A Summary of the Panel Discussion on Saturday, May 3, 2003 at the Cornell Topology Festival

May 20, 2003

This year's Cornell Topology Festival devoted about one third of its talks to the area of Geometric Group Theory, with another large cohort of speakers talking about low-dimensional topology and knot theory. In addition to their individual talks, the speakers were asked to conduct a panel discussion. In particular, each speaker was asked to prepare a description of some recent mathematics which had caught his or her interest, outside of the speaker's own work.**Karen Vogtmann**, one of the Cornell organizers of the Festival as well as a speaker, served as moderator for the discussion. She asked speakers to limit their remarks to "one blackboard's length".

In this summary of the panel discussion, a panelist's name is indicated in bold-face type, whereas names of mathematicians whose work receives significant mention are indicated in italic.

Not surprisingly, most panelists chose to remark about work that highlighted some connection between geometric group theory / low-dimensional topology and other areas of mathematics.

Dror Bar Natan of Hebrew University in Jerusalem and the University of Toronto had begun the conference with a talk on the connection between knot theory and certain algebraic structures, namely, quandles and Lie algebras. He further emphasized role that the "algebraic sciences" could play in illuminating knot theory by leading off the panel discussion with some remarks on *Khovonov's* categorification of the Jones polynomial. "Khovonovification", as Bar Natan refers to the theory, associates to a knot a complex of vector spaces which yield homological invariants stronger than the Jones polynomial. Bar Natan pointed out that despite its promise for insight into knot theory, as yet mathematicians understand very little about the subject.

Joan Birman of Columbia University then described recent work on mapping class groups by *C. Leininger* which may provide information about a conjecture of Lehmer involving certain monic polynomials. Leininger has analyzed a construction due to W. Thurston of subgroups in the mapping class group of a surface generated by positive multi-twists and has given a graph theoretic characterization of when such subgroups are free. Leininger shows that the dilatation of a pseudo-Anosov map contained in a subgroup generated by two positive multi-twists is bounded below by the constant known as Lehmer's number (which we define below). Moreover, Leininger constructs a particular pseudo-Anosov map which factors as a product of two positive multi-twists whose dilatation factor realizes Lehmer's number. Lehmer's number is the Mahler measure of a certain monic integer polynomial of degree 10, and it is conjectured that Lehmer's number is a lower bound for the Mahler measure of all such monic polynomials.

Benson Farb of the University of Chicago added to Birman's discussion by pointing out a connection between between Leininger's work and certain Coxeter groups as studied by *C. McMullen*. There is a natural surjection from a certain hyperbolic Coxter group onto the mapping class group of a genus 5 surface for which the image of the Coxeter element is Leininger's example realizing Lehmer's number, indicating some connection between the eigenvalues of elements of Coxeter groups and pseudo-Anosov dilatation factors. Farb went on to describe a deep theorem of *A. Nabutovsky* and *S. Weinberger* which provides infinitely many examples of local minimum values of the diameter functional from the space of a certain class Riemannian metrics on a manifold of dimension 5 or greater into the real numbers. Their proof gives a connection between the geometry near local minima of the diameter functional and the complexity of the word problem for groups.

Fred Cohen, from the University of Rochester gave an outline of a recent work of *Ib Madsen* and *Michael Weiss* in which they compute the integral cohomological structure of the stable mapping class group. He emphazised that this was a very promising line of research and further asked what happens to their construction if one tries to apply it to the automorphism group of a free group, as those are known to share similarities with mapping class groups.

In fact, the work of Madsen and Weiss builds on techniques previously developed by Madsen and one of the panelists,**Ulrike Tillman** of Oxford

University. Tillman added to Fred Cohen's remarks by explaining some theorems of a book by *Kiyoshi Igusa*, as well as a result by *Allen Hatcher* on the stable homology of the automorphism group of a free group. Both rely on the homotopy machinery developed for the study of high dimensional manifolds and use techniques and results from Waldhausen K-theory. In the case of the mapping class group, the stable homotopy theoretic approach highlighted in Cohen's discussion has by now completely determined the stable homotopy of mapping class groups.

Gilbert Levitt, from Université Paul Sabatier in France explained the concept of random groups, introduced by *M. Gromov*, and outlined some related questions and results. "What are the properties of a randomly chosen finitely presented group? The answer to this question of course depends on the method used for the random choice." wrote Etienne Ghys in his recent contribution on the subject for the Séminaire Bourbaki. "Gromov's work highlights the existance of finitely presented groups with astonishing properties." Levitt particularly focused on the question of groups with two generators and asked what can be said about such a group for "most" choices of relations. He gave a result of Gromov which characterizes when the group is trivial or else word hyperbolic, according to a fixed "density" value in the chosen "randomness" or density model.

John Meier from Lafayette College shared a few quotes about group theory, including the following excerpt from an article in "The Morning Call", a local newspaper, on April 20, 2003: "Topology is the study of geometric groups." Meier continued the discussion of finitely presented groups with astonishing properties, by emphasizing work by *Daniel Farley*. Starting from a very simple diagram, Farley can describe fairly complicated groups such as Thompson's groups. Moreover Farley highlights some geometrical properties of those groups, called diagram groups, such as non-positive curvature. A recent theorem of *Victor Guba* and *Mark Sapir* gives a new approach to the study of diagram groups, as well as a topological interpretation.

Justin Roberts of the University of California at San Diego described recent work by *Y. Taylor* and *C. Woodward* on quantum $6j$-symbols, which are the simplest example of quantum invariants of three-dimensional manifolds and can be thought of as somehow being invariants of the tetrahedra which make up the manifold. To quote Woodward himself, their work sheds light on "the relationship between non-Euclidean geometry...and the representation theory of quantum groups". Taylor and Woodward use spherical tetrahedra to produce an explicit formula for these $6j$-symbols. Roberts also noted that they also have a version of the formula in the case of hyperbolic tetrahedra. Taylor and Woodward's formulas generalize Roberts' own work on classical $6j$-symbols.

Up to this point in the panel discussion, various speakers had alluded to the recent announcement by *G. Perelman* of a possible solution to the famous *Geometrization Conjecture of W. Thurston*, but had declined to address the issue. However, **Dylan Thurston** of Harvard University took the plunge and began with a short description of the conjecture itself as well as a very general overview of how Perelman's possible solution builds on *R. Hamilton's* program for using Ricci flows as an approach to the Geometrization Conjecture. Thurston then asked the audience to entertain the "If Perelman is correct" scenario, and presented some problems for mathematicians in the field to consider in that case. For example, he suggested a search for a combinatorial model for Ricci flow. He also described a connection with the Nabotovsky-Weinberger result mentioned previously in the panel discussion by Farb and suggested that the methods used in the 5 (and higher)-dimensional case ought to be compared to the 3-manifold situation. Finally, he asked the following: what is the complexity of recognizing the 3-sphere? Is it NP-complete? In support of this guess, he cited a theorem due to *I. Agol, J. Hass*, and *W. Thurston* which states that the problem of finding the genus of a knot is NP-hard.

Finally, **Alain Valette** of Université de Neuchâtel outlined recent work by *Yehuda Shalom* in which he proves that several well-known algebraic or analytic features such as Betti numbers are actually geometric when restricted to some classes of amenable groups, in the sense that they are quasi-isometry invariants. Conversely, Shalom also showed that some features of certain groups which are, a priori, geometric in nature, such as having a uniformly embedded amenable group, yield a lower bound on the rational cohomological dimension.

Karen Vogtmann then thanked the speakers for their participation in the panel, and declined her share of the blackboard because of the length of the session. She proposed that further discussion take place at the Topology Festival picnic, which received the unanimous approval of the audience.