A = \tfrac{1}{2} \\
y_p &= \tfrac{t}{2}\cos 2t - \tfrac{t}{4}\sin 2t \\
y &= \tfrac{t}{2}\cos 2t - \tfrac{t}{4}\sin 2t + c_1\cos 2t + c_2\sin 2t \\
y(0) = 3 &= c_1 \\
y' &= \tfrac{1}{2}\cos 2t - t\sin 2t - \tfrac{1}{4}\sin 2t \\
&\qquad - \tfrac{t}{2}\cos 2t - 2c_1\sin 2t + 2c_2\cos 2t \\
y'(0) = 4.5 &= \tfrac{1}{2} + 2c_2 \\
4 &= 2c_2 \\
2 &= c_2 \\
y &= \tfrac{t}{2}\cos 2t - \tfrac{t}{4}\sin 2t + 3\cos 2t + 2\sin 2t
\end{aligned}
$$

For problems 19–21, use the fact that since the period of the motion is $T = 2\pi\sqrt{\frac{m}{k}}$, if you have two systems, the periods are related by

$$\frac{T_1}{T_2} = \sqrt{\frac{m_2 k_1}{m_1 k_2}}.$$

§6.1 Homework Assignment:

Read: 252–262

Exercises: 3,6,11,14,16–21

6.2. **Spring Problems 2.**

6.2.1. *Damped free oscillation.*

The equation of motion for free oscillation with damping is

$$my'' + cy' + ky = 0.$$

The characteristic equation is

$$mr^2 + cr + k = 0,$$

which gives

$$r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}.$$

Since m and k are always positive, the system depends entirely on the size of the damping factor c, which determines the sign of the discriminant $c^2 - 4mk$.

Intuitively, you might expect a system with very little damping to be very oscillatory, and a system with extremely high damping to have very little. Think of a pendulum swinging about a hinge which may be tightened so as to provide more friction. If the hinge is loose, the pendulum swings for a very long time. If the hinge is cranked down very hard, the pendulum will swing down extremely slowly and may not even oscillate at all. We can confirm this intuition by studying the three cases for $c^2 - 4mk$.

case (i) $c^2 - 4mk < 0$. In this case, the roots are the complex conjugates

$$r = -\frac{c}{2m} \pm \frac{\sqrt{4mk - c^2}}{2m} i$$

and we say that the system is *underdamped*.

The frequency of this system will be given by the imaginary exponent when we plug this into Euler's Formula, so we define

$$\omega_1 := \frac{\sqrt{4mk-c^2}}{2m}$$

that we may write the general solution as

$$y = c_1 e^{-ct/2m}(c_1 \cos \omega_1 t + c_2 \sin \omega_1 t).$$

Again, it is more useful to write this in the form

$$y = Re^{-ct/2m}\cos(\omega_1 - \varphi),$$

where

$$R = \sqrt{c_1^2 + c_2^2}, c_1 = R\cos\varphi, \text{ and } c_2 = R\sin\varphi.$$

Definition. $Re^{-ct/2m}$ is the *time-varying amplitude* of the motion, as the oscillation is bounded

$$|y(t)| \leq Re^{-ct/2m}, t > 0.$$

$\omega_1 = \sqrt{4mk - c^2}/2m$ is called the *frequency*.

$T = 2\pi/\omega_0$ is the period of the cosine term, and is hence called the *quasi-period*. ("quasi" because the system does not return to the exact same state after the period has elapsed, it only returns to the analogous state for a lower energy level.) The quasi-period is the time that elapses between successive minima or maxima.

Note that $\lim_{t\to\infty} Re^{-ct/2m} = 0$, so the oscillation eventually tends to 0.

case (ii) $c^2 - 4mk > 0$. In this case, the roots are distinct real numbers $r_1 < r_2 < 0$, given by

$$r = -\frac{c}{2m} \pm \frac{\sqrt{c^2-4mk}}{2m}$$

and we say that the system is *overdamped.*

The general solution is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

We have $\lim_{t\to\infty} y(t) = 0$ here also, but note that we cannot have $y(t) = 0$ for more than one value of t unless $c_1 = c_2 = 0$

case (iii) $c^2 - 4mk = 0$. In this case, there is a repeated real root $r_1 = r_2 = -c/2m < 0$ and we say that the system is *critically damped.*

The behavior of this type of system is identical to the overdamped case. However, it is the limiting case of overdamping behavior as any slight decrease in the damping factor would result in oscillations.

6.2.2. *Damped forced oscillation.*

The equation of motion for forced oscillation with damping is

$$my'' + cy' + ky = F(t).$$

We consider the case when

$$my'' + cy' + ky = F_0 \cos \omega t.$$

The combination of damping and a forcing function has the effect of driving the system toward mimicking the external force. That is, the initial conditions become insignificant for large values of t, when the displacement is closely approximated by

$$y(t) = R\cos(\omega t - \varphi),$$

where the amplitude R depends on $m, c, k,$ F_0, and ω. We want to find out what value of ω will produce the largest amplitude R, assuming everything else remains constant, and what this largest R will be. To this end, we solve the differential equation

$$my'' + cy' + ky = F_0 \cos \omega t.$$

Since $\cos \omega t$ does not satisfy the complementary equation, we can obtain a particular solution in the form

$$\begin{aligned} y_p &= A \cos \omega t + B \sin \omega t \\ y_p' &= -A\omega \sin \omega t + B\omega \cos \omega t \\ y_p'' &= -A\omega^2 \cos \omega t - B\omega^2 \sin \omega t \end{aligned}$$

Then

$$\begin{aligned} F_0 \cos \omega t &= my_p'' + cy_p' + ky_p \\ &= m(-A\omega^2 \cos \omega t - B\omega^2 \sin \omega t) \\ &\qquad + c(-A\omega \sin \omega t + B\omega \cos \omega t) \\ &\qquad + k(A \cos \omega t + B \sin \omega t) \\ &= ((k - m\omega^2)A + c\omega B) \cos \omega t \\ &\qquad + (-c\omega A + (k - m\omega^2)B) \sin \omega t \end{aligned}$$

So we solve the system

$$\begin{aligned} (k - m\omega^2)A + c\omega B &= F_0 \\ -c\omega A + (k - m\omega^2)B &= 0 \end{aligned}$$

to obtain

$$y_p = \frac{F_0}{(k - m\omega^2)^2 + c^2\omega^2} \left((k - m\omega^2) \cos \omega t + c\omega \sin \omega t \right),$$

which can be written in phase-amplitude form as

$$y_p = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \cos(\omega t - \varphi),$$

where

$$\cos\varphi = \frac{k - m\omega^2}{(k - m\omega^2)^2 + c^2\omega^2} \quad \text{and} \quad \sin\varphi = \frac{c\omega}{(k - m\omega^2)^2 + c^2\omega^2}.$$

Finally, since

$$k - m\omega^2 = m\left(\tfrac{k}{m} - \omega^2\right) = m(\omega_0^2 - \omega^2),$$

(where $\omega_0 = \sqrt{k/m}$ is the natural frequency of the undamped, unforced system), we can rewrite the phase-amplitude form as

$$y_p = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2}} \cos(\omega t - \varphi).$$

In the case when $\omega = \omega_0$, this reduces to

$$y_p = \frac{F_0}{c\omega} \cos(\omega t - \varphi).$$

Compare this to the expressions we found in §6.1:

$$\begin{aligned} y_p &= \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t - \varphi) & \qquad (\omega \neq \omega_0) \\ y_p &= \frac{F_0 t}{2m\omega_0} \sin(\omega_0 t) & \qquad (\omega = \omega_0) \end{aligned}$$

In particular, note that the presence of a damping term (the c) algebraically ensures that our general formula for y_p works in all cases; we do not have an extra t arising from resonance when $\omega = \omega_0$!

The solution of the IVP

$$my'' + cy' + ky = F_0 \cos\omega t, \quad y(0) = y_0, \quad y'(0) = v_0$$

is of the form $y = y_p + y_c$, where y_c is the complementary solution and will be of the form

$$\begin{aligned} y_c &= e^{-ct/2m}(c_1 \cos \omega_1 t + c_2 \sin \omega_1 t), \\ y_c &= e^{-ct/2m}(c_1 + c_2 t), \text{ or} \\ y_c &= c_1 e^{-r_1 t} + c_2 e^{-r_2 t} \qquad (r_1, r_2 > 0). \end{aligned}$$

Note that for each of these,

$$\lim_{t\to\infty} y_c(t) = 0.$$

For this reason, the complementary solution y_c is sometimes also called the *transient component* of the solution. The behavior of y for large t is given by y_p, which we call the *steady-state component* of y. Thus, for large t, the motion of the system is almost identical to

$$y_p = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2}} \cos(\omega t - \varphi).$$

Note that the amplitude is bounded for all time:

$$R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2}} \qquad \text{is finite, regardless of } t > 0.$$

Let's find the value $\omega_{\max}$of ω for which R is maximized. This is the value of ω for which the denominator

$$\begin{aligned} \varrho(\omega) &= m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2 \\ &= m^2(\omega_0^4 + \omega^4) + (c^2 - 2m^2\omega_0^2)\omega^2 \end{aligned}$$

attains its minimum value. If

$$c \geq \sqrt{2m^2\omega_0^2} = \sqrt{2mk},$$

then this last term will be positive and the entire expression for $\varrho(\omega)$ will increase as ω^2 does. Thus, $\varrho(\omega)$ will be minimized for $\omega_{\max} = 0$.

So what happens when $\omega = \omega_{\max} = 0$? From the earlier equation

$$\cos\varphi = \frac{k - m\omega^2}{(k - m\omega^2)^2 + c^2\omega^2},$$

we see that $\omega = 0$ implies $\cos\varphi = 1$, so $\varphi = 0$.

To find the maximal amplitude, we evaluate y_p for this case:

$$R_{\max} = \frac{F_0}{\sqrt{m^2\omega_0^4}}\cos(0t - 0) = \frac{F_0}{m^2\omega_0^2} = \frac{F_0}{k}.$$

This is consistent with Hooke's law: if we subject a spring to a constant force F_0, displacement should approach a constant y_p such that $ky_p = F_0$.

All this is for the case when the damping factor c satisfies

$$c \geq \sqrt{2mk},$$

but what about when

$$c < \sqrt{2mk}?$$

In this case, the last term of

$$\varrho(\omega) = m^2(\omega_0^4 + \omega^4) + (c^2 - 2m^2\omega_0^2)\omega^2$$

is negative and we need to use the first derivative test to locate the minimum of $\varrho(\omega)$:

$$\begin{aligned}\varrho'(\omega) &= 4m^2\omega^3 + 2(c^2 - 2m^2\omega_0^2)\omega \\ &= 2\omega\left(2m^2\omega^2 + c^2 - 2m^2\omega_0^2\right),\end{aligned}$$

This derivative is 0 when $\omega = 0$ or $2m^2\omega^2 + c^2 - 2m^2\omega_0^2 = 0$. We want to minimize $\varrho(\omega)$, so choose the latter (by the second derivative test).

$$\begin{aligned} 2m^2\omega^2 + c^2 - 2m^2\omega_0^2 &= 0 \\ 2m^2\omega^2 &= 2m^2\omega_0^2 - c^2 \\ \omega^2 &= \omega_0^2 - \tfrac{c^2}{2m^2} \\ \omega &= \sqrt{\omega_0^2 - \tfrac{c^2}{2m^2}} \\ \omega_{\max} &= \sqrt{\tfrac{k}{m}\left(1 - \tfrac{c^2}{2km}\right)}. \end{aligned}$$

Then the maximal amplitude is

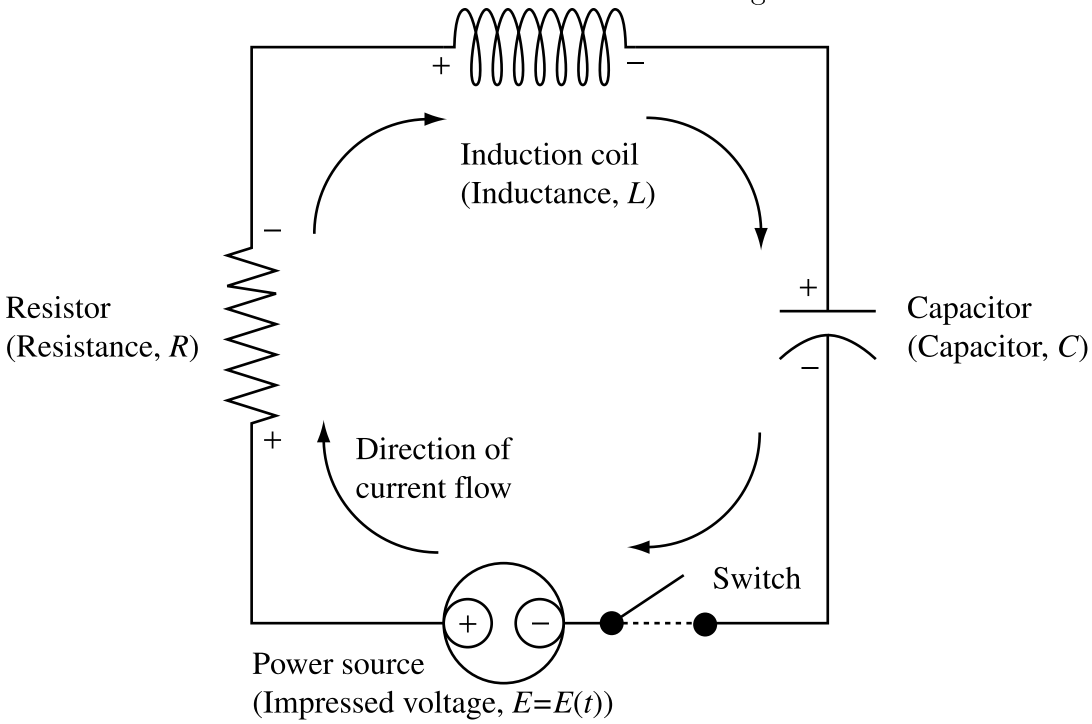
$$\begin{aligned} R_{\max} &= \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2}} \\ &= \frac{F_0}{\sqrt{m^2\left(\omega_0^2 - \tfrac{k}{m}\left(1 - \tfrac{c^2}{2km}\right)\right)^2 + \tfrac{c^2 k}{m}\left(1 - \tfrac{c^2}{2km}\right)}} \\ &= \frac{2mF_0}{c\sqrt{4mk - c^2}} \end{aligned}$$

§6.2 Homework Assignment:
Read: 263–271
Exercises: 8,15,18,20,25

6.3. **The RLC Circuit.**

Definition. A circuit with resistors, inductors, and capacitors is called an RLC circuit. It is the electrical analogue of a mass-spring system, as is evinced by the nearly identical relevant equation.

FIGURE 6.3.1. An RLC circuit diagram.



Differences in electrical potential cause the current to flow. The power source is essentially just responsible for creating a difference in electrical potential $E = E(t)$. We adopt the convention that $E(t) > 0$ if the potential at the positive terminal is greater than the potential at the negative terminal. E is the *impressed voltage*.

The current $I = I(t)$ is always the same at all points of the circuit. We say $I(t) > 0$ when current is flowing from positive to negative.

Differences in potential occur at each of the components: resistor, inductor, and capacitor. This difference is called the *voltage drop*.

The voltage drop across a resistor is given by

$$V_R = IR,$$

where I is current and R is a positive constant called the *resistance* of the resistor.

The voltage drop across the induction coil is given by

$$V_I = L\frac{dI}{dt} = LI',$$

where L is a positive constant called the *inductance* of the coil.

A capacitor stores charge $Q = Q(t)$, which is related to the current in the circuit by the equation

$$Q(t) = Q_0 + \int_0^t I(s)\,ds,$$

where $Q_0 = Q(t_0)$ is the initial charge on the capacitor. The voltage drop across the capacitor is given by

$$V_C = \frac{Q}{C},$$

where C is a positive constant called the *capacitance* of the capacitor.

According to Kirchhoff's Law, the sum of the voltages drops in a closed RLC circuit equals the impressed voltages. Thus,

$$LI' + RI + \frac{1}{C}Q = E(t).$$

Since the equation for Q gives $Q' = I$ (current is the rate at which charge is changing), so we can rewrite this as

$$LQ'' + RQ' + \frac{1}{C}Q = E(t).$$

Except for notation, this is identical to

$$my'' + cy' + ky = F(t).$$

In fact, even some physical aspects of the two systems are parallel: resistance=damping, impressed voltage=external force, etc.

As before, we say the circuit is in *free oscillation* if $E \equiv 0$. In this case, the characteristic polynomial has roots

$$r = \frac{-R \pm \sqrt{R^2 - 4L/C}}{2L},$$

and we have the same three cases as before:

1. The oscillation is *underdamped* for $R < \sqrt{4L/C}$.
   Most actual RLC circuits are underdamped. In this case, the general solution is
   $$\begin{aligned} Q(t) &= e^{-Rt/2L}(c_1 \cos \omega_1 t + c_2 \sin \omega_1 t) \\ &= Ae^{-Rt/2L} \cos(\omega_1 t - \varphi), \end{aligned}$$
   where
   $$\omega_1 = \tfrac{4L/C - R^2}{2L}, \quad A = \sqrt{c_1^2 + c_2^2},$$
   $$A \cos \varphi = c_1, \text{ and } \quad A \sin \varphi = c_2,$$
   as before. In the idealized case when $R = 0$, this reduces to
   $$Q(t) = A \cos\left(\tfrac{t}{\sqrt{LC}} - \varphi\right).$$
   (This situation doesn't actually occur in the real world, as there is always some slight resistance to the circuit.)

2. The oscillation is *overdamped* for $R > \sqrt{4L/C}$.
   In this case, the general solution is
   $$Q(t) = c_1 e^{-r_1 t} + c_2 e^{-r_2 t}, \quad \text{with } r_1, r_2 > 0.$$

3. The oscillation is *critically damped* for $R = \sqrt{4L/C}$.
   In this case, the general solution is
   $$Q(t) = c_1 e^{-rt} + c_2 t e^{-rt}, \quad \text{with } r = \tfrac{R}{2L}.$$

Example 6.3.1. Assuming $E \equiv 0$, find the current in the RLC circuit with $R = 2$ ohms, $L = 0.05$ henrys, and $C = 0.01$ farads, if $Q_0 = 2$ coulombs and $I_0 = -2$ amps.

We have to solve the IVP

$$\tfrac{1}{20}Q'' + 2Q' + 100Q = 0, \quad Q(0) = 2, \quad Q'(0) = -2.$$

Since the characteristic polynomial

$$p(r) = \tfrac{1}{20}r^2 + 2r + 100$$

has roots

$$r = \frac{-2 \pm \sqrt{4 - 20}}{1/10} = (-2 \pm 4i) \cdot 10 = -20 \pm 40i,$$

the general solution is

$$Q(t) = e^{-20t}(c_1 \cos 40t + c_2 \sin 40t).$$

To plug in the initial conditions, we compute

$$\begin{aligned} I(t) &= Q'(t) \\ &= -20e^{-20t}(c_1 \cos 40t + c_2 \sin 40t) \\ &\qquad + e^{-20t}(-40c_1 \sin 40t + 40c_2 \cos 40t) \end{aligned}$$

$$\begin{aligned} Q(0) = 2 &= c_1 \\ Q'(0) = -2 &= -20c_1 + 40c_2 \\ &= -40 + 40c_2 \\ 42 &= 40c_2 \\ c_2 &= \tfrac{21}{20} \\ Q(t) &= e^{-20t}(2 \cos 40t + \tfrac{21}{20} \sin 40t) \\ I(t) &= e^{-20t}(2 \cos 40t - 101 \sin 40t) \end{aligned}$$

If $R \neq 0$ (as is always the case in the real world) then the exponentials in each case are negative, and for $E \equiv 0$, the solution of any homogeneous IVP

$$LQ'' + RQ' + \tfrac{1}{C} = 0, \quad Q(0) = Q_0, \quad Q'(0) = I_0,$$

goes to 0 exponentially as $t \to \infty$. Thus, all solutions are transient.

If $E \not\equiv 0$, then the solution has the form

$$Q = Q_c + Q_p,$$

where the transient solution Q_c satisfies the complementary equation and vanishes exponentially for any initial conditions; and the *steady-state charge* Q_p depends only on E and is independent of the initial conditions.

Definition. Since $I = Q' = Q_c' + Q_p'$ and Q_c' also tends to 0 exponentially as $t \to \infty$,[6] we say that $I_c = Q_c'$ is the *transient current* and $I_p = Q_p'$ is the *steady-state current*. We are usually just interested in the steady-state charge and current.

Example 6.3.2. Find the steady-state current in the circuit described by the equation

$$\tfrac{1}{10}Q'' + 3Q' + 100Q = 5\cos 10t - 5\sin 10t.$$

The steady-state current is independent of the transient complementary solution, so we just need to find a particular solution. Since this is a constant coefficient equation with nice forcing function, try

[6] To see this, differentiate Q from any of the three cases of damping discussed above.

undetermined coefficients.

$$\begin{aligned} Q_p &= A\cos 10t + B\sin 10t \\ Q_p' &= -10A\sin 10t + 10B\cos 10t \\ Q_p'' &= -100A\cos 10t - 100B\sin 10t \end{aligned}$$

Then

$$\begin{aligned} \tfrac{1}{10}Q'' + 3Q' + 100Q &= -10A\cos 10t - 10B\sin 10t \\ &\quad - 30A\sin 10t + 30B\cos 10t \\ &\quad + 100A\cos 10t + 100B\sin 10t \\ &= (90A + 30B)\cos 10t + (-30A + 90B)\sin 10t \\ &= 5\cos 10t - 5\sin 10t = E(t) \end{aligned}$$

gives

$$\begin{aligned} 90A + 30B &= 5 \\ -30A + 90B &= -5 \\ -90A + 270B &= -15 \\ 300B &= -10 \\ B &= -\tfrac{1}{30} \\ 90A - 1 &= 5 \\ A &= \tfrac{6}{90} = \tfrac{1}{15} \end{aligned}$$

The steady-state charge is $Q_p = \frac{1}{15}\cos 10t - \frac{1}{30}\sin 10t$, and the steady-state current is

$$I_p = Q_p' = -\tfrac{2}{3}\sin 10t - \tfrac{1}{3}\cos 10t.$$

§6.3 Homework Assignment:

Read: 273–278

Exercises: 1–3,6–8,11–12

6.4. **Motion Under a Central Force.**

Definition. A *central force* is a force whose magnitude depends only on the distance to the origin; it is radially symmetric about the origin.

If we represent a central force in polar coordinates,

$$x = r\cos\theta, \qquad \text{and} \qquad y = r\sin\theta,$$

so that the force is

$$\mathbf{F}(r,\theta) = f(r)(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}),$$

where $f(r)$ is continuous for $r > 0$. Then the magnitude of $\mathbf{F}$ is

$$\begin{aligned} |\mathbf{F}| &= \sqrt{f^2(r)\cos^2\theta + f^2(r)\sin^2\theta} \\ &= \sqrt{f^2(r)}\sqrt{\cos^2\theta + \sin^2\theta} \\ &= |f(r)| \cdot 1, \end{aligned}$$

and so depends only on the distance to the origin.

Definition. Central forces include gravitation and electromagnetism, and thus the curves traversed by objects under central forces are called *orbits*.

We will see how Kepler's laws of planetary motion (which Kepler deduced by astronomical observation) can be derived from Newton's calculus, and show that the orbit of an object moving under a central force is given by

$$r(\theta) = \frac{1}{u(\theta)},$$

where u is a solution of

$$\tfrac{d^2u}{d\theta^2} + u = -\tfrac{1}{mh^2u^2}f(1/u),$$

where m is the mass of the object and h is a constant that depends on the initial conditions.

By equating our expressions for the force on the object, using Newton's Second law, we get

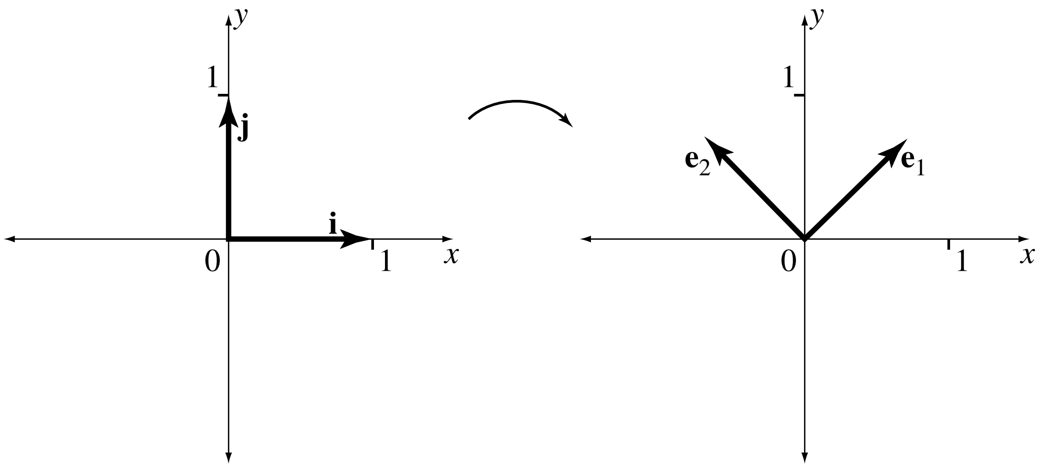
$$\begin{aligned}\mathbf{F} = m\mathbf{a} &= m(r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j})'' \\ &= f(r)(r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}) = \mathbf{F}(r,\theta).\end{aligned}$$

To deal with this equation, we introduce new unit vectors

$$\mathbf{e}_1 = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} \quad \text{and} \quad \mathbf{e}_2 = \sin\theta\mathbf{i} + \cos\theta\mathbf{j},$$

so that $\mathbf{e}_1$ is the vector appearing above.

FIGURE 6.4.1. Change of basis vectors.



To understand this change of basis better, note that

$$\frac{d\mathbf{e}_1}{d\theta} = \mathbf{e}_2, \quad \text{and} \quad \frac{d\mathbf{e}_2}{d\theta} = -\mathbf{e}_1.$$

The dot product also shows $\mathbf{e}_1 \perp \mathbf{e}_2$:

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = \cos\theta(-\sin\theta) + \sin\theta\cos\theta = 0.$$

Also, we have

$$\frac{d\mathbf{e}_1}{dt} = \mathbf{e}_1' = \theta'\mathbf{e}_2 = \frac{d\theta}{dt} be2$$
$$\frac{d\mathbf{e}_2}{dt} = \mathbf{e}_2' = -\theta'\mathbf{e}_1 = -\frac{d\theta}{dt} be1$$

Now returning the equation for $\mathbf{F} = m\mathbf{a}$, we can write

$$m(r\mathbf{e}_1)'' = f(r)\mathbf{e}_1.$$

Using the above expressions for $\mathbf{e}_1'$ and $\mathbf{e}_2'$,

$$(r\mathbf{e}_1)'' = (r'' - r(\theta')^2)\mathbf{e}_1 + (r\theta'' + 2r'\theta')\mathbf{e}_2,$$

so we get

$$m(r'' - r(\theta')^2)\mathbf{e}_1 + m(r\theta'' + 2r'\theta')\mathbf{e}_2 = f(r)\mathbf{e}_1.$$

Equating coefficients, this yields the system

$$m(r'' - r(\theta')^2) = f(r)$$
$$m(r\theta'' + 2r'\theta') = 0.$$

From the second eqn, multiplying by r gives

$$r^2\theta'' + 2rr'\theta' = (r^2\theta')' = 0,$$

which indicates that $r^2\theta'$ is a constant. Since its values are the same for all time, we use the initial conditions to define

$$h := r^2(0)\theta'(0) = r^2(t)\theta'(t), \; \forall t > 0.$$

This is Kepler's Second Law. In geometrical terms, it means that the position vector of an object moving under a central force sweeps out equal areas in equal times.

Theorem 6.4.1. *If $\theta(t_1) \le \theta(t_2)$, then the area of the sector*

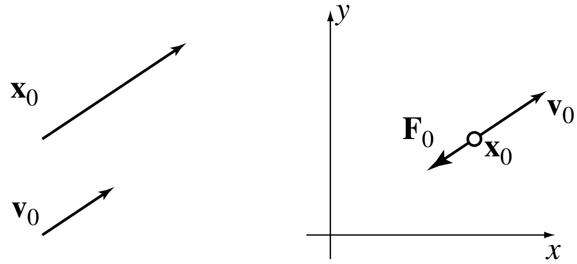
$$\{(x, y) = (r\cos\theta, r\sin\theta) : 0 \le r \le r(0), and \theta(t_1) \le \theta \le \theta(t_2)\}$$

is given by

$$A = h(t_2 - t_1)/2.$$

If the initial position and velocity vectors are parallel, the subsequent motion will be either toward or away from the origin, as in §4.3.

FIGURE 6.4.2. Parallel initial position and velocity vectors.



Since the initial position and velocity vectors are

$$\mathbf{x}_0 = r(0)\mathbf{e}_1(0)$$
$$\mathbf{v}_0 = r'(0)\mathbf{e}_1(0) + r(0)\theta'\mathbf{e}_2(0),$$

the assumption that these are not parallel implies that $\theta'(0) \neq 0$, in which case $h \neq 0$ (so we may divide by h).

Define $u := 1/r$. Then $u^2 = \theta'/h$ and for $r = 1/u$,

$$\tfrac{dr}{dt} = r' = -\tfrac{u'}{u^2} = -h\left(\tfrac{u'}{\theta'}\right) = -h\tfrac{du}{dt}\tfrac{dt}{d\theta} = -h\tfrac{du}{d\theta}$$
$$r'' = -h\tfrac{d}{dt}\tfrac{du}{d\theta} = -h\tfrac{d^2u}{d\theta^2}\theta' = -h^2u^2\tfrac{d^2u}{d\theta^2} \quad (\theta' = hu^2)$$

So from the second equation of that linear system (remember, we only used the first equation so far), we get

$$m(r'' - r(\theta')^2) = f(r)$$
$$m\left(-h^2u^2\tfrac{d^2u}{d\theta^2} - \tfrac{1}{u}h^2u^4\right) = f\left(\tfrac{1}{u}\right)$$
$$\tfrac{d^2u}{d\theta^2} + u = -\tfrac{1}{mh^2u^2}f\left(\tfrac{1}{u}\right),$$

as promised.

6.4.1. *Motion under an inverse square law force.* In the case of gravity, and electromagnetism, one often encounters inverse square laws:

Newton's law:

$$F = g\frac{m_1 m_2}{r^2} \qquad (g \text{ is the gravitation constant}).$$

Coulomb's law: the force between two point charges is inversely proportional to the square of the distance between them.

$$F = \tfrac{1}{4\pi\varepsilon}\left(\tfrac{Q_1 Q_2}{r^2}\right) \qquad (\varepsilon \text{ is environmental permittivity})$$

For gravity, $f(r) = -mk/r^2 = -mku^2$ (where k is some constant of proportionality), so that the equation of motion becomes

$$\tfrac{d^2u}{d\theta^2} + u = \tfrac{k}{h^2}.$$

The complementary equation has the solution

$$u_c = A\cos(\theta - \varphi),$$

as we've seen before. Since

$$u_p = k/h^2$$

is a particular solution, the general solution is

$$u = A\cos(\theta - \varphi) + \tfrac{k}{h^2}.$$

Thus, the orbit is given by

$$r = \left(A\cos(\theta - \varphi) + \tfrac{k}{h^2}\right)^{-1} = \frac{\varrho}{1 + \varepsilon\cos(\theta - \varphi)},$$

where $\varrho = h^2/k$ and $\varepsilon = A\varrho$.

Definition. $\varepsilon > 0$ is the *eccentricity* of the orbit.
If $\varepsilon = 0$, the orbit is a circle.
If $\varepsilon < 1$, the orbit is an ellipse.

If $\varepsilon = 1$, the orbit is parabola.
If $\varepsilon > 1$, the orbit is hyperbola.

For ellipses, the minimum and maximum distance from the origin are given by

$$r_{\min} = \frac{\varrho}{1+\varepsilon} \qquad \text{(when } \theta = \varphi\text{), and}$$
$$r_{\max} = \frac{\varrho}{1-\varepsilon} \qquad \text{(when } \theta = \varphi + \pi\text{),}$$

which are called the *perihelion distance* and *aphelion distance*, respectively. The point of the orbit corresponding to $r = r_{\min}$ is called the *perigee* and the point corresponding to $r = r_{\max}$ is called the *apogee*.

§6.4 Homework Assignment:

Read: 279–285
Exercises: 1–5

Note: #3 has a typo — “aphelion” and “perihelion” are interchanged.