

Cornell Topology Festival Panel Discussion

May 2006

The 2006 Topology Festival had a concentration of speakers in algebraic topology with approximately half the talks being in this area, and two of the speakers, Jesper Grodal and Gunnar Carlsson, giving introductory workshops on the subject.

The speakers featured in a panel discussion on Saturday 20th May in which they presented some recent results, not their own, that had caught their attention. Karen Vogtmann, one of the Cornell organisers, acted as moderator.

Noel Brady (*University of Oklahoma*) on work of D.Groves & J.Manning [4] and D.Osin [10].

A celebrated result of W.P.Thurston is his hyperbolic Dehn surgery theorem: if a knot admits a complete finite volume hyperbolic complement then all but finitely many Dehn fillings on the knot are also hyperbolic. In particular, Dehn filling along sufficiently long curves gives a hyperbolic manifold, and the same is true for Dehn filling on links. D.Groves & J.Manning [4] and D.Osin [10] proved analogous results in the setting of hyperbolic groups, continuing the theme of results and techniques in 3-manifold theory inspiring work in geometric group theory.

The following formulation is that of Groves & Manning. It concerns any torsion-free group G that is a relatively hyperbolic with respect to finitely generated subgroups $\{P_1, \dots, P_n\}$, and a finite generating set S for G such that $S \cap P_i$ generates P_i for all i . To say G is hyperbolic with respect to $\{P_1, \dots, P_n\}$ means that if in the Cayley graph of (G, S) one suitably *cones off* every coset of every P_i then the resulting graph is hyperbolic in the sense of Gromov. The conclusion is that there is a constant C such that if $\{K_1, \dots, K_n\}$ is a family of subgroups such that $K_i \triangleleft P_i$ and the minimal length of a non-trivial element of K_i (using the word metric on P_i with respect to $S \cap P_i$) is at least C then for all i , the natural map $P_i/K_i \xrightarrow{\iota_i} G/\langle\langle K_1, \dots, K_n \rangle\rangle$ is injective and $G/\langle\langle K_1, \dots, K_n \rangle\rangle$ is hyperbolic relative to $\{\iota_1(P_1/K_1), \dots, \iota_n(P_n/K_n)\}$.

Thomas Farrell (*Binghamton University*) on work of J.Grunewald [5] and R.Ramos [11].

Let $R_\alpha[t]$ denote the twisted polynomial ring associated to a ring R and automorphism $\alpha : R \rightarrow R$. As the inclusion map $R \rightarrow R_\alpha[t]$ is a section of the map

$R_\alpha[t] \rightarrow R$ given by mapping t to 0, one can express $K_1(R_\alpha[t])$ as $K_1R \oplus Nil_\alpha(R)$. One refers to $Nil_\alpha(R)$ as the Bass–Heller-Swan group.

It is a conjecture of Farrell that if $Nil_\alpha(R)$ is non-zero then it is not a finitely generated group. This is open even in the case where R is a group ring $\mathbb{Z}G$. In his thesis, Ramos proves the conjecture in the case where α^n is the identity for some n . Additionally, he shows that $Nil_\alpha(\mathbb{Z}G) = 0$ if G is a finite group with $|G|$ square-free. Grunewald [5] has independently proved the first result.

Soren Galatius (*Stanford University*) on work of K. Cieliebak & J.Latschev.

Cieliebak & Latschev relate the string topology of any orientable manifold M to the contact homology of the unit cotangent bundle $S(T^*M)$. In so doing, they form a bridge between two important new developments in mathematics: Chas & Sullivan’s string topology, in which the homology of the free loop space is endowed with extra structure, and the symplectic field theory of Eliashberg, Givental and Hofer.

Robert Ghrist (*University of Illinois at Urbana-Champaign*) on work of A.Ames & S.Sastry [2] and M.Farber & S.Yuzvinsky [9].

Many physical systems can be modeled by *hybrid systems* – that is, cell complexes in which each cell is (compatibly) equipped with a flow that carries a particle around. A challenge is to identify rogue features known as *Zeno orbits*: orbits where particles visit infinitely many cells in finite time.

Ames & Sastry [2] formalise hybrid systems using the language of category theory and define what they call a *hybrid homology*. They prove (roughly speaking) that if the first hybrid homology of a hybrid system vanishes then there are no Zeno orbits.

The work of Farber & Yuzvinsky [9] concerns robot motion planning. Consider the problem of programming the movement of N robots on a factory floor X (or more abstractly, a path-connected topological space) from one set of positions to another without incurring any collisions. Idealising the robots as points, this amounts to selecting a path between any two given points in the configuration space $Y := C^N X$, i.e., choosing a section σ of the map $PY \rightarrow Y \times Y$ from the path space of configurations that sends a path to a pair consisting of its initial and terminal points.

Ideally, one might hope for σ to be continuous, but this can only be the case when Y is contractible. Farber & Yuzvinsky define the of topological complexity $TC(Y)$ to be the minimal k such that there is a cover $\{U_i\}_{i=1}^k$ of $Y \times Y$ by open

contractible sets, in such a way that a σ can be found that is continuous on every U_i . They calculate $\text{TC}(C^N\mathbb{R}^m)$ for $m = 2$ and all odd m , they show that $\text{TC}(Y)$ depends only on the homotopy type of Y , and they give inequalities relate $\text{TC}(Y)$ to the Lusternik–Schnirelmann category of Y .

Jesper Grodal (*University of Chicago*) on work of N.Castellana, J.A.Crespo & J.Scherer [3].

This work addresses what kind of H -spaces (spaces with a multiplication in the homotopy category) can exist. Castellana, Crespo & Scherer’s result concerns H -spaces X such that $H^*(X; \mathbb{F}_p)$ is finitely generated as an algebra over the Steenrod algebra. Note that both finite complexes and spaces with finitely many homotopy groups satisfy this finiteness assumption. Castellana, Crespo & Scherer show that the classification of such H -spaces in some sense can be reduced to that of finite H -spaces. More precisely, they show that for X as above, $X \simeq_p Y$ for some Y in an H -fibration $P \rightarrow Y \rightarrow K$ in which K is a finite H -space and P is a finite Postnikov system.

Maurice Herlihy (*Brown University*) on work of S.Rajsbaum & E.Gafari.

Many combinatorial problems of interest to computer scientists can be reduced to questions about colorability of graphs. Herlihy reported on the following theorem of Rajsbaum & Gafari. Let S be an n -simplex with $(n - 1)$ -dimensional faces S_0, \dots, S_n . Let $\sigma(S)$ be a subdivision of S which is $(n + 1)$ - colorable such that

- each vertex in $\sigma(S)$ is colored with the color of a vertex in its carrier
- each n -simplex in $\sigma(S)$ is colored with $n + 1$ distinct colors
- $\sigma(S)$ is symmetric on the boundary, i.e. $\sigma(S_i) \equiv \sigma(S_j)$ for all i, j

Let B be the boundary complex of $\sigma(S)$, and let M be any pseudomanifold whose boundary complex is B . Suppose you can color each vertex of M with a black or white pebble such that:

- the pebbling is symmetric on B
- every n -simplex of M has at least one black pebble and at least one white pebble.

Then M is not orientable.

Tara Holm (*Connecticut and Cornell Universities*) on work of A.Björner and T.Ekedahl [1].

An appeal of the following result of Björner & Ekedahl is that it is a theorem in combinatorics but their proof is topological – it uses results from intersection homology and properties of the cohomology of Kac-Moody Schubert varieties.

Let (W, S) be a crystallographic Coxeter system. For $w \in W$ define

$$f_i^w := \#\{u \in W \mid u \leq w, \ell(u) = i\}$$

where the order on W is the Bruhat order. Björner & Ekedahl prove that $f_i^w \leq f_{\ell(w)-i}^w$ for $i < (\ell(w)/2)$, and $f_0^w \leq f_1^w \leq \dots \leq f_{\lfloor \ell(w)/2 \rfloor}^w$. Moreover, equalities $f_i^w = f_{\ell(w)-i}^w$ for $i = 0, \dots, k$ imply the vanishing of the first $k + 1$ coefficients of the Kazhdan-Lusztig polynomial. Equality for $i = 0, \dots, \ell(w)/2$ reproduces the Carrel-Peterson criterion for rational smoothness.

Martin Kassabov (*Cornell University*) on work of Y.Shalom.

For $\mathrm{SL}_n(\mathbb{Z})$ and $\mathrm{SL}_n(\mathbb{F}_p[t])$, there is a sharp dichotomy between higher rank (that is, $n \geq 3$), where the groups enjoy Property T and rigidity and arithmeticity results apply, and rank 1 (that is $n = 2$), where some of these results break down. One can ask whether a similar sharp transition occurs if we replace \mathbb{Z} and $\mathbb{F}_p[t]$ by other infinite rings R .

Shalom proved that $\mathrm{SL}_n(\mathbb{Z}[x_1, \dots, x_k])$ has Property T if $n \geq k + 3$. A similar result applies for $\mathrm{SL}_n(R)$ when n is at least 2 greater than the stable range of R .

This raises the question of whether variants of rigidity and arithmeticity results apply in this higher-rank range and where the cut-offs lie.

Lee Mosher (*Rutgers University*) on work of V.Guirardel [6] .

Suppose T_1 and T_2 are \mathbb{R} -trees (e.g. simplicial trees) and G is a group acting on T_1 and T_2 by isometries. Assume that the minimal invariant subtree of T_i is dense and the actions of G on T_1 and on T_2 are not refinements of a common action. Guirardel shows that there exists a unique minimal subset, the *core*, of $T_1 \times T_2$ which is G -invariant, closed, T_1 -convex, T_2 -convex, and connected.

This geometric construction generalizes and unifies a number of concepts: intersection number for pairs of curves or pairs of measured foliations on a surface, Scott's intersection number for splittings, the apparition of surfaces in Fujiwara-Papasoglus construction of JSJ.

Nathalie Wahl (*University of Chicago*) on work of K.Igusa [7].

Building on the concept of Riedemeister torsion, Igusa defined in [8] a family of *torsion invariants* $\tau_{2k} \in H^{4k}(B, \mathbb{R})$ of bundles E of manifolds M fibering as $M \rightarrow E \rightarrow B$.

Recently [7] Igusa has been able to simplify the situation dramatically. Assume $\pi_1 B$ acts trivially on $H_*(M)$. He defines the following *Axioms for Torsion*.

1. *naturality* with respect to maps $B' \rightarrow B$.
2. *additivity*: if $E = E_1 \cup E_2$ then $\tau_k(E) = \tau(DE_1) + \tau(DE_2)$, where DE_i denotes the double of E_i along its boundary.
3. *transfer*: given fibrations $M \rightarrow E \rightarrow B$ and $S^n \rightarrow D \rightarrow E$, we find $\tau_B(D) = \tau_B(E)\chi(S^n) + tr_B^E \tau_E(D)$.

Igusa proves that these axioms determine τ up to two scalars.

References

- [1] A. Björner and T. Ekedahl. On the shape of Bruhat intervals.
arXiv:math.CO/0508022.
- [2] A. D. Ames and S. Sastry. A homology theory for hybrid systems: hybrid homology. In M. Morari and L. Thiele, editors, *Hybrid Systems: Computation and Control, 8th International Workshop, Proc.*, volume 3414 of *Lecture Notes in Computer Science*, pages 86–102. Springer-Verlag, 2005.
- [3] N. Castellana, J. A. Crespo, and J. Scherer. Deconstructing Hopf spaces.
arXiv:math.AT/0404031.
- [4] D. Groves and J. F. Manning. Dehn filling in relatively hyperbolic groups.
arXiv:math.GR/0601311.
- [5] J. Grunewald. Non-finiteness results for nil-groups.
<http://wwwmath1.uni-muenster.de/reine/u/grunewald/>.
- [6] V. Guirardel. Core and intersection number for group actions on trees.
arXiv:math.GR/0407206.
- [7] K. Igusa. Axioms for higher torsion invariants of smooth bundles.
arXiv:math.KT/0503250.

- [8] K. Igusa. *Higher Franz-Reidemeister torsion*, volume 31 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI, 2002.
- [9] M. Farber and S. Yuzvinsky. Topological Robotics: Subspace Arrangements and Collision Free Motion Planning. [arXiv:math.AT/0210115](https://arxiv.org/abs/math/0210115).
- [10] D. Osin. Peripheral fillings of relatively hyperbolic groups. [arXiv:math.GR/0510195](https://arxiv.org/abs/math/0510195).
- [11] R. Ramos. *A characterization for the twisted Nil group $NK_1^a \mathbf{ZG}$* . PhD thesis, IMU Morelia, Mexico, 2006.