Cornell Topology Festival 2016: Panel Discussion

The following is a brief report on the panel discussion of the 52^{nd} Topology Festival held at Cornell University, May 13^{th} through May 15^{th}, 2016. The eleven speakers of the Topology Festival were asked to present in no more than five minutes and on at most one blackboard an interesting recent result or open problem. Below are outlines of their presentations.

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Erdős' Distinct Distances Problem

Florian Frick, Cornell University

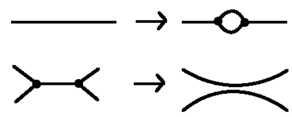
Recently Guth and Katz showed that N points in $\mathbb{R}^2$ determine at least $C\frac{N}{\log N}$ distinct distances. Let $\mathcal{P} = \{p_1, p_2, p_3, p_4 : |p_1 - p_2| = |p_3 - p_4|\}$. Then it suffices to show that $|\mathcal{P}| \leq CN^3 \log(N)$. Quadruples of points determining two lines of equal length are in one-to-one correspondence with orientation preserving rigid motions in $\mathbb{R}^2$ (moving p_1 to p_3 and rotating so that p_2 and p_4 agree). In the right coordinates rigid motions can be parametrized by $\mathbb{R}^3$ and moreover the family of rigid motions mapping x to y turns out to be a straight line in these coordinates.
This reduces the problem to incidences of lines in $\mathbb{R}^3$: given N^2 lines in $\mathbb{R}^3$ with at most N lines intersecting in a common point and at most $O(N)$ lines on any plane or regulus, there are at most $O(N^3 \log N)$ pairs of intersecting lines. For this it suffices to prove that given N^2 lines in $\mathbb{R}^3$ with at most $O(N)$ on any plane or regulus, any set of points S where each point is incident to at least k lines has cardinality $|S| \leq N^3/k^2$. To prove this use the polynomial ham sandwich theorem to partition $\mathbb{R}^3$ into cells. If most points are in the interior of cells proceed by Szemerédi–Trotter, whereas if most points are in the boundary of cells, and thus in the zero set of a polynomial of sufficiently low degree, proceed via the polynomial method.

Surfaces from Planar Graphs

Emmy Murphy, Massachusetts Institute of Technology

Let $G \subseteq S^2$ be a planar graph of valence 3. We can define a surface in $\mathbb{R}^3$ in the following way: over each face put two parallel planes, where edges of the graph correspond to intersections of two planes in a line, and vertices correspond to D_4^+-singularities.

The Gauss map then yields a smooth embedded surface $\Sigma \subseteq \mathbb{R}^3 \times S^2$ of Euler characteristic $\chi(\Sigma) = 4 - \#\{\text{vertices}\}$. Surfaces constructed in this way are smoothly isotopic if they have the same genus. The Legendrian isotopy type keeps track of some structure, for example whether the graph contains bridges. We allow two moves that modify the graph:



Using these moves, we cannot get from the graph with two vertices and three edges between them to a graph with a bridge.

Car Crash Theorem

Anders Björner, Royal Institute of Technology (KTH)

A regular cell decomposition of the 2-sphere subdivides the 2-sphere into countries (the 2-cells), separated by borders (the 1-cells). Each country has a patrol car that patrols its border with positive orientation.

Theorem (Car Crash Theorem, Klyachko)**.** *There will be a car crash. Further, if there is no crash on an edge then all cars will crash simultaneously at a vertex.*

Counting Triangulations of S^n

Greg Kuperberg, University of California at Davis

The following is an open problem which is important in physics. Let $N_n(t)$ denote the number of triangulations of S^n with t simplices.

Problem. *How well can we describe the asymptotics of $N_n(t)$ for varying t and fixed dimension n?*

It is simple to bound $N_n(t)$ between exponential and factorial: $2^{\Omega(t)} \leq N_n(t) \leq 2^{O(t(\log t))}$. In dimension $n = 2$ it is known to be exponential, but it remains open in all other dimensions. One thing that is known is that the bounds are monotonic, up to a constant factor, so that if it is factorial in a given dimension n, then it is factorial in all dimensions greater than n.

The Shuffle Conjecture, $n!$ Conjecture and Macdonald Polynomials

Patricia Hersh, North Carolina State University

In 2001 Mark Haiman proved the $n!$ conjecture. Erik Carlsson and Anton Mellit recently proved the shuffle conjecture using Macdonald polynomials.
Macdonald polynomials are certain polynomials $P_\lambda(X, q, t)$. The related Kostka polynomials $K_{\lambda,t}(q, t)$ are conjectured to be in $\mathbb{N}[q, t]$. This is equivalent to conjecturing that there exists a bigraded S_n-module with these as Frobenius characters.

Theorem (Haiman)**.** *This is a Frobenius character of particular S_n-modules.*

Corollary. *The dimension of the modules is $n!$.*

There are related S_n-modules which Haiman proved are $(n + 1)^{n-1}$-dimensional. The shuffle conjecture gives an explicit description of the Frobenius characters of these $(n + 1)^{n-1}$ modules in a monomial basis.

$$DR_n = \mathbb{C}[x + 1, \ldots, x_n, y_1, \ldots, y_n]/I$$

where I is the ideal of polynomials invariant under the diagonal action of S_n on the variables.

Four Color Theorem

Ciprian Manolescu, University of California at Los Angeles

There has been an attempt recently by Kronheimer and Mrowka to present a *readable* proof of the four color theorem. One equivalent formulation of the theorem is to count distinct edge colorings of a trivalent graph, known as Tait colorings. For a given trivalent graph G, let $\tau(G)$ be the number of Tait colorings of G. If G is planar and bridgeless, this gives a map $\pi_1(S^3 \backslash G) \longrightarrow SO(3)$.
Define a group $J^\#(G)$, associated to any $G \subset S^3$ trivalent, as the homology of a certain chain complex with generators representations $\pi_1 \longrightarrow SO(3)$ and with boundary map counting the Yang-Mills equations with singularities along G.

Theorem. $\dim J^\#(K) > 0$ *if and only if K is spacially bridgeless.*

Conjecture. $\tau(G) = \dim J^\#(G)$ *if G is planar.*

This truth of this conjecture would imply the four-color theorem. Some evidence in this direction is that the smallest counterexample has no bigons, triangles, or squares. Similar results have been studied replacing $SO(3)$ with $SU(3)$. In this case, the conjecture has been proven, but the theorem has not.

Random Symplectic Manifolds

Bena Tshishiku, Stanford University

Is there a random model of symplectic 4-manifolds?

Theorem (Donaldson)**.** *(X, ω) a symplectic 4-manifold. There exists*

$$\begin{array}{ccc} X' & \xrightarrow{\beta} & X \\ \downarrow{\scriptstyle\pi} & & \\ \mathbb{C}P^1 & & \end{array}$$

such that

- *β is a blow-up at finitely many points*
- *π is a Lefschetz fibration: $\pi^{-1}(x) \simeq S_g$ for a non-singular point $x \in \mathbb{C}P^1$, and for the singular points $\{z_1, \ldots, z_m\}, \pi^{-1}(z_i) = S_g/C_i$, where C_i is a curve on S_g.*

A Lefschetz fibration is determined by its monodromy $\pi_1(\mathbb{C}P^1 \backslash \{z_1, \ldots, z_n\}) \longrightarrow \text{Mod}(S_g)$, $\gamma_i \mapsto T_{C_i}$. In this way, random Lefschetz fibrations correspond to random walks on the complex of curves of S_g.

As an example, one might ask how the signature behaves for random 4-manifolds. There is the following conjecture:

Conjecture ($\frac{11}{8}$-conjecture)**.** *X^4 spin, $b_2(X) \geq \frac{11}{8}|\text{sig}(X)|$*

The Steinhaus–Banach–Grünbaum–Hadwiger–Ramos Problem

Günter M. Ziegler, Freie Universität Berlin

Given m masses in $\mathbb{R}^d$, is it possible to find h hyperplanes cutting each mass into 2^h equal pieces? For $h = 1$ hyperplane and $m = d$ this can be done and is known as ham sandwich theorem. For $m = 1$ mass and $h = d$ hyperplanes this is possible if $d \leq 3$, while there are counterexamples for $h = d > 4$. There is a general lower bound for the dimension d:

Theorem (Ramos)**.** *The condition $dh \geq m(2^h - 1)$ is necessary.*

Despite great efforts the question whether this lower bound is tight has remained open apart from certain special cases.

Points in Convex Position

Matthew Kahle, The Ohio State University

It is a result due to Erdős and Szekeres from 1935 that for any n there is a sufficiently large number $f(n)$ such that any set of $f(n)$ points in $\mathbb{R}^2$, no three of them colinear, contains n points in convex position.

We have that $f(4) = 5$: the convex hull of the five points is either a pentagon with any four of the points in convex position, a quadrilateral where the vertices are in convex position, or a triangle. In the last case, the two points in the interior of the triangle and two of the vertices of the triangle work.
A recursive construction yields the lower bound $f(n) \geq 2^{n-2} + 1$. It was known that $f(n) \leq \binom{2n-4}{n-2} + 1$, and in 1998 this bound was improved by one. A new paper on the arXiv by Suk claims that $f(n) \leq 2^{n+O(n)}$. Indeed, for large enough n, $f(n) \leq 2^{n+2n^{3/4}}$.

c-Arrangements and Matroids

Karim Adiprasito, Hebrew University of Jerusalem

A c-arrangement is a collection of subspaces of codimension c, such that any intersection of them has codimension divisible by c. Any c-arrangement defines a matroid. Some non-representable matroids, like the non-Pappus matroid, are represented by c-arrangements, whereas the Vámos matroid does not come from a c-arrangement.

Question 1. *What matroids come from c-arrangements?*

Question 2. *Which rank functions can be approximated with c-arrangements?*

Hyperfields

Laura Anderson, Binghamton University

A hyperfield is an algebraic structure F with two operations $\boxplus$ and $\odot$, similar to a field, but with set-valued addition: $\boxplus\colon F \times F \to 2^F \setminus \emptyset$. The multiplication $\odot$ is still single-valued. The axioms are similar to the field axioms, except the axiom about the existence of additive inverses is replaced by $\forall x \exists -x : 0 \in x \boxplus -x$.
One example of a hyperfield is the sign hyperfield $\mathbb{S}$, with elements 0, $+$ and $-$, where for example $+ \boxplus + = \{+\}$, $+ \boxplus - = \{0, +, -\}$ etc., according to what the possible sign of the result of adding or multiplying numbers with the given signs can be.
An oriented matroid is a subset of $\mathbb{S}^n$, and a matroid is a subset of $\{0, \neq 0\}$. One can define Grassmannians, $\mathrm{Gr}(r, F^n) \subseteq \mathbb{P}F^{\binom{n}{r}-1}$. It is not clear what the topology should be to make $\boxplus$ continuous.