Cornell Topology Festival 2019 Panel Discussion

This is a report on the panel discussion of the 55th Topology Festival at Cornell University, which ran from May 10th to May 12th, 2019. Each of the speakers was given approximately 5 minutes to outline either a particularly interesting recent result, or an open problem in the field. Summaries of their presentations follow. Reported by: Frederik De Keersmaeker, Brandon Shapiro, Shruthi Sridhar, and Chaitanya Tappu.

1. Mona Merling

Question: (Smooth Poincaré Conjecture) If M is a closed smooth manifold homeomorphic to S^n, is M also diffeomorphic to S^n?

The answer is yes for $n = 3$ (Moise), unknown for $n = 4$, and not always for $n \geq 5$. For instance it is false for $n = 7$. In other words, for which n does S^n admit a unique differentiable structure?

Let Θ_n be the group of homotopy n-spheres under connected sum up to h-cobordism. For $n \neq 4$, $\Theta_n = 0$ if and only if S^n has a unique differentiable structure. By a result of Kervaire and Milnor, for $n = 4k + 3$ (k ¿ 0) Θ_n is never 0.

By a result of Guozhen Wang and Zhouli Xu, the case of $n = 4k + 1$ depends on the existence of manifolds of Kervaire invariant 1. Let Θ_n^{bp} be the subgroup of Θ_n consisting of homotopy spheres bounding parallelizable manifolds. Then there is an exact sequence of the form

$$0 \to \Theta_{4k+2} \to \pi_{4k+2}^S/J \xrightarrow{\phi_{4k+2}} \mathbb{Z}/2 \to \Theta_{4k+1}^{bp} \to 0$$

where ϕ is the Kervaire invariant, and J the J-homomorphism. This tells us, using only the Kervaire invariant, that the only odd dimensional spheres that could possibly have a unique smooth structure are those of dimension 1, 3, 5, 13, 29, 61, and 125. This was known to be the case in dimensions 1, 3, 5 and not the case in dimensions 13, 29. Wang and Xu go on to show that S^{61} has a unique smooth structure and S^{125} does not, so the only such odd spheres are S^1, S^3, S^5, and S^{61}.

2. Dave Futer

Suppose $f : S \looparrowright M$ is an immersion of a closed and orientable surface S into a compact orientable 3-manifold M.

The Simple Loop Conjecture states that if $\ker(f_* : \pi_1 S \to \pi_1 M)$ is non trivial, then there is a simple loop γ on S realising a nontrivial element of $\ker f_*$.

This conjecture is known to be true under any of the following additional hypotheses:

(a) f is an embedding (Papakyriakopoulos).

(b) $M = \Sigma_g \times I$ is the product of a closed orientable surface of genus g with an interval (Gabai).

(c) Graph manifolds (Rubinstein-Wang).

(d) M admits one of the eight Thurston geometries (Hass, Zemke for Sol, and more recently, Markovic for hyperbolic space).

What's left? The case when M is a 3-manifold built out of the eight geometric manifolds.

3. Chris Kapulkin

Question: What is an elementary ∞-topos?

In 1-category theory, we have two notions of *topos*:

- A Grothendieck topos is a category that can be presented as a category of generalized sheaves
- An elementary topos is a category with the structure necessary for categorical logic

Lurie has developed Grothendieck ∞-toposes, an analogue to Grothendieck toposes for ∞-categories (defined in this context as quasicategories), but how this might extend to a notion of elementary ∞-topos remains open. A definition of elementary ∞-toposes would ideally include:

1. All Grothendieck ∞-toposes

2. The *classifying category* (or *syntactic category*) of homotopy type theory (HoTT): the objects are lists of dependent types in the syntax of HoTT, a morphism between two sequences is a way of constructing a term of each codomain type from a term of each domain type, and composition is by substitution.

The motivation for (1) is that for 1-categories every Grothendieck topos is an elementary topos. The converse is not true, as the category of finite sets is an elementary topos and not a Grothendieck topos, but this is often seen as an anomaly suggesting that the definition of elementary topos could be improved. It is hoped that a notion of elementary ∞-topos would suggest what such an improvement should look like.

The idea behind (2) is that the logical structure present in elementary toposes is described by "extensional" type theory, and elementary ∞-toposes are intended to have the same relationship with HoTT, an "intensional" type theory.

Rasekh, Anel-Joyal, and Shulman have each proposed a definition, but none of them are known to include (2) or come with a definition for morphisms of ∞-toposes. However, a recent result of Shulman characterizing injective fibrations shows that any Grothendieck ∞-topos can be presented by a Quillen model category in which HoTT can be interpreted, so this seems like a good sign.

4. Ruth Charney

In this short talk we present a result of Jonas Beyrer, Elia Fioravanti, and Merlin Incerti-Medici of the following flavor: to what extent does the boundary of a space determine determine that space?

Cat(0) cube complexes are at the intersection between geometry and combinatorics. Each hyperplanes divides the space into a positive and negative half space. Cat(0) cube complexes come equipped with two natural metrics:

1. the Cat(0) metric measures the shortest path between two points;
2. the L^1 metric or edge path metric measures the distance between points in the 1-skeleton. (This metric is technically only defined on the 1-skeleton.) It counts the number of hyperplanes you cross on a shortest path in the Cat(0) metric.

The two metrics are different but quasi-isometric.

There are different notion of boundary of a Cat(0) cube complex. Without going into the details, we use the notion of Roller boundary here. It keeps track of the hyperplanes (or half spaces) you cross as you go to infinity.

The cross-ratio of a tuple (a, b, c, d) of four points of a Cat(0) cube complex is defined as

$$\text{crossratio}(a, b, c, d) = \binom{\text{number of hyperplanes you}}{\text{cross going from } a \text{ to } c} - \binom{\text{number of hyperplanes you}}{\text{cross going from } b \text{ to } d}$$

This definition still makes sense when we let the points go to infinity. The notion of cross-ratio therefore extends to the boundary.

Theorem. [Beyrer-Fioravanti-Incerti-Medici] Let X and Y be Cat(0) cube complexes and

$$f : \partial X \to \partial Y$$

a map between their Roller boundaries that preserves cross-ratios. Then f extends to a cubical isomorphism $X \to Y$.

5. Inna Zakharevich

Cayley showed in that there are 27 lines on a cubic surface over $\mathbb{C}$. On a real cubic surface, the number of real lines can only be 27, 15, 7, or 3, and these lines can be classified into two types called *elliptic* and *hyperbolic*. Results of Finashin-Kharlamov ('13) and Okonek-Teleman ('14) show that in each case the number of hyperbolic lines is exactly 3 more than the number of elliptic ones.

Kass and Wickelgren use ideas from A^1 homotopy theory to generalize the above result to lines on a smooth cubic surface over an arbitrary field k with characteristic other than 2. They define a higher version of the Euler trace: for L the field of definition of a line on the surface, $Tr_{L/k} : GW(L) \to GW(k)$, where $GW(L)$ is the Gromov-Witt group of nondegenerate symmetric bilinear forms over L. Letting $\langle a\rangle \in GW(L)$ send (x, y) to axy, to each line is associated a *type*, $\langle h\rangle$. They prove that the lines on a smooth cubic surface over k satisfy

$$\sum_L \Big((\#\textit{hyperbolic lines}) \cdot Tr_{L/k}(\langle 1\rangle) + \sum_{h\in L^*/(L^*)^2} (\#\textit{elliptic lines of type } h) \cdot Tr_{L/k}(\langle h\rangle) \Big)$$

$$= 15 \cdot \langle 1\rangle + 12 \cdot \langle -1\rangle$$

which specializes to the results above when k is $\mathbb{C}$ or $\mathbb{R}$.

6. Alexander Kupers

Let $BTop(n)$ be the classifying space for fiber bundles with fiber $\cong \mathbb{R}^n$. Let $B\mathcal{O} = \text{colim}_n B\mathcal{O}(n)$ and $BTop = \text{colim}_d BTop(n)$. We know that $H^*(B\mathcal{O}, \mathbb{Q})$ is the polynomial ring $\mathbb{Q}[p_1, p_2 \ldots]$ generated by the Pontryagin classes p_i where $|p_i| = 4i$ and that the cohomology ring $H^*(B\mathcal{O}(2n), \mathbb{Q})$ is the polynomial ring $\mathbb{Q}[p_1, \ldots p_n]$ where the pontryagin classes come from restricting the respective classes of $B\mathcal{O}$.

It is known that $H^*(BTop, \mathbb{Q}) \cong H^*(B\mathcal{O}, \mathbb{Q})$. Michael S. Weiss in a 2015 paper titled: "Dalian notes on Pontryagin classes" shows a surprising result that this is not true for $BTop(2n)$. Specifically, we have the following theorem.

Theorem: There exist positive constants c_1 and c_2 such that, for all positive integers n and k where $n \geq c_1$ and $k < 5n/4 - c_2$, the class p_{n+k} is nonzero in $H^{4n+4k}(BTop(2n), \mathbb{Q})$.

We would like to study these "exotic" pontryagin classes further.

Question 1: Are there any n such that all p_{n+k} are non zero?

Question 2: Are there any other values for n and k other than the bounds in the above theorem for which the theorem remains true?

Question 3: What are these exotic pontryagin classes explicitly?

7. Alan Reid

The intersection form is a key concept in the theory of 4-manifolds. Let M be a closed orientable 4-manifold. Its intersection form is the symmetric bilinear form

$$Q_M : H_2(M; \mathbb{Z}) \times H_2(M; \mathbb{Z}) \to \mathbb{Z}$$

defined by

$$Q_M([S_\alpha], [S_\beta]) = S_\alpha \cdot S_\beta,$$

where S_α and S_β are submanifolds of dimension 2 and the product on the right is the intersection product. Recall that any class in $H_2(M; \mathbb{Z})$ can be represented by a submanifolds.

The intersection form Q_M is even if $Q_M(\alpha, \alpha)$ is even for all $\alpha \in H_2(M; \mathbb{Z})$. Otherwise it is odd. The signature of the intersection form of a hyperbolic manifold is necessarily zero. Actually, the intersection form is necessarily of one of the following two types:

$$\oplus_k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad \oplus_k \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

If M admits a spin structure, then Q_M is even. The following theorem shows that there are hyperbolic 4-manifolds that are not spin.

Theorem [Martelli, Riolo, Slavich] There exist closed orientable hyperbolic 4-manifolds whose intersection form is odd.

The same authors proved that the result still holds in higher dimensions.

Theorem [Martelli, Riolo, Slavich] In all dimensions $n \geq 4$, there exist closed orientable hyperbolic 4-manifolds whose intersection form is odd.

8. Jonathan Barmack

Fix a word $w \in F_2$, the free group on 2 generators. An element $r \in F_2$ is called a *normal root* of w if $w \in N(r)$, where $N(r)$ is the normal closure of r. For example, $w = x^5yx^4y^{-1}$ has normal root $r = x$ because $w = x^5 \cdot yx^4y^{-1}$, Given that conjugates of r are also normal roots, we will refer to the whole conjugacy class of r by r when we refer to a normal root. The following facts are known:

- There exists elements w such that the only normal root is w itself. An easy example are the generators x, y of $F_2 = \langle x, y \rangle$. In fact any primitive element (element that can be completed to a free basis of F_2) satisfies this.
- If $w \in F_2'$ (the derived subgroup), then it has infinitely many normal roots. For one, all primitive elements will be normal roots.

In a sense, these cases are opposite ends of the spectrum. An open question posed by Magnus in 1930:

Question: Are there elements in $F_2 \setminus F_2'$ that have infinitely many normal roots?

Further work has shown some negative answers to this question. More precisely, elements of the following forms have been shown to have finitely many primitive roots

- x^py^p where p is prime (Magnus)

- $x^p y^q$ where p and q are primes (Steinberg)
- $x^n y^m$ (McCool)

9. Anthony Genevois

A classical problem in the theory of groups is to construct new examples of finitely presented (infinite) simple groups. One class of examples, the Röver-Nekrashevych groups are defined as follows: Take two complete n-ary trees T_n so that its boundary is a Cantor set C.

Choose from each a complete subtree, with the same number of (finitely many) leaves, and a bijection between these sets of leaves. Also let $G \leq \mathrm{Isom}(T_n)$ be a *self-similar* subgroup. For each leaf in the first tree, map the complete subtree rooted at that leaf to the corresponding complete subtree rooted at its image via an element of G. This induces a homeomorphism between the boundaries of the trees T_n. The *Röver-Nekrashevych group* $V_n(G)$ is the subgroup of $\mathrm{Homeo}(C)$ consisting of all such homeomorphisms.

Theorem (Skipper-Witzel-Zaremsky, 2018)**.** For each integer $k \geq 2$ and $n \geq 3$, there is a self-similar $G \leq \mathrm{Isom}(T_n)$ such that $V_n(G)$ is virtually simple of type F_k but not of type F_{k+1}.

Recall that a group is of type F_k if it has a classifying space containing finitely many k-cells.

Note that $V_n(G)$ is a very big group, it contains an embedded copy of $\mathbb{Z}^\infty$.

Problems to think about: Construct simple groups which are (a) type F (b) CAT(0) or (c) cubulable.

10. Steve Ferry

Gromov argued using rational homotopy theory that if a manifold of *complexity* (think volume) A bounds another manifold, then that manifold has complexity at most $F(A)$. Gromov's upper estimate for $F(A)$ is a tower of exponentials in A.

Theorem (Chambers-Dotterrer-Manin-Weinberger, 2018)**.** $F(A)$ is bounded by a polynomial in A whose degree depends on $\dim M$.

In fact, Gromov conjectured that $F(A)$ is linear in A.

The following result states that F comes close to achieving this goal.

Theorem (Manin-Weinberger)**.** In fact $F(A)$ is $O(A^{1+\varepsilon})$ for every $\varepsilon > 0$.