

MATH 121 — GAME THEORY REVIEW

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I have tried to follow some conventions:

- The row player in a 2-person game is called “**Rose**” and the column player is called “**Colin**”, for obvious reasons.
- The payoff for pure strategies is denoted using $\pi(\sigma, \tau)$. In the extension to expected payoffs for mixed strategies, I use $P(\mathbf{x}, \mathbf{y})$.
Also, a payoff $\pi(\sigma, \tau)$ or $P(\mathbf{x}, \mathbf{y})$ is actually a vector — the k^{th} component (i.e., the payoff to the k^{th} player) is denoted $\pi_k(\sigma, \tau)$ or $P_k(\mathbf{x}, \mathbf{y})$. In some places, I may have neglected to include the subscript when it’s fairly clear what is meant.
- Vectors are bold, their components are not: $\mathbf{x} = (x_1, x_2, \dots, x_n)$.
- The letters u, v are reserved for discussing payoffs.
- Stars represent optimal things: $\mathbf{x}^*, \mathbf{y}^*, u^*, v^*$, etc.

Disclaimer: this document was prepared by the TA and not the professor, so it should **not** be considered as an exhaustive list of the material covered in the course. It is just a collection of the most important ideas for help with studying for the final. Also, notation may differ slightly from the professor’s — ask me if you have questions: epearse@math.ucr.edu.

1. DEFINITIONS

1.1. Non-cooperative Games.

extensive form: The game represented as a tree, where each branch point indicates a choice made by one of the players (or by chance/nature). The “leaves” (final nodes) are the final outcomes of one round of play, and have associated payoffs for the players.

pure strategy: A prior, comprehensive list of the choices to be made at each decision point the player might encounter during a play of the game.

mixed strategy: A probability vector $\mathbf{x} = (x_1, x_2, \dots, x_n)$ where x_i indicates the likelihood with which the player will play the i^{th} pure strategy.

probability vector: $\mathbf{x} \in \mathbb{R}^n$ satisfying

- (i) $0 \leq x_i \leq 1$, for all $i = 1, 2, \dots, n$.
- (ii) $\sum_{i=1}^n x_i = 1$.

worthwhile: A pure strategy (in a matrix game) is called *worthwhile* iff it appears with positive (i.e., nonzero) probability in some optimal strategy.

information set: For a game in extensive form, two or more branch points/nodes are in the same *information set* if the player making the decision at that point cannot tell the nodes apart.

perfect information: A game is said to have perfect information iff all the information sets consist only of single points, i.e., at any point in the game, both players know everything that has occurred up to that point.

normal form: For 0-sum 2-player games, the normal form is the game represented as a matrix, where the entries in the matrix are the payoffs to Rose. Note: More generally (i.e. for n -player games which aren't necessarily 0-sum), the normal form is a function from the cross product of the strategy spaces to \mathbb{R}^n . In the case of two players, this just looks like

$$\pi : \{\sigma\}_{i=1}^m \times \{\tau\}_{j=1}^m \rightarrow \mathbb{R}^2, \quad \text{with } \pi(\sigma_i, \tau_j) = (\pi_1(\sigma_i, \tau_j), \pi_2(\sigma_i, \tau_j)).$$

so it can be represented as a matrix with entries $(\pi_1(\sigma_i, \tau_j), \pi_2(\sigma_i, \tau_j))$.

expected value: The *expected value* of getting payoffs a_1, a_2, \dots, a_k with respective probabilities p_1, p_2, \dots, p_k is $p_1 a_1 + p_2 a_2 + \dots + p_k a_k$. Written as a payoff vector $\mathbf{a} = (a_1, a_2, \dots, a_k)$ and a probability vector $\mathbf{p} = (p_1, p_2, \dots, p_k)$, the expected value is $\mathbf{a}^T \mathbf{p}$.

expected payoff: For mixed strategies $\mathbf{x} = (\xi_1, \xi_2, \dots, \xi_m)$, $\mathbf{y} = (\eta_1, \eta_2, \dots, \eta_n)$, in an $m \times n$ matrix game A whose entry a_{ij} is the payoff $\pi(\sigma_i, \tau_j)$, the *expected payoff* is

$$P(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T A \mathbf{y} = \sum_{i=1}^m \sum_{j=1}^n \xi_i \pi(\sigma_i, \tau_j) \eta_j.$$

optimal strategy: An *optimal strategy* for Rose is \mathbf{x}^* such that $\min_y \pi(\mathbf{x}^*, \mathbf{y}) = V$. Similarly, an optimal strategy for Colin is \mathbf{y}^* such that $\max_x \pi(\mathbf{x}, \mathbf{y}^*) = V$.

upper value, lower value: Consider a matrix game $A = [\pi(\sigma_i, \tau_j)]$. For pure strategies, the *lower value* is

$$\underline{v} = \max_{\sigma} \min_{\tau} \pi(\sigma, \tau),$$

and the *upper value* is

$$\bar{v} = \min_{\tau} \max_{\sigma} \pi(\sigma, \tau).$$

For mixed strategies, the *lower value* is

$$\underline{V} = \max_{\mathbf{x}} \min_{\mathbf{y}} P(\mathbf{x}, \mathbf{y}),$$

and the *upper value* is

$$\bar{V} = \min_{\mathbf{x}} \max_{\mathbf{y}} P(\mathbf{x}, \mathbf{y}).$$

value of the game: If $\underline{v} = \bar{v}$, then this common value is V , the *value of the game*.

Otherwise, define the *value of the game* V to be the common value $\underline{V} = \bar{V}$.

Note: it is a *theorem* that $\underline{V} = \bar{V}$.

Note: in the case when $\underline{v} = \bar{v}$, there a couple of important relevant facts:

- It is a *theorem* that $\underline{v} = \bar{v}$ iff there is a saddle point.
- It is a *theorem* that if $\underline{v} = \bar{v}$, then $\underline{v} = \bar{v} = V = \underline{V} = \bar{V}$.
- When $\underline{v} = \bar{v}$, we also say that the game has a *solution in pure strategies*.

Note: it is a *theorem* that value of the game is the expected payoff when optimal strategies compete (or more generally, when an optimal strategy competes against a worthwhile strategy), i.e.,

$$V = P(\mathbf{x}^*, \mathbf{y}^*).$$

solution in mixed strategies: In a matrix game, a *solution in mixed strategies* is a triple $(\mathbf{x}^*, \mathbf{y}^*, V)$ where x^* is an optimal strategy for the first player and y^* is an optimal strategy for the second player, and V is the value of the game.

domination: Strategy σ_1 dominates strategy σ_2 iff

$$\pi_1(\sigma_1, \tau_j) \geq \pi_1(\sigma_2, \tau_j), \forall j.$$

Similarly, strategy τ_1 dominates strategy τ_2 iff

$$\pi_2(\sigma_i, \tau_1) \geq \pi_2(\sigma_i, \tau_2), \forall i.$$

If all the inequalities are strict, the strategy is said to be *strictly dominated*.

Note: when the game is 0-sum, τ_1 dominates strategy τ_2 iff

$$\pi_1(\sigma_i, \tau_1) \leq \pi_1(\sigma_i, \tau_2), \forall i.$$

saddle point: In a matrix game $A = [\pi(\sigma_i, \tau_j)]$, a *saddle point* is a pair of **pure** strategies (σ^*, τ^*) such that

$$\pi(\sigma, \tau^*) \leq \pi(\sigma^*, \tau^*) \leq \pi(\sigma^*, \tau), \text{ for all other pure strategies } \sigma, \tau.$$

equilibrium pair: In a matrix game $A = [\pi(\sigma_i, \tau_j)]$, a *equilibrium pair* is a pair of (pure or mixed) strategies $(\mathbf{x}^*, \mathbf{y}^*)$ such that

$$\pi_1(\mathbf{x}, \mathbf{y}^*) \leq \pi_1(\mathbf{x}^*, \mathbf{y}^*) \text{ and } \pi_2(\mathbf{x}^*, \mathbf{y}) \leq \pi_2(\mathbf{x}^*, \mathbf{y}^*) \text{ for all other strategies } \mathbf{x}, \mathbf{y}.$$

Note: for 0-sum games, this can be written

$$\pi_1(\mathbf{x}, \mathbf{y}^*) \leq \pi_1(\mathbf{x}^*, \mathbf{y}^*) \leq \pi_1(\mathbf{x}^*, \mathbf{y}), \text{ for all other strategies } \mathbf{x}, \mathbf{y}.$$

maximin strategies: The optimal strategies for the 0-sum games e_1 and e_2^T obtained from the original payoff matrix by only considering one player's payoffs.

Note: for Colin, remember to take the transpose.

maximin values: The values of the 0-sum games e_1 and e_2^T .

1.2. Cooperative 2-person Games.

closed: A set $K \subseteq \mathbb{R}^n$ is *closed* iff it contains its boundary.

bounded: A set $K \subseteq \mathbb{R}^n$ is *bounded* iff it is contained in some disk of finite radius, i.e.,

$$\exists R < \infty \text{ such that } K \subseteq B(0, R) = \{\mathbf{x} : |\mathbf{x}| \leq R\}.$$

convex: A set $K \subseteq \mathbb{R}^n$ is *convex* iff any line joining two points of the set lies entirely within the set, i.e.,

$$\mathbf{x}, \mathbf{y} \in K \implies t\mathbf{x} + (1-t)\mathbf{y} \in K, \forall 0 \leq t \leq 1.$$

Note: $t\mathbf{x} + (1-t)\mathbf{y}$ is called a *convex combination* of \mathbf{x} and \mathbf{y} when $0 \leq t \leq 1$.

continuous: A function f is *continuous* iff it maps nearby points to nearby points, i.e.,

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } |\mathbf{x} - \mathbf{y}| < \delta \implies |f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon.$$

Pareto optimal: The pair of strategies $(\mathbf{x}_1, \mathbf{y}_1)$ is *Pareto optimal* iff there does NOT exist $(\mathbf{x}_2, \mathbf{y}_2)$ such that

$$\pi_1(\mathbf{x}_2, \mathbf{y}_2) \geq \pi_1(\mathbf{x}_1, \mathbf{y}_1) \quad \text{and} \quad \pi_2(\mathbf{x}_2, \mathbf{y}_2) \geq \pi_2(\mathbf{x}_1, \mathbf{y}_1),$$

where at least one of the inequalities is strict.

A pair of payoffs (u, v) is *Pareto optimal* iff there does not exist (u', v') such that

$$u' \geq u \quad \text{and} \quad v' \geq v,$$

where at least one of the inequalities is strict.

(noncooperative) payoff region: The set of payoffs which can be obtained by players using mixed strategies (where each player is randomizing separately).

cooperative payoff region: The set of payoffs which can be obtained through cooperative play. In jargon, it's the image of the payoff function (general normal form, as above) when the players are using jointly randomized strategies.

Note: it is a *theorem* that the cooperative payoff region is the convex closure (or convex hull) of the payoff region of the non-cooperative game.

frontier set: (u, v) is in the *frontier set* F iff there is no other point (u', v') in the cooperative payoff region with $u \leq u'$ and $v \leq v'$.

negotiation set: (Also called the von Neumann- Morgenstern negotiation set or bargaining set) The subset $N \subseteq F$ of the frontier set satisfying $u \geq u_0$ and $v \geq v_0$, where (u_0, v_0) is the status quo point, i.e., $(u, v) \in N$ iff

- (i) $(u, v) \in C$
- (ii) $u \geq u_0$ and $v \geq v_0$
- (iii) (u, v) is Pareto optimal in C .

status quo point: A pair of payoffs (u_0, v_0) which the players will receive if they cannot agree to cooperate.

maximin bargaining solution: The payoff (u^*, v^*) which is a result of applying the Nash arbitration procedure with the maximin value (or security levels) as the status quo point. *Note:* The method for obtaining the Nash maximin bargaining solution is described below.

threat bargaining solution: The payoff (u^*, v^*) which is a result of applying the Nash arbitration procedure with the threat outcomes as the status quo point. *Note:* The method for obtaining the Nash threat bargaining solution is also described below.

1.3. Cooperative n -person Games (in coalitional form).

coalition: In an n -player game, the set of players is $N = \{1, 2, \dots, n\}$ and a *coalition* is any subset of N . (Consider it to be a team formed when certain players choose to work together). The *empty coalition* is \emptyset . The *grand coalition* is N . The set of all coalitions is the power set of N , denoted $\mathcal{P}(N)$ and contains every subset of N as an element.

characteristic function: An n -player game represented as a function $v : \mathcal{P}(N) \rightarrow \mathbb{R}$, where $v(S)$ represents the maximum that the coalition $S \subseteq N$ can guarantee for itself, regardless of what the coalition $N - S$ could do to thwart it:

$$v(S) = \max_{x \in X_S} \min_{y \in Y_{N-S}} \sum_{i \in S} \pi_i(\mathbf{x}, \mathbf{y}).$$

superadditive: A set function f is *superadditive* iff

$$v(S \cup T) \geq v(S) + v(T),$$

whenever $S \cap T = \emptyset$.

infinitely divisible: The rewards of a game are said to be *infinitely divisible* if they can be split up among the players in any way.

side payments: A transfer of payoffs from one player to another.

imputations: A reasonable share-out of the payoffs. For an n -player game v , the set of *imputations* is

$$\mathcal{E}(v) = \{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n x_i = v(N), \text{ and } x_i \geq v(\{i\}), \forall i = 1, 2, \dots, n\}.$$

domination: An imputation \mathbf{x} *dominates* \mathbf{y} over S iff

(i) $x_i > y_i, \forall i \in S$, and

(ii) $\sum_{i \in S} x_i \leq v(S)$.

An imputation \mathbf{x} *dominates* \mathbf{y} iff it dominates \mathbf{y} over some S .

core: For an n -player game v , the core is the set of imputations which are not dominated for any coalition.

Note: see the Core Characterization Theorem below, for a more useful way to compute the core.

Shapley value: A function $\varphi(v)$ which assigns each player a number/value indicating the relative power of that player in the game (average marginal power).

2. THEOREMS AND MAIN IDEAS

It is not necessarily important to know the names of these items, I just included them as memory aids for the most part. Items marked with a $\boxed{\star}$ symbol are ones that might be good to study in case you are asked to prove them on the exam.

Expected Value Principle: If you were to know your opponent is playing a given mixed strategy, and will continue to play it regardless of what you do, you should play your strategy which has the largest expected value.

Minimax Theorem: (von Neumann, 1928) Every $m \times n$ matrix game has a solution. That is, there is a unique number v called the *value of the game*, and there are optimal mixed strategies for Rose and Colin. I.e.,

- (1) if Rose plays her optimal strategy, Rose's expected payoff will be $\geq v$, no matter what Colin does, and
- (2) if Colin plays his optimal strategy, Rose's expected payoff will be $\leq v$, no matter what Rose does.

Principle of Domination: A rational player should never play a strictly dominated strategy. Moreover, removing a strictly dominated strategy from the game will not change the solution to the game.

Principle of Higher-Order Domination: The Principle of Domination may be extended to the resulting smaller game.

I.e., after applying the Principle of Domination, the resulting smaller game may contain dominated strategies, even though these strategies weren't dominated in the original game. The Principle of Higher Order Domination says that these strategies should also be removed from the game, and that players should only play strategies which survive this multi-stage process.

$\boxed{\star}$ **Worthwhile Strategies Lemma:** When a worthwhile strategy plays against an optimal strategy in a matrix game, the payoff is the value of the game. (p. 37)

$\boxed{\star}$ **Equilibrium Pair Lemma:** If $(\mathbf{x}_1, \mathbf{y}_1)$ and $(\mathbf{x}_2, \mathbf{y}_2)$ are equilibrium pairs in a matrix game, then $P(\mathbf{x}_1, \mathbf{y}_1) = P(\mathbf{x}_2, \mathbf{y}_2)$. (p. 43)

Equilibrium Pair Theorem: (x, y) is an equilibrium pair in a matrix game if and only if $(x, y, e(x, y))$ is a solution to the game.

Core Characterization Theorem: For an n -player game v , the core is

$$\mathcal{C}(v) = \{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n x_i = v(N), \text{ and } \sum_{i \in S} x_i \geq v(S), \forall S \subseteq N\}.$$

Note: In comparison, the set of imputations is

$$\mathcal{E}(v) = \{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n x_i = v(N), \text{ and } \sum_{i \in S} x_i \geq v(S), \forall S = \{i\}, i \in N\},$$

so the only difference between core and imputations is that to be an *imputation*, \mathbf{x} need only satisfy the condition

$$\sum_{i \in S} x_i \geq v(S) \tag{*}$$

when S is a singleton $\{i\}$. To be in the *core*, \mathbf{x} must satisfy (*) for **every** $S \subseteq N$.

3. TECHNIQUES

3.1. Tips to remember when solving 2-player 0-sum games:

- (1) Check for saddle points.
- (2) Check for dominated strategies.
- (3) Solve for mixed strategies.

3.2. Solving for mixed strategies in 2-player 0-sum games:

- (1) Try to reduce an $m \times n$ game to a $2 \times n$ or $m \times 2$ game by using domination.
- (2) If you have a $2 \times n$ or $m \times 2$ game, use the graphical technique for ascertaining which strategies are worthwhile. Remember:
 - (a) when you have 2 column strategies, you are minimaxing, so find the lowest point on the upper envelope, and
 - (b) when you have 2 row strategies, you are maximinizing, so find the highest point on the lower envelope.
- (3) Determine the optimal strategies by equalizing expectations (in accordance with the Expected Value Principle). For example, let $\mathbf{x} = (\xi, 1 - \xi)$ and $\mathbf{y} = (\eta, 1 - \eta)$ and consider:

$$\begin{pmatrix} 2 & -3 \\ -1 & 3 \end{pmatrix}$$

The optimal strategy for Rose is found by setting $P(\mathbf{x}, \tau_1) = 2\xi + (-1)(1 - \xi)$ equal to $P(\mathbf{x}, \tau_2) = -3\xi + 3(1 - \xi)$ and solving to get $\mathbf{x} = (\frac{4}{9}, \frac{5}{9})$. The optimal strategy for Colin is found by setting $P(\sigma_1, \mathbf{y}) = 2\eta + (-3)(1 - \eta)$ equal to $P(\sigma_2, \mathbf{y}) = (-1)\eta + 3(1 - \eta)$ and solving to get $\mathbf{y} = (\frac{2}{3}, \frac{1}{3})$.

3.3. Finding equilibrium pairs in 2-person nonzero-sum games.

This is known as the Swastika method. Consider the matrix game

$$A = \begin{bmatrix} (a_{11}, b_{11}) & (a_{12}, b_{12}) \\ (a_{21}, b_{21}) & (a_{22}, b_{22}) \end{bmatrix}.$$

1. Find the expected value of the mixed strategies $\mathbf{x} = (\xi, 1 - \xi)$, $\mathbf{y} = (\eta, 1 - \eta)$ for each player:

$$\begin{aligned} P_1(\mathbf{x}, \mathbf{y}) &= \xi\eta a_{11} + \xi(1 - \eta)a_{12} + (1 - \xi)\eta a_{21} + (1 - \xi)(1 - \eta)a_{22} \\ &= \xi(\eta(a_{11} - a_{12} - a_{21} + a_{22}) + (a_{12} - a_{22})) + \eta(a_{21} - a_{22}) + a_{22}. \end{aligned}$$

and similarly,

$$\begin{aligned} P_2(\mathbf{x}, \mathbf{y}) &= \xi\eta b_{11} + \xi(1 - \eta)b_{12} + (1 - \xi)\eta b_{21} + (1 - \xi)(1 - \eta)b_{22} \\ &= \eta(\xi(b_{11} - b_{12} - b_{21} + b_{22}) + (b_{21} - b_{22})) + \xi(b_{12} - b_{22}) + b_{22}. \end{aligned}$$

Don't worry about memorizing the formulas here and in the next couple of steps, just remember the technique.

2. Find the values of η which makes

$$\eta(a_{11} - a_{12} - a_{21} + a_{22}) + (a_{12} - a_{22}) < 0.$$

On the unit square, plot the points $(0, \eta)$ for these values of η , and plot the points $(1, \eta)$ for those values of η which reverse the inequality. Also draw the horizontal line $y = \eta$ for the critical value of η which makes

$$\eta(a_{11} - a_{12} - a_{21} + a_{22}) + (a_{12} - a_{22}) = 0,$$

and connects the other two pieces.

3. Find the values of ξ which makes

$$\xi(b_{11} - b_{12} - b_{21} + b_{22}) + (b_{21} - b_{22}) < 0.$$

On the unit square, plot the points $(\xi, 0)$ for these values of ξ , and plot the points $(\xi, 1)$ for those values of ξ which reverse the inequality. Also draw the vertical line $x = \xi$ for the critical value of ξ which makes

$$\xi(b_{11} - b_{12} - b_{21} + b_{22}) + (b_{21} - b_{22}) = 0,$$

and connects the other two pieces.

4. The equilibrium pairs are any places where the graphs from 2 and 3 intersect. You're done!

Note: For steps 2 & 3, you may find it easier to solve

$$\eta(a_{11} - a_{12} - a_{21} + a_{22}) + (a_{12} - a_{22}) = 0$$

and

$$\xi(b_{11} - b_{12} - b_{21} + b_{22}) + (b_{21} - b_{22}) = 0$$

first and obtain the horizontal and vertical lines, and then figure out how to complete the figure based on the inequalities *afterwards*. Remember, however, that the right way to figure out how to complete the figure is to solve

$$\eta(a_{11} - a_{12} - a_{21} + a_{22}) + (a_{12} - a_{22}) < 0$$

and

$$\xi(b_{11} - b_{12} - b_{21} + b_{22}) + (b_{21} - b_{22}) < 0,$$

as described above. I'm mentioning this because students often seem to make bizarre mistakes with the swastika method by finding the horizontal and vertical lines, and then filling in the rest of the figure arbitrarily (or perhaps based on some mistaken assumption). So use the inequality!

3.4. Finding the Nash maximin bargaining solution.

Consider the noncooperative matrix game A with entries

$$a_{ij} = (\pi_1(\sigma_i, \tau_j), \pi_2(\sigma_i, \tau_j)) = (u_{ij}, v_{ij}).$$

1. Plot the entries (u_i, v_j) of the payoff matrix in (u, v) -space, where u is the payoff to Rose and v is the payoff to Colin. Then the cooperative payoff region C is the convex hull of the points (u, v) .
2. Locate the frontier set by finding the points in C which are Pareto optimal. (u, v) is **not** Pareto optimal in C if there are any points above or to the right of it, which are also in C , i.e. if there is $u', v' \in C$ with

$$u' \geq u \text{ or } v' \geq v.$$

Obtain something of the form

$$F = \{(u, v) : au + v = b, c \leq u \leq d\}.$$

Note that F may consist of two or more pieces (line segments with different slope), e.g.,

$$F = \{(u, v) : a_1u + v = b_1, c_1 \leq u \leq d_1\} \cup \{(u, v) : a_2u + v = b_2, c_2 \leq u \leq d_2\}.$$

or even a single point: $F = \{(u, v)\}$.

Note: if $F = \{(u, v)\}$, a single point, then you are done. The Nash solution is (u, v) .

3. Get the negotiation/bargaining set by finding the status quo point (u_0, v_0) . u_0 is the value of the 0-sum game $e_1 = [u_{ij}]$ obtained by considering only the payoffs to Rose from the matrix A . v_0 is the value of the 0-sum game $e_2 = [v_{ij}]^T$ obtained by considering only the payoffs to Colin from the matrix A , and then transposing the matrix.

4. Obtain the negotiation set N by taking N to be those points of F which are above and to the right of (u_0, v_0) :

$$N = \{(u, v) : au + v = b, c' \leq u \leq d'\},$$

where

$$[c', d'] = [\max\{c, u_0\}, \max\{-ad + b, v_0\}].$$

Don't bother remembering the formula — just the idea.

5. Maximize the function $f(u, v) = (u - u_0)(v - v_0)$ on N by rewriting it as $f(u) = (u - u_0)(-au + b - v_0)$ and applying the first derivative test. This gives u^* . Then $v^* = -au^* + b$. The Nash maximin bargaining solution is

$$(u^*, v^*).$$

3.5. Finding the Nash threat bargaining solution. Consider the same noncooperative matrix game A with entries

$$a_{ij} = (\pi_1(\sigma_i, \tau_j), \pi_2(\sigma_i, \tau_j)) = (u_{ij}, v_{ij}).$$

1. Find the frontier set F as outlined in 1–2 above.

$$F = \{(u, v) : au + v = b, c \leq u \leq d\}.$$

2. Write down the game $ae_1 - e_2$, where e_1, e_2 are as described above in step 3. a is the coefficient of u in the equation for the line in the frontier set. Then find the value w^* of the game $ae_1 - e_2$.

3. Using w^* and the constant term b from the equation for the line in the frontier set, get

$$u^* = \frac{1}{2a}(b + w^*) \quad \text{and} \quad v^* = \frac{1}{2}(b - w^*).$$

4. This (u^*, v^*) is the threat bargaining solution.

Note: although they are not required in finding the threat bargaining solution, the optimal threats are just the optimal strategies \mathbf{x}^* and \mathbf{y}^* of the game $ae_1 - e_2$. Thus, the threat status quo point is just $(P_1(\mathbf{x}^*, \mathbf{y}^*), P_2(\mathbf{x}^*, \mathbf{y}^*))$.

3.6. Examples with complex negotiation sets. It is possible that the frontier set consists of two or more line segments, so that the negotiation set can be quite complex. Here is an example of how to find the maximin bargaining solution and threat bargaining solution of such a game.

$$\begin{bmatrix} (2, 1) & (3, 2) & (0, 4) \\ (0, 1) & (4, 0) & (2, 1) \end{bmatrix}$$

The maximin bargaining solution.

- (1) The cooperative payoff region is

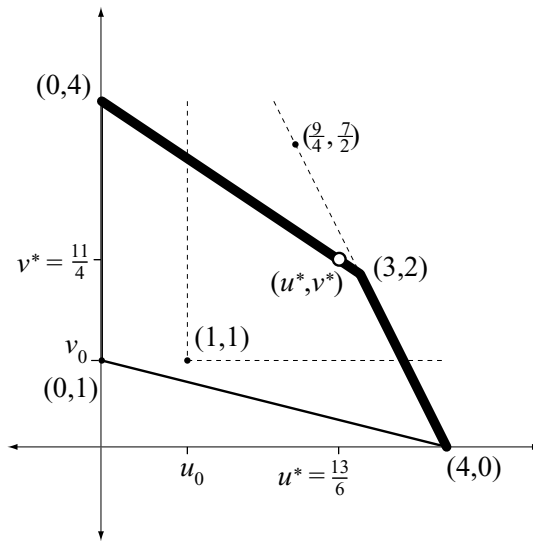


FIGURE 1. Cooperative payoff region and maximin status quo point.

- (2) The frontier set is

$$F = \{(u, v) : \frac{2}{3}u + v = 4, 0 \leq u \leq 3\} \cup \{(u, v) : 2u + v = 8, 3 \leq u \leq 4\}.$$

- (3) The status quo point is $(1, 1)$ when you find the values of the games

$$\begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 4 & 1 \end{bmatrix}.$$

- (4) The negotiation set is

$$N = \{(u, v) : \frac{2}{3}u + v = 4, 1 \leq u \leq 3\} \cup \{(u, v) : 2u + v = 8, 3 \leq u \leq \frac{7}{2}\}.$$

- (5) The maximum of $f(u, v) = (u - u_0)(v - v_0)$ on N proceeds in two steps.

- (a) For the first set, maximize $f(u) = (u - 1)(4 - \frac{2}{3}u - 1)$ for $1 \leq u \leq 3$ and obtain $(u^*, v^*) = (\frac{11}{4}, \frac{13}{6})$.
- (b) For the second set, maximize $f(u) = (u - 1)(8 - 2u - 1)$ for $3 \leq u \leq \frac{7}{2}$ and obtain $(u^*, v^*) = (\frac{9}{4}, \frac{7}{2})$.

The second solution is not within the negotiation set, and the closest we can get to it within the second component of the negotiation set is the point $(3, 2)$, which is also in the first component. Moreover, the max of f on the first component is at $(\frac{11}{4}, \frac{13}{6})$, so this is our Nash bargaining solution.

The threat bargaining solution.

IMPORTANT NOTE: You are not responsible for finding threat bargaining solutions for complex negotiation sets, as described in this section — Dr. Gretsky recently informed me that this is beyond the scope of the class, so there will be no problems like this on the final exam. I have included the following material here only because I promised I would, but you don't have to worry about it. ☹

- (1) Find the cooperative payoff region as before. This time, make use of Lemma 2.2 (p. 79) to find the threat solution.

Lemma 1. If (u^*, v^*) is the Nash bargaining solution for status quo point (u_0, v_0) , and the negotiation set at (u^*, v^*) is a straight line with (u^*, v^*) not at an end point, then the slope of the line joining u_0, v_0 to (u^*, v^*) is the negative of the slope of the negotiation set at (u^*, v^*) .

- (2) Find the threat status quo point by determining the optimal strategies $\mathbf{x}^*, \mathbf{y}^*$ of the game $ae_1 = e_2$, as described above (“Finding Nash threat bargaining solutions”).
- (3) Draw a lines with negative slope as indicated in Figure 2. If the threat status quo is in the white region, like t_3 , then the threat solution is T_3 . If the threat status quo

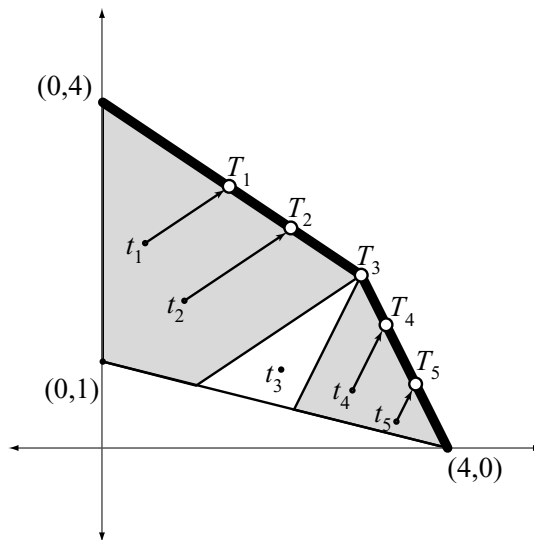


FIGURE 2. Threat solutions for different threat status quos.

is in the upper shaded region like t_1 or t_2 , then follow the line with slope $\frac{2}{3}$ to the boundary. The intersection will be the threat solution.

If the threat status quo is in the lower shaded region like t_4 or t_5 , then follow the line with slope 2 to the boundary. The intersection will be the threat solution.

4. PRACTICE EXERCISES

- (1) **Domination.** Find all cases of domination in the following game:

$$\begin{pmatrix} 3 & -6 & 2 & -4 \\ 2 & 1 & 0 & 1 \\ -4 & 3 & -5 & 4 \end{pmatrix}$$

- (2) **Higher-Order Domination.** Reduce this game using the principle of higher-order domination:

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 0 \end{pmatrix}$$

- (3) **Saddle Points.** Find all saddle points in the following games.

(a)

$$\begin{pmatrix} 3 & 2 & 4 & 2 \\ 2 & 1 & 3 & 0 \\ 2 & 2 & 2 & 2 \end{pmatrix}$$

(b)

$$\begin{pmatrix} -2 & 0 & 4 \\ 2 & 1 & 3 \\ 3 & -1 & -2 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 4 & 3 & 8 \\ 9 & 5 & 1 \\ 2 & 7 & 6 \end{pmatrix}$$

- (4) The first game in the previous exercise shows that a saddle point may appear in a dominated strategy. The Domination Principle says we shouldn't play these strategies. Show that the Domination Principle cannot come into direct conflict with the Saddle Point Principle by showing that if Row A dominates Row B, and Row B contains a saddle point entry, then the entry a in the same column of Row A is also a saddle point.
- (5) Prove (or at least make arguments for) the following:
- If a is a saddle point entry, then the row containing a is a maximin row, the column containing a is a minimax column, and $\text{maximin} = a = \text{minimax}$.
 - If $\text{maximin} = \text{minimax}$, then the intersection of the maximin row and the minimax column is a saddle point.
- (6) Prove this theorem: Any two saddle points in a matrix game have the same value. Furthermore, if the Rows player and Columns player both play strategies which contain a saddle point outcome, then the result will always be a saddle point.

- (7) Consider a general 2×2 game:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The game will have a saddle point unless the two largest entries are diagonally opposite each other, so suppose the two largest entries are a and d . Suppose Colin plays his strategies C_1 and C_2 with probabilities x and $(1 - x)$.

- (a) Show that the value of x which will equalize Rose's expectations for Rose A and Rose B is

$$x = \frac{d - b}{(a - c) + (d - b)}.$$

- (b) Show that the value of the game is

$$v = \frac{ad - bc}{(a - c) + (d - b)}.$$

- (8) Solve the following games:

- (a)

$$\begin{pmatrix} -3 & 5 \\ -1 & 3 \\ 2 & -2 \\ 3 & -6 \end{pmatrix}$$

- (b)

$$\begin{pmatrix} -2 & 5 \\ 1 & 2 \\ 0 & -2 \\ 0 & 4 \end{pmatrix}$$

- (c)

$$\begin{pmatrix} -4 & 2 & 0 & 3 & -2 \\ -4 & -1 & 0 & -3 & 1 \end{pmatrix}$$

- (9) Some games have more than one solution: the value of the game is fixed, but the players may have several different strategies which ensure this value.

- (a) Draw the graph for the following game. What happens?

$$\begin{pmatrix} -2 & 0 & 2 \\ 3 & 1 & -1 \end{pmatrix}$$

- (b) Show that there are two different optimal strategies for Colin, corresponding to the solutions for the two different 2×2 subgames. The third 2×2 subgame does not yield a solution. In the graph, what is different about that subgame?

(10) Solve the following games:

(a)

$$\begin{pmatrix} 3 & 0 & 1 \\ -1 & 2 & 2 \\ 1 & 0 & -1 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 5 & 2 & 1 \\ 4 & 1 & 3 \\ 3 & 4 & 3 \\ 1 & 6 & 2 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 4 & -3 & 2 & -4 \\ 4 & -4 & 4 & -2 \\ -5 & 2 & -7 & 2 \\ 3 & -2 & 2 & -2 \end{pmatrix}$$

5. SOLUTIONS

(1) C_2 dominates C_4 and C_3 dominates C_1 .

(2) The reduced game involves only R_1, R_4 and C_3, C_5 :

$$\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$$

(3) (a) Four saddle points: R_1 or R_4 and C_2 or C_4 .

(b) R_2, C_2

(c) none

(4) Since b is largest in its column, $b \geq a$. Since R_1 dominates R_2 , $a \geq b$. Hence $a = b$ and a is also the largest entry in its column. To show that a is a smallest entry in its row, consider any other entry c in R_1 and let d be the corresponding entry in R_2 . Then $c \geq d \geq b = a$. The first inequality holds by dominance, and the second because b is a saddle point.

(5) (a) Since a is smallest in its row, it is the row minimum for its row. Since it is the largest in its column, the other row minima cannot be larger. Hence it is the row maximin. Similarly, it is the column minimax.

(b) Let a be the maximin row I and the minimax column J . Then (the minimum of Rose I) $\leq a \leq$ (the maximum of Colin J). Since we are given that the two

extreme numbers in the inequality are the same, the inequality is in fact an equality. Hence, a is smallest in its row and largest in its column.

- (6) Suppose that a and b are saddle point entries in a matrix game, and c and d are the other entries in the corners of a rectangle containing a and b :

$$\begin{array}{ccc} a & \dots & c \\ \vdots & & \vdots \\ d & \dots & b \end{array}$$

Since a is the smallest entry in its row and b is the largest entry in its column, we get $a \leq c \leq b$. Since b is a smallest entry in its row and a is a largest entry in its column, we get $b \leq d \leq a$. Together, this shows that all the inequalities must actually be equalities, so that all four numbers are the same. Hence, c and d are also largest in their columns and smallest in their rows, and thus are saddle points.

- (7) See page 42, derivation of 2.42 and 2.41.

- (8) (a) $R^* = (0, \frac{1}{2}, \frac{1}{2}, 0)$, $C^* = (\frac{5}{8}, \frac{3}{8})$, $v = \frac{1}{2}$.
 (b) Saddle point at R_2, C_1 . $v = 1$.
 (c) $R^* = (\frac{4}{9}, \frac{5}{9})$, $C^* = (0, 0, 0, \frac{1}{3}, \frac{2}{3})$, $v = -\frac{1}{2}$.

- (9) (a) The three lines intersect at one point.
 (b) The value is $\frac{1}{2}$, and Rose's optimal strategy is $R^* = (\frac{1}{2}, \frac{1}{2})$. Colin can play $C_1^* = (\frac{3}{8}, 0, \frac{5}{8})$ or $C_2^* = (0, \frac{3}{4}, \frac{1}{4})$ or any mixture of these. C_1 and C_2 don't yield a solution because both of these lines slant to the left; the solution to that 2×2 subgame would be a saddle point.

- (10) (a) R_3 is dominated, then C_3 is dominated. Solution:
 $R^* = (\frac{1}{2}, \frac{1}{2}, 0)$, $C^* = (\frac{1}{3}, \frac{2}{3}, 0)$, $v = 1$.
 (b) Saddle point at R_3, C_3 , $v = 3$.
 (c) Eliminate first C_1 , then R_1 , then C_4 . Solve the resulting 4×2 game to get:
 $R^* = (0, 0, \frac{1}{2}, 0, \frac{1}{2})$, $C^* = (0, \frac{5}{8}, \frac{3}{8}, 0)$, $v = -\frac{1}{2}$.