Symplectic versus hyperkahler geometry

Cornell Topology Festival

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SYMPLECTIC MANIFOLDS

Closed nondegenerate 2-form ω on M^{2n}

Local model : $\sum dp_i \wedge dq_i$ on $\mathbb{R}^{2n}$

EXAMPLES

$\mathbb{C}^n$, cotangent bundles T^*N

coadjoint orbit of G (flag manifold)

Kähler manifolds:

metric g, complex structure $I : TM \to TM$

$$\omega(X,Y) = g(IX,Y).$$

HYPERKAHLER MANIFOLDS

Metric g, complex structures I, J, K with quaternionic relations

$$IJ = K = -JI \quad \text{etc}$$

Now three symplectic forms

$$\omega_1(X,Y) = g(IX,Y) \ : \ \omega_2(X,Y) = g(JX,Y)$$

$$\omega_3(X,Y) = g(KX,Y)$$

Riemannian data ; no local model

Examples?

MOMENT MAPS AND SYMPLECTIC REDUCTION

(M^{2n}, ω) with S^1 action: Killing field X

$$1 - \text{form} \quad \omega(X, .) = \iota_X \omega$$

$$0 = L_X \omega = d\iota_X \omega + \iota_X d\omega = d\iota_X \omega.$$

In good cases:

$$i_X \omega = d\mu$$

where

$$\mu : M \to \mathbb{R}$$

is S^1-invariant.

μ is the *moment map*

Now $\mu^{-1}(\epsilon)/S^1$ is a symplectic manifold of dimension $\dim M - 2$.

Example. Flat $\mathbb{C}$ with standard action of S^1

$$z \mapsto e^{i\theta} z.$$

Moment map is

$$\phi : z \mapsto |z|^2$$

Only onto half-line. Trivial fibration away from origin. Collapse at origin.

For action of general G, μ takes values in $Lie(G)^*$ and is G-equivariant. Take ϵ in centre; dim of symplectic quotient is $\dim M - 2 \dim G$.

HYPERKAHLER QUOTIENTS

Action of G on M hyperkähler. Moment map

$$\mu = (\mu_1, \mu_2, \mu_3) : M \to Lie(G)^* \otimes \mathbb{R}^3$$

Hyperkähler quotient is

$$\mu^{-1}(\epsilon_1, \epsilon_2, \epsilon_3)/G$$

of dimension $\dim M - 4 \dim G$

Example. Flat $\mathbb{H}$ with standard action of S^1

$$(z, w) \mapsto (e^{i\theta} z, e^{-i\theta} w)$$

Moment map is

$$\phi : (z, w) \mapsto (\frac{1}{2}(|z|^2 - |w|^2), \text{Re } zw, \text{Im } zw).$$

Difference of squares.

Surjective . Hopf fibration over $\mathbb{R}^3$. Collapse at origin.

Singular ϵ-locus typically codim 3 in target; no wall-crossing.

SYMPLECTIC CUTS (E. Lerman 1995).

M symplectic with circle action. Consider $M \times \mathbb{C}$ with action

$$(m, z) \mapsto (e^{i\theta}.m, e^{-i\theta}z)$$

New moment map is

$$\widehat{\mu} : (m, z) \mapsto \mu(m) - |z|^2$$

What does $\widehat{\mu}^{-1}(\epsilon)/S^1$ look like ?

For $\mu(m) - |z|^2 = \epsilon$, need

$$\mu(m) \geq \epsilon.$$

If $\mu(m) > \epsilon$, use circle action to rotate z to real $+\sqrt{\mu(m) - \epsilon}$.

If $\mu(m) = \epsilon$, then $z = 0$ so can use action to change m.

Symplectic quotient $\widehat{\mu}^{-1}(\epsilon)/S^1$ can be identified with

$$\{m \in M : \mu(m) > \epsilon\} \cup \mu^{-1}(\epsilon)/S^1$$

Ie we have cut out "half" the manifold and quotiented the boundary of what is left.

This is the SYMPLECTIC CUT of M at level ϵ.

HYPERKAHLER ANALOGUE ? (D-Swann)

Hyperkähler M with S^1 action. Consider $M \times \mathbb{H}$ with action

$$(m, (z, w)) \mapsto (e^{i\theta}.m, (e^{-i\theta}z, e^{i\theta}w)).$$

New moment map is

$$\widehat{\mu} : (m, (z, w)) \mapsto \mu(m) - \phi(z, w)$$

where ϕ was moment map for action on $\mathbb{H}$.

What does $M_{\text{mod}} = \widehat{\mu}^{-1}(\epsilon)/S^1$ look like?

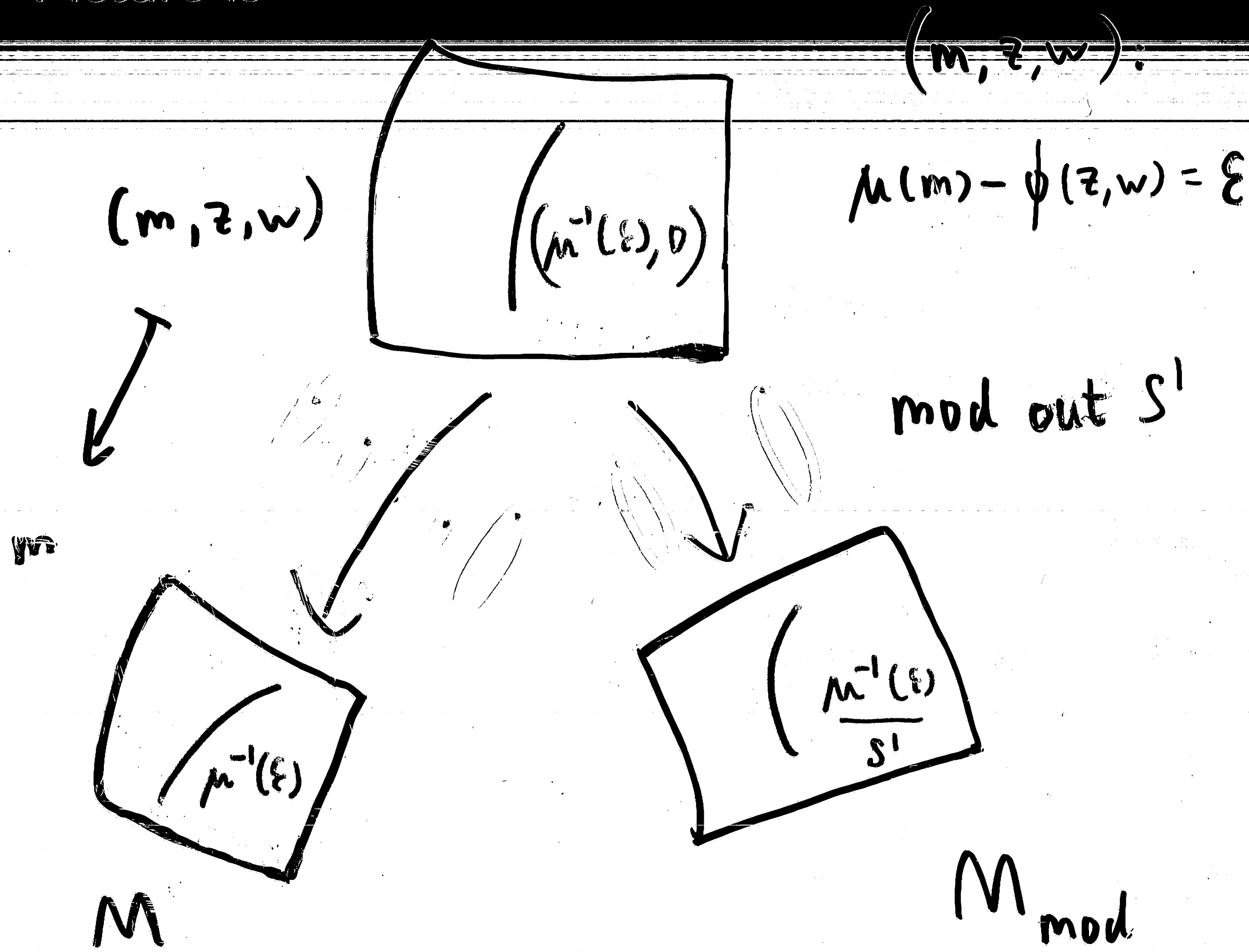
$$\mu(m) - \phi(z, w) = \epsilon$$

But ϕ is now ONTO $\mathbb{R}^3$ so this is always solvable for z, w. No restriction on m, so do not cut out any part of M.

Over $\mu^{-1}(\epsilon)$, fibre of ϕ is just 0, so we get a copy of $\mu^{-1}(\epsilon)/S^1$ in M_{mod}.

CANNOT now identify $M - \mu^{-1}(\epsilon)$,with $M_{\text{mod}} - \mu^{-1}(\epsilon)/S^1$ as ϕ gives a NONTRIVIAL (Hopf) fibration.

Picture is



When we form M_{mod} from M we are adding a new "brane" : codimension 4 hyperkähler submanifold $\mu^{-1}(\epsilon)/S^1$:

Also, the long-range topology of M is getting a twist (unlike the symplectic case).

Nonabelian cuts?

Reduce $M \times X$ by G where X has $G \times G$ action and $\dim X = 2\dim G$ (resp. $4\dim G$).

Moment geometry of X determines geometry of cut

Weitsman: $G = U(n)$

$$X = \mathrm{hom}(\mathbb{C}^n, \mathbb{C}^n) \quad : \quad A \mapsto UAV^{-1}$$

$$\phi_{\mathrm{Right}} : A \mapsto A^* A$$

trivial $U(n)$-fibration over nonnegative matrices, collapsing on boundary.

Fibres are $U(n)_{\mathrm{Left}}$ orbits. Polar decomposition gives section.

Gives description of cutting M, analogous to $n = 1$.

Hyperkähler case?

$X = \text{hom}(\mathbb{C}^n, \mathbb{C}^n) \oplus \text{hom}(\mathbb{C}^n, \mathbb{C}^n)$

Action is: $(A, B) \mapsto (UAV^{-1}, VBU^{-1})$

$\phi_{\text{Right}} : (A, B) \mapsto (A^*A - BB^*, BA)$

Some fibres are not $U(n)_{\text{Left}}$-orbits (contrast with $n = 1$)

$\phi^{-1}(0,0) : A = UP, B = PV$, rank $P \leq \frac{1}{2}n$.

Implosion

Implosion of M is "abelianisation" :

$M//_\lambda G = M_{\text{impl}}//_\lambda T$

Universal example: $M = T^*G$ so want

$M_{\text{impl}}//_\lambda T = \mathcal{O}_\lambda$ coadjoint orbit

Example. $G = SU(2)$, $\mathcal{O}_\lambda = S^2$ or $*$, so

$M_{\text{impl}} = \mathbb{C}^2$.

In general: take $G \times \mathfrak{t}^*_+$ and collapse by commutator of stabiliser of $t \in \mathfrak{t}^*_+$.

eg for $SU(2)$ take $SU(2) \times [0, \infty)$ and collapse by $SU(2)$ at origin, to obtain $\mathbb{C}^2$.

Algebro-geometric description

$G_{\mathbb{C}}//N$: N maximal unipotent

eg for $SU(2)$ we have $SL(2,\mathbb{C})//N$

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \mapsto \begin{pmatrix} x_{11} & x_{12}+nx_{11} \\ x_{21} & x_{22}+nx_{21} \end{pmatrix}$$

Invariants x_{11}, x_{21}, so $SL(2,\mathbb{C})//N = \mathbb{C}^2$.

HK version? (Kirwan-D) work in progress:

$M = T^*G_{\mathbb{C}}$. Want

$M_{\text{impl}}///_\lambda T \sim$ complex coadjoint orbit

Consider complex-symplectic quotient by N ; this is

$(G_{\mathbb{C}} \times \mathfrak{b})//N$

Top stratum :

open set in $G_{\mathbb{C}} \times_N \mathfrak{b}$

$=$ open set in $G_{\mathbb{C}} \times \mathfrak{t}_{\mathbb{C}}$

Bielawski monopole spaces

eg $G = SU(2)$ get $\mathbb{C}^4 = \mathbb{H}^2$. Correct !

For $SU(n)$, link with quiver varieties.

In general torus reductions will give :

Kostant varieties (level sets of the collection of invariant polynomials)

eg regular semisimple orbits

in general, closure of a regular orbit. Union of orbits. Semisimple orbit is lowest stratum

eg nilpotent variety : semisimple stratum is zero matrix