Floer Theory for Lagrangian submanifolds

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(M, ω) (closed) symplectic manifold

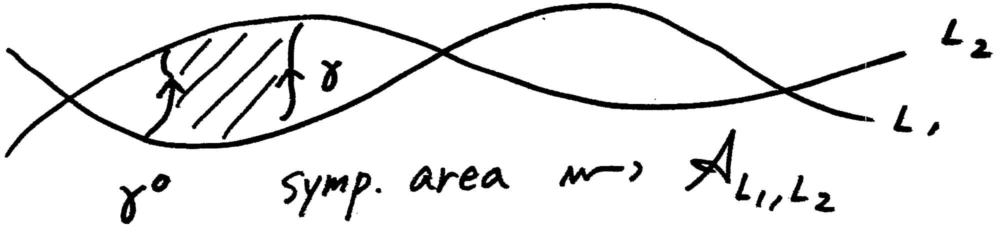
$L_1, L_2 \subset M$ embedded Lagrangian submanifolds

$L_1 \pitchfork L_2$

$$\mathcal{P}(L_1, L_2) := \{ \gamma : [0,1] \longrightarrow M \mid \gamma(0) \in L_1,\ \gamma(1) \in L_2 \}$$

- $$\begin{tikzcd} \tilde{\mathcal{P}}(L_1, L_2) \arrow[r, "\mathcal{A}_{L_1,L_2}"] \arrow[d, "\pi"'] & \mathbb{R} \\ \mathcal{P}(L_1, L_2) & \end{tikzcd}$$ $\mathrm{Crit}\,\mathcal{A}_{L_1,L_2} = \pi^{-1}(L_1 \cap L_2)$

- $\mathrm{Crit}(\mathcal{A}_{L_1,L_2}) \longrightarrow \mathbb{Z}$ Maslov-Viterbo index



$CF^{\cdot}(L_1, L_2)$ graded "free module" generated by $\mathrm{Crit}\,\mathcal{A}_{L_1,L_2}$

Need to take completion w.r.t. $\mathcal{A}_{L_1,L_2}$

Pick J almost complex structure compatible with ω

"gradient flow lines" $\gamma : \mathbb{R} \longrightarrow \overset{(\sim)}{\Phi}(L_1, L_2)$

$$\updownarrow$$

$$u : \underset{(\tau}{\mathbb{R}} \times \underset{,\ t)}{[0,1]} \longrightarrow M$$

$$\begin{cases} \dfrac{\partial u}{\partial \tau} + J(u)\dfrac{\partial u}{\partial t} = 0 \\ u(\tau, 0) \in L_1, \quad u(\tau, 1) \in L_2 \\ u(\tau, t) \longrightarrow p^{\pm} \quad (\tau \to \pm\infty) \end{cases}$$

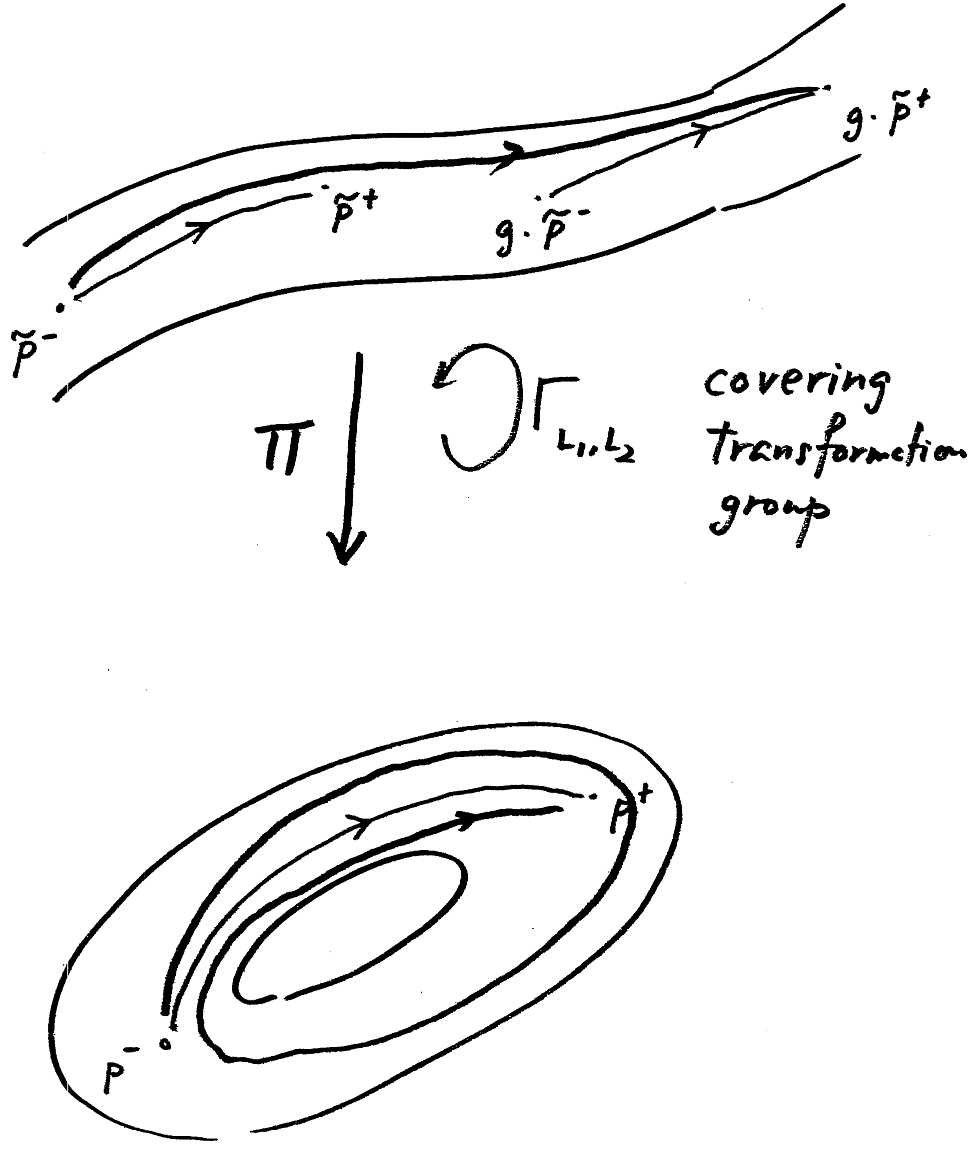
$$p^{\pm} \in L_1 \cap L_2$$

$$\delta : CF^{\cdot}(L_1, L_2) \longrightarrow CF^{\cdot+1}(L_1, L_2)$$

$$\delta\, \tilde{p}^- := \sum \# \mathcal{M}_J(\tilde{p}^-; \tilde{p}^+)\, \tilde{p}^+$$

$\mathcal{M}_J(\tilde{p}^-; \tilde{p}^+)$

= moduli space of "gradient flow lines" from $\tilde{p}^-$ to $\tilde{p}^+$



- group ring of Γ_{L_1, L_2}

$$\Big\downarrow \text{ completion}$$

Λ_{L_1, L_2} Novikov ring

Hope :

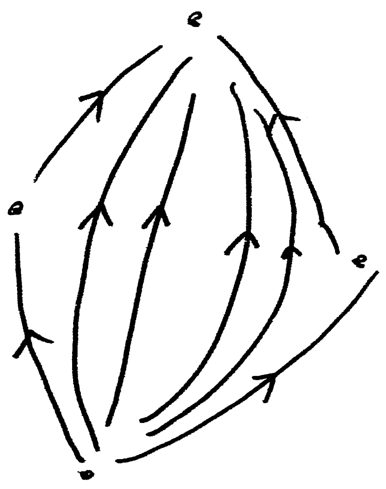
- $\delta \circ \delta = 0$

  Then define $HF^{\cdot}(L_1, L_2) = \mathrm{Ker}\,\delta / \mathrm{Im}\,\delta$

- $HF^{\cdot}(L_1, L_2)$ independent of J, perturbation, etc.
- invariance under Hamiltonian deformations.

Need :

compactness up to splitting phenomena



Bubbling - off phenomena :

- sphere bubbles "complex codim 1"

  we can exclude such phenomena
  using multi-valued perturbation in Kuranishi str.

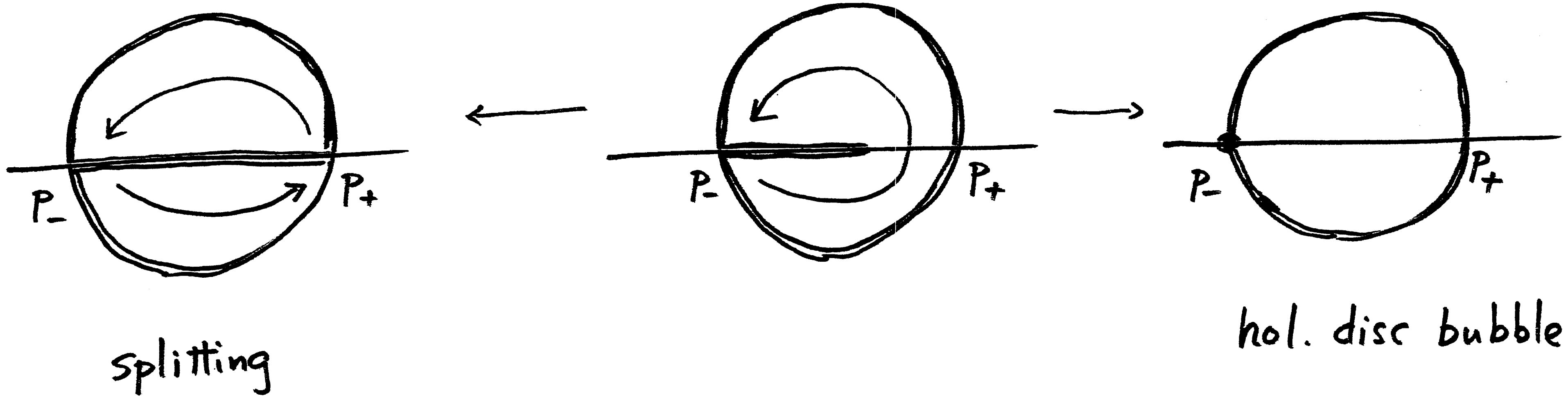
- disc bubbles "real codim 1"

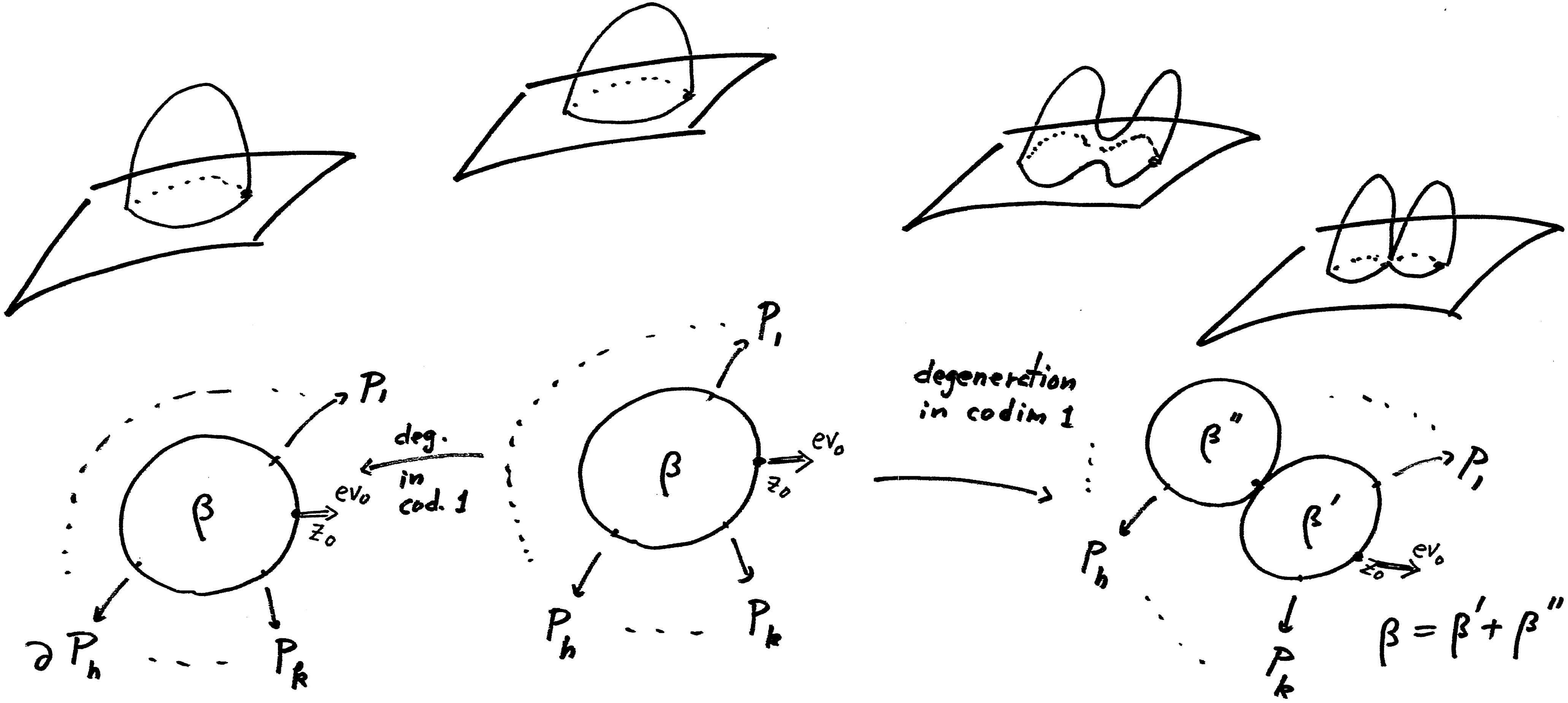
$\rightsquigarrow$ systematic study on holomorphic discs

$\rightsquigarrow$ filtered A_∞- algebras
filtered A_∞- bimodules

- multi-valued perturbation → need orientation on moduli spaces of hol. discs

  relative spin structure


P_
P+
splitting
P_
P+
P_
P+
hol. disc bubble



- $m_{1,0} \; (= \overline{m}_1) = \pm \partial$

$$\sum_{\beta=\beta'+\beta''} \sum_{1 \leqq i \leqq j \leqq k} m_{k-j+i-1,\beta'}(P_1, \cdots, m_{j-i,\beta''}(P_{i+1}, \cdots, P_j), \cdots, P_k) = 0$$

$(\beta'' = 0 \Rightarrow i < j)$

$(C^\cdot, d, \cdot)$ differential graded algebra

$$d : C^p \longrightarrow C^{p+1}$$
$$\cdot : C^p \otimes C^q \longrightarrow C^{p+q}$$

- $\star$ $d^2 = 0$
- $\star$ $d(a \cdot b) = (da) \cdot b + (-1)^{\deg a} a \cdot db$
- $\star$ $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

$C[1]^\cdot$ $\quad C[1]^p := C^{p+1}$ degree shift $\quad \deg' = \deg - 1$

$$d \rightsquigarrow \overline{m}_1(a) := (-1)^{\deg a} da$$
$$\cdot \rightsquigarrow \overline{m}_2(a, b) := (-1)^{\deg a (\deg b + 1)} a \cdot b$$

Extend d, $\cdot$ to graded coderivation on $BC[1] := \bigoplus_{k=0}^{\infty} \underbrace{C[1] \otimes \cdots \otimes C[1]}_{k}$

$$\widehat{m}_1(x_1, \cdots, x_k) := \sum (-1)^{\sum_{j=1}^{i-1} \deg' x_j} x_1 \otimes \cdots \otimes x_{i-1} \otimes \overline{m}_1(x_i) \otimes x_{i+1} \otimes \cdots \otimes x_k$$
$$\widehat{m}_2(x_1, \cdots, x_k) := \sum (-1)^{\sum_{j=1}^{i-1} \deg' x_j} x_1 \otimes \cdots \otimes x_{i-1} \otimes \overline{m}_2(x_i, x_{i+1}) \otimes x_{i+2} \otimes \cdots \otimes x_k$$

$$d \circ d = 0 \qquad \Longrightarrow \qquad \overline{m}_1 \circ \overline{m}_1 = 0$$

$$d(a \cdot b) = (da) \cdot b + (-1)^{\deg a} a \cdot db$$

$$\Longrightarrow \quad \overline{m}_1 \circ \overline{m}_2 (a \otimes b) + \overline{m}_2 \circ (\overline{m}_1 \otimes 1)(a \otimes b) + \overline{m}_2 \circ (1 \otimes \overline{m}_1)(a \otimes b) = 0$$

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

$$\Longrightarrow \quad \overline{m}_2 \circ (\overline{m}_2 \otimes 1)(a \otimes b \otimes c) + \overline{m}_2 \circ (1 \otimes \overline{m}_2)(a \otimes b \otimes c) = 0$$

$$\mathbb{d} := \widehat{m}_1 + \widehat{m}_2 \ : \ BC[1] \longrightarrow BC[1]$$

$$\Longrightarrow \quad \mathbb{d}^2 = 0 .$$

A_∞- algebra

$\bar{C}^\bullet$ graded module. $\bar{C}[1]^p := \bar{C}^{p+1}$

$\bar{m}_k : \bar{C}[1]^\bullet \otimes \cdots \otimes \bar{C}[1]^\bullet \longrightarrow \bar{C}[1]^\bullet$ (deg 1) $k = 1, 2, \cdots$

$(C^\bullet, \{\bar{m}_k\}_{k=1,2,\cdots})$ A_∞- algebra

$\underset{\text{def.}}{\Longleftrightarrow}$ $\hat{d} \circ \hat{d} = 0$

$\hat{d} = \sum_{k=1}^{\infty} \widehat{\bar{m}}_k$, $\widehat{\bar{m}}_k : BC[1]^\bullet \longrightarrow BC[1]^\bullet$ the extension of $\bar{m}_k$ as coderivation

- intersection theory in chain level:

transversality fails

$\bar{m}_1$ = usual boundary operator (up to sign)

$\bar{m}_2$ = intersection of chains <u>after perturbation</u>

associativity fails, but holds up to homotopy

A few formulae

$\overline{\mathfrak{m}}_1 \circ \overline{\mathfrak{m}}_1 = 0$

$\overline{\mathfrak{m}}_1 \circ \overline{\mathfrak{m}}_2 + \overline{\mathfrak{m}}_2 \circ (\overline{\mathfrak{m}}_1 \otimes id. \pm id. \otimes \overline{\mathfrak{m}}_1) = 0$

$\overline{\mathfrak{m}}_1 \circ \overline{\mathfrak{m}}_3 + \overline{\mathfrak{m}}_2 \circ (\overline{\mathfrak{m}}_2 \otimes id. \pm id. \otimes \overline{\mathfrak{m}}_2) + \overline{\mathfrak{m}}_3 \circ (\overline{\mathfrak{m}}_1 \otimes id. \otimes id. \pm id. \otimes \overline{\mathfrak{m}}_1 \otimes id. \pm id. \otimes id. \otimes \overline{\mathfrak{m}}_1) = 0$

...

filtered case: $1 \in \Lambda_{0,nov} \subset B(C[1] \otimes \Lambda_{nov})$

$\mathfrak{m}_1 \circ \mathfrak{m}_0 = 0$

$\mathfrak{m}_1 \circ \mathfrak{m}_1 + \mathfrak{m}_2 \circ (\mathfrak{m}_0(1) \otimes id. \pm id. \otimes \mathfrak{m}_0(1)) = 0$

$\mathfrak{m}_1 \circ \mathfrak{m}_2 + \mathfrak{m}_2 \circ (\mathfrak{m}_1 \otimes id. \pm id. \otimes \mathfrak{m}_1) + \mathfrak{m}_3 \circ (\mathfrak{m}_0(1) \otimes id. \otimes id. \pm id. \otimes \mathfrak{m}_0(1) \otimes id. \pm id. \otimes id. \otimes \mathfrak{m}_0(1)) = 0$

....

$\overline{m}_1(P) = \pm \partial P$

$\overline{m}_2(P_1, P_2) =$ intersection of perturbed P_1, P_2

$\overline{m}_k(P_1, \cdots, P_k)$ is defined by parametrized family of perturbation of the diagonal depending on $P_1, \cdots P_k$

The resulting A_∞-algebra is "homotopy equivalent" to (degree shifted) de Rham DGA.

filtered A_∞-algebra

$$\Lambda_{nov} := \{ \sum a_i e^{n_i} T^{\lambda_i} \mid a_i \in \mathbb{Q},\ n_i \in \mathbb{Z},\ \lambda_i \to +\infty \}$$

$$\Lambda_{0,nov} := \{ \sum a_i e^{n_i} T^{\lambda_i} \in \Lambda_{nov} \mid \lambda_i \geqq 0 \}$$

$$\Lambda_{+,nov} := \{ \sum a_i e^{n_i} T^{\lambda_i} \in \Lambda_{nov} \mid \lambda_i > 0 \}$$

grading $\deg e = 2 \qquad \deg T = 0$

(In the second part of this talk, we omit e.)

$$C := \bar{C}^{\cdot} \otimes \Lambda_{0,nov}$$

$m_k : C[1]^{\cdot} \otimes \cdots \otimes C[1]^{\cdot} \longrightarrow C[1]^{\cdot}$ (degree 1), $k = \underline{\underline{0}}, 1, 2, \ldots$

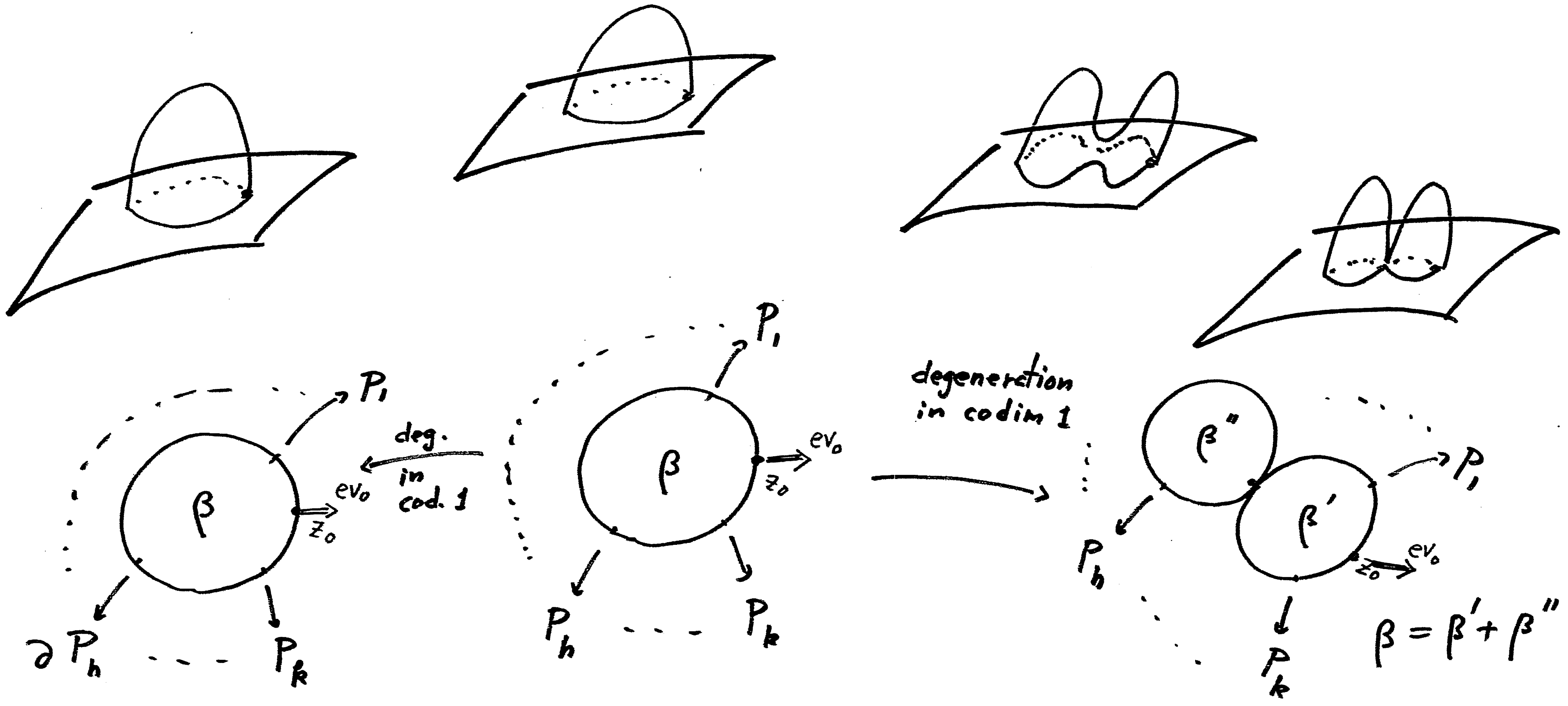
$m_0(1) \in \bar{C}^{\cdot} \otimes \Lambda_{\underline{\underline{+}},nov}$

[In fact, we assume "gapped conditions" in the argument.]

$\widehat{m}_k : BC[1] \longrightarrow BC[1]$

$(C, \{m_k\}_{k=0,1,2,\cdots})$ filtered A_∞-algebra $\underset{def}{\Longleftrightarrow}$ $\hat{d} \circ \hat{d} = 0$

$$\hat{d} = \sum_{k=}^{\infty} \widehat{m}_k$$



- $\mathfrak{m}_{1,0} \ (= \overline{\mathfrak{m}}_1) = \pm \partial$

$$\sum_{\beta = \beta' + \beta''} \sum_{1 \leqq i \leqq j \leqq k} \mathfrak{m}_{k-j+i-1, \beta'}(P_1, \cdots, \mathfrak{m}_{j-i, \beta''}(P_{i+1}, \cdots, P_j), \cdots, P_k) = 0$$

$(\beta'' = 0 \Rightarrow i < j)$

A_∞- relations

$$\Rightarrow \quad m_1 \circ m_0(1) = 0$$

$$m_1 \circ m_1(P) \pm m_2(m_0(1), P) \pm m_2(P, m_0(1)) = 0$$

When $m_0(1) \neq 0$, $\quad m_1 \circ m_1 = 0$ may not hold.

- Maurer - Cartan equation

$$b \in (\overline{C}[1]^{\cdot} \otimes \Lambda_{+,nov})^0 \qquad e^b = 1 + b + b \otimes b + b \otimes b \otimes b + \cdots$$

$$\hat{d}(e^b) = 0 \quad (\Longleftrightarrow \underbrace{m_0(1) + m_1(b) + m_2(b,b) + \cdots}_{m(e^b)} = 0,$$

$\star$ b sol. of M.-C. eq. $\Rightarrow$ can deform m_k to m_k^b
so that $m_0^b(1) = 0$.

$$\Rightarrow \quad m_1^b \circ m_1^b = 0$$

$b \in (C[1]^{\cdot} \otimes \Lambda_{+,nov})^{0}$

$$\begin{tikzcd}
BC[1] \arrow[r, "\widehat{m}_k"] \arrow[d, "\Phi^b"'] & BC[1] \arrow[d, "\Phi^b"'] \\
BC[1] \arrow[r, "\widehat{m}_k^b"] & BC[1]
\end{tikzcd}$$

$$m_k(P_1, \cdots, P_k) := \sum m_{k+\ell}(b, \cdots, b, P_1, b, \cdots \cdots, b, P_i, b, \cdots \cdots, b, P_k, b, \cdots, b)$$

$$m_k^b \rightsquigarrow \widehat{m}_k^b \rightsquigarrow \hat{d}^b$$

$\hat{d}^b \circ \hat{d}^b = 0$ deformation of filtered A_∞-str.

- (filtered) A_∞- homomorphism.
- homotopy theory based on "model $[0,1] \times C^\bullet$".
- Whitehead type theorem
- canonical model theorem

  ⋮

- deformation of (filtered) A_∞-algebra — governed by a certain L_∞-algebra.
  (DGLA)

- unit

$\mathbb{e} \in C[1]^{-1} \ (= C^0)$ unit

$$\Longleftrightarrow \begin{cases} \mathfrak{m}_2(\mathbb{e}, P) = (-1)^{\deg' P} \mathfrak{m}_2(P, \mathbb{e}) = P \\ \mathfrak{m}_k(\cdots, \mathbb{e}, \cdots) = 0 \qquad (k \neq 2) \end{cases}$$

- homotopy unit

$\mathbb{e}' \in C[1]^{-1}$ homotopy unit

$\Longleftrightarrow \quad C^+ = C \oplus \langle \mathbb{e}, \mathbb{f} \rangle$ filtered A_∞-alg.

$$\begin{cases} \mathbb{e} \text{ unit in } C^+ \\ \delta^+ \mathbb{f} = \mathbb{e} - \mathbb{e}' \quad \cdots \end{cases}$$

- $\mathcal{M}((C,\{m_k\})) = \{\text{sol. of M.C. eq.}\} \Big/ \underset{\text{“gauge eq”}}{\sim}$

<u>unital case</u>

- $\mathcal{M}_{\text{weak}}((C,\{m_k\})) = \{b \in (C[1] \otimes \Lambda_{+,nov})^0 \mid m(e^b) = {}^{\exists}\underset{A \in \Lambda_{+,nov}}{A} \cdot \mathbf{e}\} \Big/ \sim$

$$b \in \mathcal{M}_{\text{weak}}((C,\{m_k\})) \xrightarrow{\mathfrak{PO}} \Lambda_+ \qquad \text{potential function}$$

$$\mathfrak{PO}(b) \in \Lambda_{+,nov} \;:\; m(e^b) = \mathfrak{PO}(b) \cdot \mathbf{e}$$

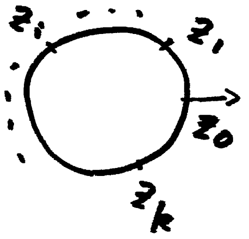
$L^n \subset M^{2n}$ embedded Lagrangian submanifold
equipped with rel. spin str.

$C^{\cdot}(L)$ suitable subcomplex of singular chain complex of L

$C^p(L) := C_{n-p}(L) \subset S_{n-p}(L)$

$\beta \in \mathrm{Im}(\pi_2(M,L) \to H_2(M,L))$

“$\mathcal{M}_{k+1}(\beta) = \Big\{$



$\xrightarrow{u} M \;\Big|\; \begin{matrix} J\text{-hol} \\ u(\partial D) \subset L \end{matrix} \Big\}$”

bordered stable maps of genus 0

$P_1, \cdots, P_k \in C^{\cdot}(L)$

$$m_{k,\beta}(P_1, \cdots, P_k) = \underset{\substack{\uparrow \\ \text{with appropriate sign}}}{\pm}(ev_0 : \mathcal{M}_{k+1}(\beta) \times_{L \times \cdots \times L} (P_1 \times \cdots \times P_k) \longrightarrow L)$$

$$\mathfrak{m}_k := \sum_{\beta} \mathfrak{m}_{k,\beta}$$

$$\rightsquigarrow \widehat{\mathfrak{m}}_k$$

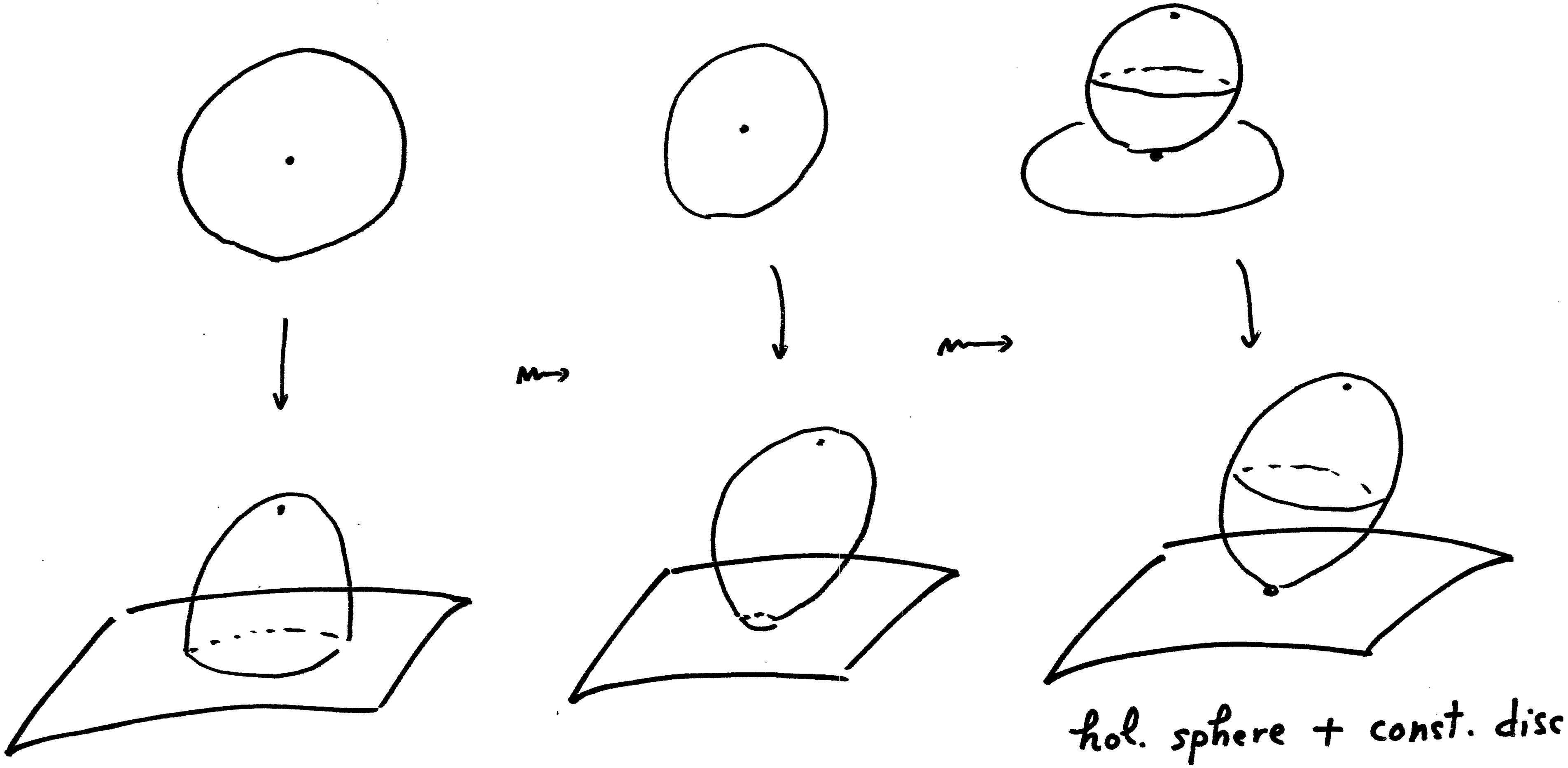
<u>Thm</u> : $(C(L), \{\mathfrak{m}_k\}_{k=0,1,2\cdots})$ filtered A_∞-algebra.

- homotopy type is uniquely determined.
- In canonical model, P.D. $[L]$ is a unit.

<u>Thm</u>: $L \subset M$ rel. spin embedded Lagr. submfd

$H^*(M : \mathbb{Q}) \longrightarrow H^*(L : \mathbb{Q})$ surjective

$\Longrightarrow \widehat{\mathcal{M}}_{\text{week,def}}(L) \neq \phi$



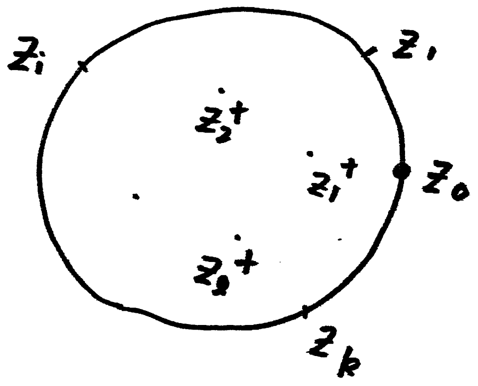
The unit disc with one interior point is not stable.
no

Bulk / boundary deformation (infinitesimal deformation)

- $b \in (C(L)[1] \otimes \Lambda_{+,nov})^0 \rightsquigarrow \mathfrak{m}_k^b$

- $C_1, \cdots, C_\ell$ chains in M

$$\mathfrak{q}(C_1, \cdots, C_\ell ; P_1, \cdots, P_k) := \pm ev_0 \mathcal{M}_{k+1,\ell}(\beta) \times_{M \times \cdots \times M \times L \times \cdots \times L} (Q_1 \times \cdots \times Q_\ell \times P_1 \times \cdots \times P_k)$$



$\mathbf{b} \in C(M) \otimes \Lambda_{+,nov}$ cycle

$$\mathfrak{m}_k^{\mathbf{b}}(P_1, \cdots, P_k) := \sum \mathfrak{q}(\mathbf{b}, \cdots, \mathbf{b} ; P_1, \cdots, P_k)$$

$b \in (C(L)[1] \otimes \Lambda_{+,nov})^{0}$, $\mathbb{b} \in C(M) \otimes \Lambda_{+,nov}$ cycle

$$\mathfrak{m}_{k}^{\mathbb{b},b}(P_1, \cdots, P_k) = \mathfrak{q}(e^{\mathbb{b}}, e^{b}, P_1, e^{b}, \cdots, e^{b}, P_k, e^{b})$$

- $\mathfrak{m}_{0}^{\mathbb{b},b} = 0 \iff \mathfrak{q}(e^{\mathbb{b}}, e^{b}) = 0$

$$\widehat{\mathcal{M}}_{def.} = \{ (\mathbb{b}, b) \mid \mathfrak{m}_{0}^{\mathbb{b},b} = 0 \}$$

$$\mathfrak{PO} : \widehat{\mathcal{M}}_{weak,def} \longrightarrow \Lambda_{0,nov}$$

by $\mathfrak{m}_{0}^{\mathbb{b},b}(1) = \mathfrak{PO}(\mathbb{b}, b) \cdot \mathbf{e}$

- $\psi : (M, L) \longrightarrow (M', L')$ symplectomorphism

$$\psi_{*} : \widehat{\mathcal{M}}_{weak,def}(L) \longrightarrow \widehat{\mathcal{M}}_{weak,def}(L')$$

$$\mathcal{M}_{(weak)}(L) \longrightarrow \mathcal{M}_{(weak)}(L')$$

<u>Thm</u>: $L \subset M$ rel. spin embedded Lagr. submfd

$H^*(M:\mathbb{Q}) \longrightarrow H^*(L:\mathbb{Q})$ surjective

$\Longrightarrow \widehat{\mathcal{M}}_{weak,def}(L) \neq \phi$

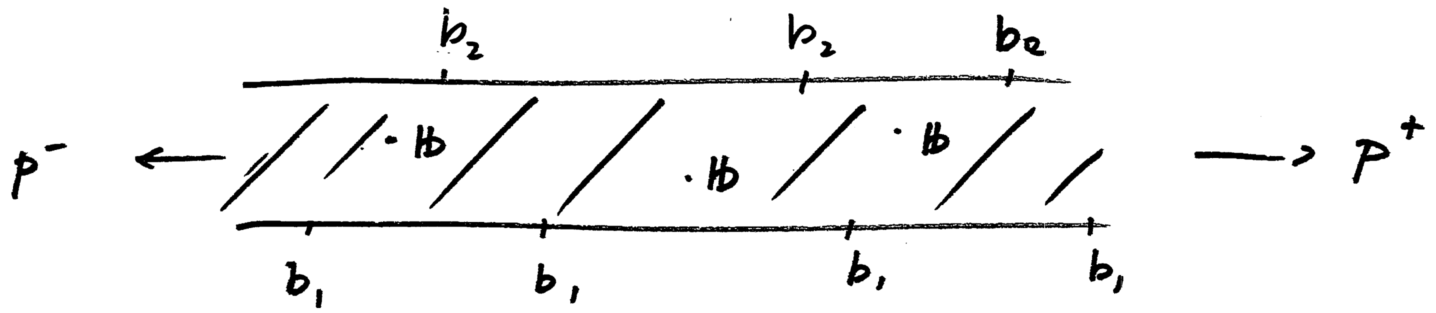
(L_1, L_2) rel. spin pair of Lag. submfds
$L_1 \pitchfork L_2$ (or clean intersection)

$(\mathfrak{b}, b_1) \in \hat{\mathcal{M}}_{\mathrm{weck,def}}(L_1)$

$(\mathfrak{b}, b_2) \in \hat{\mathcal{M}}_{\mathrm{weak,def}}(L_2)$

$$\mathfrak{PO}(\mathfrak{b}, b_1) = \mathfrak{PO}(\mathfrak{b}, b_2)$$

$$\Rightarrow \quad \delta^{(\mathfrak{b},b_1),(\mathfrak{b},b_2)} : CF^{\cdot}(L_1, L_2) \longrightarrow CF^{\cdot}(L_1, L_2)$$



- $\delta^{(\mathfrak{b},b_1),(\mathfrak{b},b_2)} \circ \delta^{(\mathfrak{b},b_1),(\mathfrak{b},b_2)} = 0$

$HF^{\cdot}((L_1,(\mathfrak{b},b_1)),(L_2,(\mathfrak{b},b_2)))$ defined.
invariant under Hamiltonian deformations

Examples   Lagrangian torus fibers of Toric manifolds

$$M \xrightarrow{\mu} \mathfrak{t}^* \qquad \text{moment map}$$

cpt toric mfd

$\mu^{-1}(a) \cong T^n$ · displacable under Hamiltonian deformation ?

$(a \in \mathrm{Int}\,\Delta)$ · If displacable, how much Hofer energy necessary ?

Ex 0 $M = \mathbb{C}P^1 \longrightarrow \mathbb{R}$

Ex 1   Clifford torus $\subset \mathbb{C}P^n \longrightarrow \mathbb{R}^n$

Cho ,   Entov - Polterovich

Floer theory for Lag. submfds   quasi-morphism, quasi-state, …, partial symplectic quasi-state

Cho - Oh : · Description of holomorphic discs in toric mfd
with bdry on torus fiber.

· Transversality

(In general, toric mfd contains hol. spheres, which
are not transversal.)

Study on balanced fibers in toric Fano mfds.

$$\Delta \subset t^* \cong \mathbb{R}^n$$

$\underset{u}{\psi}$ coordinates $(u_1, \cdots, u_n)$

$$L(u) := \mu^{-1}(u)$$

$$\bigstar \quad H^1(L(u); \Lambda_0) \hookrightarrow \mathcal{M}_{weak}(L(u) : \Lambda_0)$$

$$\mathfrak{PO}\big|_{H^1(L(u): \Lambda_0)} : \underset{\substack{\text{coordinates} \\ (x_1, \cdots, x_n)}}{H^1(L(u): \Lambda_0)} \longrightarrow \Lambda_0$$

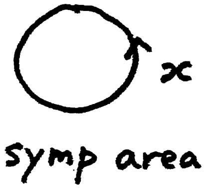
$$\mathfrak{PO}(x_1, \cdots, x_n ; u_1, \cdots, u_n) = \mathfrak{PO}^u(x_1, \cdots, x_n)$$

$H^1(L(u): \Lambda_+) \hookrightarrow \mathcal{M}_{weak}(L(u) : \Lambda_+)$ FOOO

$H^1(L(u)) \hookrightarrow \mathcal{M}_{weck}(L(u) : \Lambda_0)$ Cho

$M = \mathbb{C}P^1 \longrightarrow \mathbb{R}$

$\mathfrak{PO}(x;u) = e^{x} T^{u} + e^{-x} T^{1-u}$



Set $y := e^{x}$

$$\mathfrak{PO}^{u}(y) := \mathfrak{PO}(x;u)$$
$$= y T^{u} + y^{-1} T^{1-u}$$

$$\frac{\partial \mathfrak{PO}^{u}}{\partial y} = T^{u} - y^{-2} T^{1-u}$$

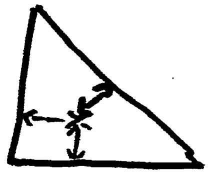
$u \neq \frac{1}{2} \Rightarrow$ y sol. of $\frac{\partial \mathfrak{PO}^{u}}{\partial y} = 0$

$$\log y = \frac{1-2u}{2} \log(\pm T) \notin \Lambda_0^{\mathbb{C}}$$

$u = \frac{1}{2} \Rightarrow$ $y = \pm 1$

$M = \mathbb{C}P^n \longrightarrow \mathbb{R}^n \qquad \Delta = \{(u_1, \cdots, u_n) \mid 0 \leqslant u_i,\ u_1 + \cdots + u_n \leqslant 1\}$

$$\mathfrak{PO}(x_1, \cdots, x_n \,;\, u_1, \cdots . u_n) = \sum_{i=1}^{m} e^{x_i} T^{u_i} + e^{-\Sigma x_i} T^{1-\Sigma u_i}$$



$$u = u_0 = \left(\frac{1}{n+1}, \cdots, \frac{1}{n+1}\right)$$

$\Rightarrow \quad \dfrac{\partial \mathfrak{PO}^{u_0}}{\partial y_i} = 0 \qquad i = 1, \cdots . n$ has solutions

$$y_1 = \cdots = y_n = \exp\left(\frac{2\pi k\sqrt{-1}}{n+1}\right) \qquad k = 0, 1, \cdots . n$$

Thm : M n-dim cpt toric mfd

$\exists$ Lagrangian torus fiber $L(u_0)$ s.T.

$$\psi(L(u_0)) \cap L(u_0) \neq \phi \quad \forall \psi \in Ham(M)$$

Furthermore, if $\psi(L(u_0)) \pitchfork L(u_0)$,

then $\# \ \psi(L(u_0)) \cap L(u_0) \geqq 2^n$

Fano case

Thm M Fano toric mfd , $u \in \mathrm{Int}\,\Delta$

$$\exists\ \psi_u : QH(M ; \Lambda) \xrightarrow{\cong} \mathrm{Jac}(\mathfrak{PO}^u)$$

$$c_1(M) \longmapsto \mathfrak{PO}^u$$

Givental, …

$$\mathrm{Jac}(\mathfrak{PO}^u) := \Lambda[y_1, \cdots, y_n, y_1^{-1}, \cdots, y_n^{-1}] \Big/ \left\langle \frac{\partial \mathfrak{PO}^u}{\partial y_i} \right\rangle_{i=1,\cdots,n}$$

Jacobian ring

Batyrev quantum cohom. ring : $QH^{\omega}(M : \Lambda)$

"linear rel"

"quantum Stanley-Reisner rel"

M cpt toric mfd $\qquad \Delta$ moment polytope

$$\mathfrak{M}(\mathcal{L}ag\, M) := \left\{ (u, \mathfrak{x}) \in \operatorname{Int}\Delta \times H^1(L(u): \Lambda_0^{\mathbb{C}}) \big/ H^1(L(u): 2\pi\sqrt{-1}\,\mathbb{Z}) \;\middle|\; \begin{array}{l} \cdot\ \mathfrak{x} \in \mathcal{M}_{\text{weak}}(L(u)) \,, \\ \cdot\ HF((L(u), \mathfrak{x}) ; \Lambda^{\mathbb{C}}) \neq \{0\} \end{array} \right\}$$

$$\operatorname{Spec}(QH(X,\Lambda))(\Lambda^{\mathbb{C}}) := \{ \varphi : QH(X,\Lambda) \longrightarrow \Lambda^{\mathbb{C}} \mid \Lambda\text{-alg hom.} \}$$

Thm $\quad M \quad$ cpt **Fano** toric mfd

$$\Rightarrow \quad \mathfrak{M}(\mathcal{L}ag\, M)) \cong \operatorname{Spec}(QH(X,\Lambda))(\Lambda^{\mathbb{C}})$$

Furthermore, if $QH(X,\Lambda)$ is semi-simple,

$$\# \mathfrak{M}(\mathcal{L}ag\, M) = \sum_d \operatorname{rk}_{\mathbb{Q}} H_d(X : \mathbb{Q}) .$$

General case

$$\mathfrak{PO} = \underset{\text{leading term}}{\mathfrak{PO}_0} + \text{"higher"} \quad (\text{in some sense})$$

$$\text{Fano} \Rightarrow \mathfrak{PO} = \mathfrak{PO}_0$$

$$\mathcal{M}_+(\mathcal{L}ag\, M) := \left\{ (u, \mathfrak{x}) \in \mathfrak{t}^* \times \left(\Lambda_0^{\mathbb{C}} / 2\pi\sqrt{-1}\,\mathbb{Z}\right)^n \;\middle|\; \frac{\partial \mathfrak{PO}^u}{\partial x_i}(\mathfrak{x}) = 0 \right\}$$

$$\mathcal{M}_{+,0}(\mathcal{L}ag\, M) := \left\{ (u, \mathfrak{x}) \in \mathfrak{t}^* \times \left(\Lambda_0^{\mathbb{C}} / 2\pi\sqrt{-1}\,\mathbb{Z}\right)^n \;\middle|\; \frac{\partial \mathfrak{PO}_0^u}{\partial x_i}(\mathfrak{x}) = 0 \right\}$$

Thm :

(1) $\mathrm{Spec}(QH^\omega(M;\Lambda))(\Lambda^{\mathbb{C}}) \cong \mathcal{M}_{+,0}(\mathcal{L}ag\, M)$

(2) M : Fano

$\Rightarrow \mathcal{M}(\mathcal{L}ag\, M) = \mathcal{M}_+(\mathcal{L}ag\, M) = \mathcal{M}_{+,0}(\mathcal{L}ag\, M)$

(3) $QH^\omega(M:\Lambda)$ semi simple $\Rightarrow \#\mathcal{M}_{+,0}(\mathcal{L}ag\, M) = \sum_d rk_{\mathbb{Q}} H_d(M:\mathbb{Q})$.

Sample results :

Thm $\forall k$ $\exists$ toric Kähler form on the k-pt blow-up of $\mathbb{C}P^2$

with exactly $(k+1)$ "<u>balanced</u>" fibers.

appoximated by a sequence of $(M, \omega_i, L(u_i))$

s.t. $HF(M, L(u_i)) \neq 0$

We also have an example of cpt toric mfd with continuous family of "balanced" fibers.

(after bulk/boundary)

- variational analysis of piecewise linear functions

  $\downarrow$

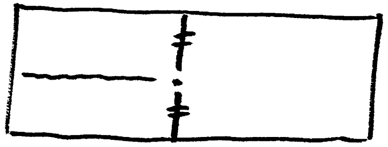
  location of undisplaceble fibers

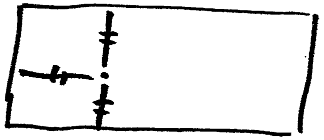
- Chekanov's proof of non-degeneracy of Hofer distance

Λ_0 - torsion of $HF^{\cdot}$ (FOOO)

$\downarrow$

lower bound for Hofer energy to displace $L(u)$ by Hamiltonian isotopy





$u \in \mathrm{Int}\, \Delta$

If $\dfrac{\partial \mathfrak{PO}^u}{\partial y_i} = 0$ has a sol. $\mathbb{y} \in \Lambda_0^{\mathbb{C}} \setminus \Lambda_+^{\mathbb{C}}$,

then $\mathbb{x} \in \log \mathbb{y} \in \Lambda_0^{\mathbb{C}}$.

$\mathbb{x} \in \mathcal{M}_{\text{week}}(L(u))$ and $H\dot{F}((L(u), \mathbb{x})) \neq 0$.

in fact $\cong \Lambda^n$.