Geometry and ∞-Poincaré inequality.

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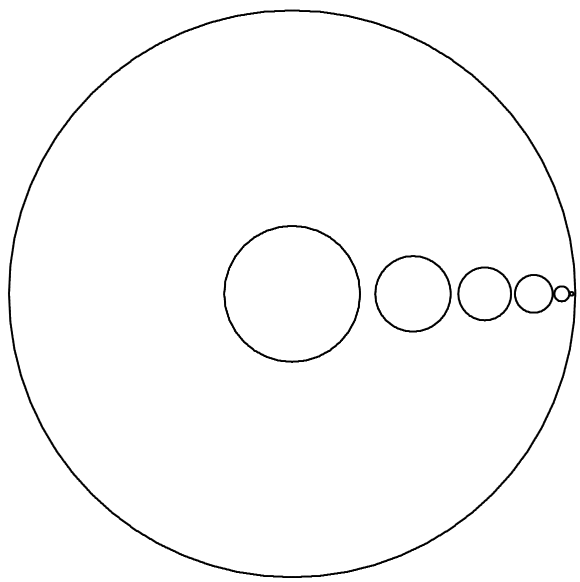
Based on joint work with Estibalitz Durand-Cartagena, Jesus Jaramillo, and Alex Williams.

Poincaré inequalities give control of the variance of a function on a ball in terms of its energy on that ball:

$$\fint_B |u - u_B| \, d\mu \leq C \, \mathrm{rad}(B) \, \left(\fint_{\tau B} |\nabla u|^p \, d\mu \right)^{1/p}.$$

Weakest of all:

$$\fint_B |u - u_B| \, d\mu \leq C \, \mathrm{rad}(B) \, \text{ess sup}_{\tau B} |\nabla u|.$$



Setting:

- (X, d, μ) complete metric measure space.

- μ Borel regular, supported on X.

- $\mu(B(x, 2r)) \le C\,\mu(B(x, r))$ all $x \in X$, $r > 0$.

- X complete.

Upper gradient g of $f : X \to \mathbb{R}$ is a Borel measurable non-negative function on X with

$$|f(y) - f(x)| \le \int_\gamma g\,ds$$

when γ rectifiable curve in X; here x, y denote end points of γ.

Given $f : B \to \mathbb{R}$,

$$\|\nabla f\|^p_{L^p(B)} := \inf_g \left(\int_B g^p \, d\mu \right)^{1/p}.$$

X supports p–**Poincaré inequality** if $\forall$ balls B and $f : \tau B \to \mathbb{R}$,

$$\fint_B |f - f_B| \, d\mu \leq C \, \frac{\mathrm{rad}(B)}{\mu(B)^{1/p}} \, \|\nabla f\|_{L^p(\lambda B)}.$$

Consequences of p-Poincaré inequality:

- Spaces that have p-Poincaré ineq., $p > 1$, support $(p - \epsilon)$-Poincaré for some $\epsilon > 0$.

- Functions with an upper gradient in $L^q(X)$ for some sufficiently large q are Hölder continuous.

- There is a weak link between Hausdorff measures and sets of capacity zero.

- X is quasiconvex.

- p-Poincaré inequality persists under Hausdorff limits.

A is **quasiconvex** if for all $x, y \in A$ can find γ_{xy} in A with $\ell(\gamma_{xy}) \leq C\,|x - y|$.

Geometric characterizations when μ is Ahlfors Q-regular:

- $p = 1$: equivalent to a relative isoperimetric inequality:

$$\min\{\mu(E \cap B), \mu(B \setminus E)\} \le C\, \mathrm{rad}(B)\, \mathrm{Per}(E, \tau B).$$

- $p = Q$, with Q a natural dimension of X: equivalent to Loewner condition: the Q-modulus of the collection of curves joining two continua has a lower bound expressed in terms of the relative separation of the two continua.

- $p > Q$: $\exists C \ge 1$ s.t. $\forall x, y \in X$,

$$\mathrm{Mod}_p(\Gamma(x, y, C)) \approx \frac{1}{d(x,y)^{1-Q/p}}.$$

Here, $\Gamma(x, y, C)$ collection of all rectifiable curves in X connecting x to y with length $\le C\, d(x, y)$.

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$$\mathrm{Mod}_p(\Gamma) = \inf_{g \in \mathcal{A}(\Gamma)} \int_X g^p \, d\mu.$$

Here $\mathcal{A}(\Gamma)$: collection of all non-negative Borel g satisfying $\int_\gamma g \, ds \geq 1$ for all $\gamma \in \Gamma$.

measures the smallest possible p-dimensional "volume" obtainable by perturbing the distance function using densities g that see each $\gamma \in \Gamma$ as of length ≥ 1.

Weakest of all the Poincaré inequalities is the ∞-Poincaré inequality:

$$\fint_B |f - f_B| \, d\mu \le C \, \mathrm{rad}(B) \, \inf_g \|g\|_{L^\infty(\tau B)}.$$

What can we infer about the geometry of X if it has ∞-Poincaré inequality?

$$\mathrm{Mod}_{\infty}(\Gamma) = \inf_{g \in \mathcal{A}(\Gamma)} \|g\|_{L^{\infty}(X)}\, d\mu.$$

[Durand-Cartagena, Jaramillo, Sh.—2008/2014]: TFAE:

- Every $f : 2\tau B \to \mathbb{R}$ with an upper gradient in $L^\infty(2\tau B)$ is Lipschitz continuous on B.

- ∞-Poincaré inequality holds.

- $\exists C \geq 1$ s.t. $\forall x, y \in X$,

$$\mathrm{Mod}_\infty(\Gamma(C, x, y)) > 0.$$

- $\exists C \geq 1$ s.t. $\forall x, y \in X$,

$$\mathrm{Mod}_\infty(\Gamma(C, x, y)) \approx \frac{1}{d(x, y)}.$$

- X is ∞-thickly quasiconvex.

Thick quasiconvexity means: for $1/2 > \epsilon > 0$, $x, y \in X$, the curves connecting $E \cap B(x, r)$ and $F \cap B(y, r)$ with length$\leq C\, d(x, y)$ has **positive** ∞-modulus, when $0 < r < \epsilon d(x, y)$,

$$\mu(E \cap B(x, r))\mu(F \cap B(y, r)) > 0.$$

However,

[Durand-Cartagena, Williams, Sh.—2009]:

- ∞-Poincaré does not imply p-Poincaré for $p < \infty$.

- There are doubling weights on $\mathbb{R}^n$ with no p-Poincaré for $p < \infty$.

- ∞-Poincaré inequality may **not** persist under Hausdorff limits.