A non-pcf fractal which makes analysis easy

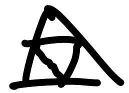
Christoph Bandt, Greifswald, Germany

Cornell University, 12 June 2014

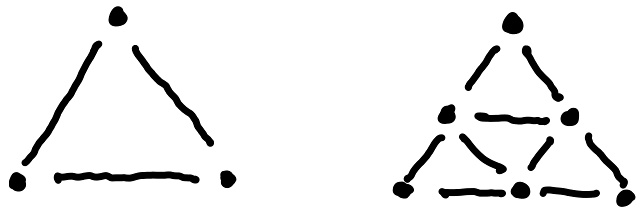
---

1. Strichartz's method of averages
2. The modified octagasket
3. Solution of the Dirichlet problem
4. Spectrum of the Laplacian

1.1 Two ways to approach a fractal

Take $A =$ 

- Increasing graphs – injective limit



$$V_0 \subset V_1 \subset V_2 \ \ldots$$

$$A = \overline{\bigcup V_i}$$

Kigami 1989, 2001

- Refining partitions – projective limit

• piece (open)

– boundary (closed)

$$X_0 \xleftarrow{\pi_1} X_1 \xleftarrow{\pi_2} X_2 \ \ldots$$

$$A = \varprojlim (X_i, \pi_i)$$

Kusuoka + Zhou 1990

Functions $f: A \to \mathbb{R}$ specified by

- values $f(x)$, $x \in V_m \subseteq A$ for injective method
- average values a_w on pieces A_w for projective method

<u>Notation</u> alphabet I, here $I = \{0, 1, 2\}$

$I^* = \bigcup_{m \geq 0} I^m$ words $w = w_1 w_2 \dots w_m$ from I

$A = \bigcup_{i \in I} \varphi_i(A)$, $A_w = \varphi_{w_1} \cdot \varphi_{w_2} \cdot \dots \cdot \varphi_{w_m}(A)$ piece

$w \sim_m w'$ words of neighboring pieces $A_w \cap A_{w'} \neq \emptyset$

μ natural measure on A: $\mu(A_w) = \frac{1}{|I|^m}$ here $\frac{1}{3^m}$

1.2 Strichartz's Laplacian (Pacific J. Math 2001)

averages $a_w = \dfrac{1}{\mu(A_w)} \displaystyle\int_{A_w} f \, d\mu$

If f is continuous and $x = \pi(w_1 w_2 \dots)$ then

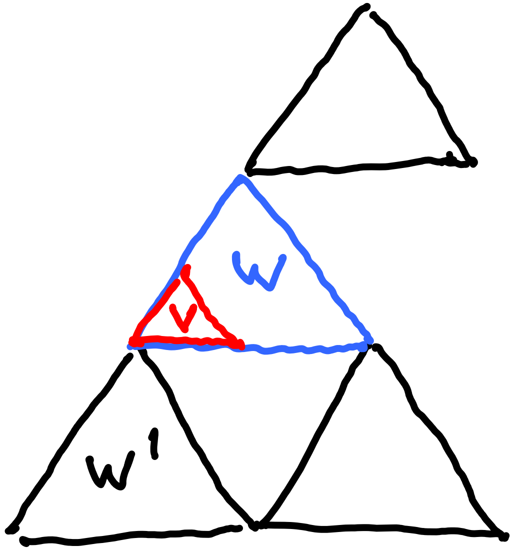
$$\lim_{m\to\infty} a_{w_1 \dots w_m} = f(x).$$

Shorthand: $a_w \to f$

---

f harmonic if $a_w = \frac{1}{3} \sum_{w' \sim w} a_{w'}$

for each word w



recursion formula for harmonic functions $a_v = \dfrac{4a_w + a_{w'}}{5}$

Let f be continuous and $b_w = \frac{3}{2} \cdot 5^m \cdot \sum_{w' \sim_m w} a_{w'} - a_w$

Th $b_w \to g$, g continuous $\Leftrightarrow$ $f \in \mathrm{dom}\,\Delta$, $\Delta f = g$
(Kigami's Δ)

So the b_w define Δ on I^* !

Th. Spectral decimation works with averages:
Let $\lambda_{m+1} = \lambda$ eigenvalue of Δ on I^{m+1} and f eigenfct., $w, w' \in I^m$, $v \in I^{m+1}$. Then we have the recursion formula for eigenfunctions

$$a_v = \frac{3}{3-\lambda} \cdot \frac{(4-\lambda)a_w + a_{w'}}{5-\lambda}$$

Ref. Ravier and Strichartz, arXiv 1308.0079

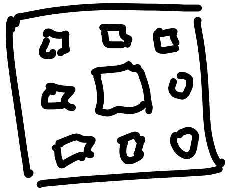
Why I like the averages

(i) Fractals are projective limits

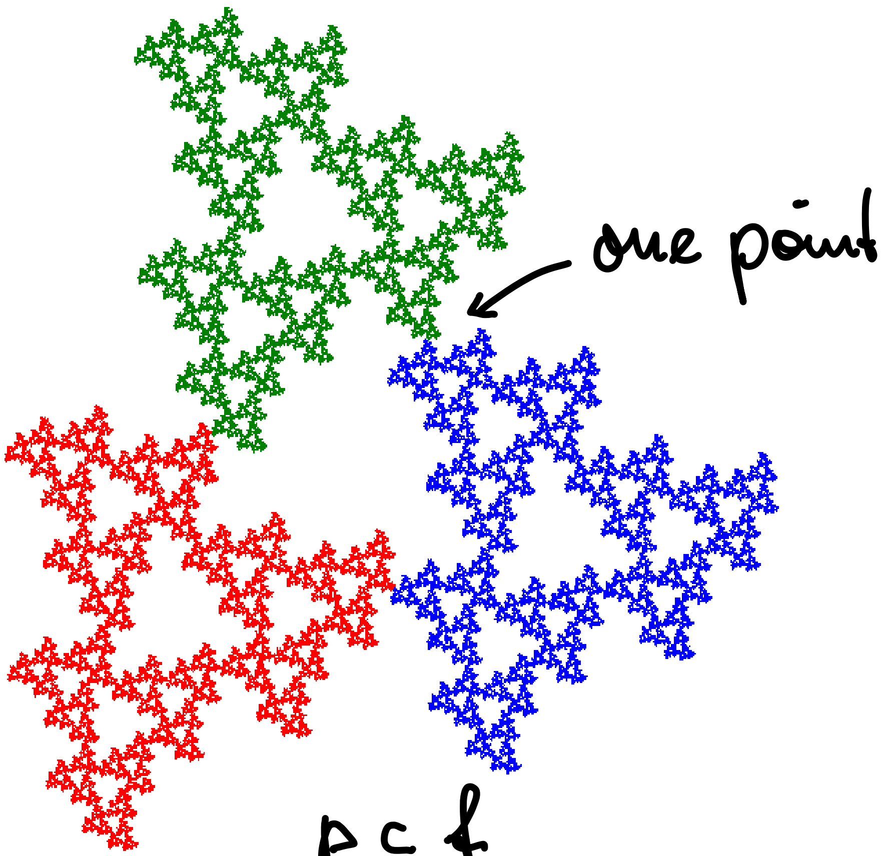
(ii) For potential applications (multiscale models of climate, flow of information, money, drugs...) averages seem much more reasonable than single values.

(iii) For fractals without pcf property, only averages can work.

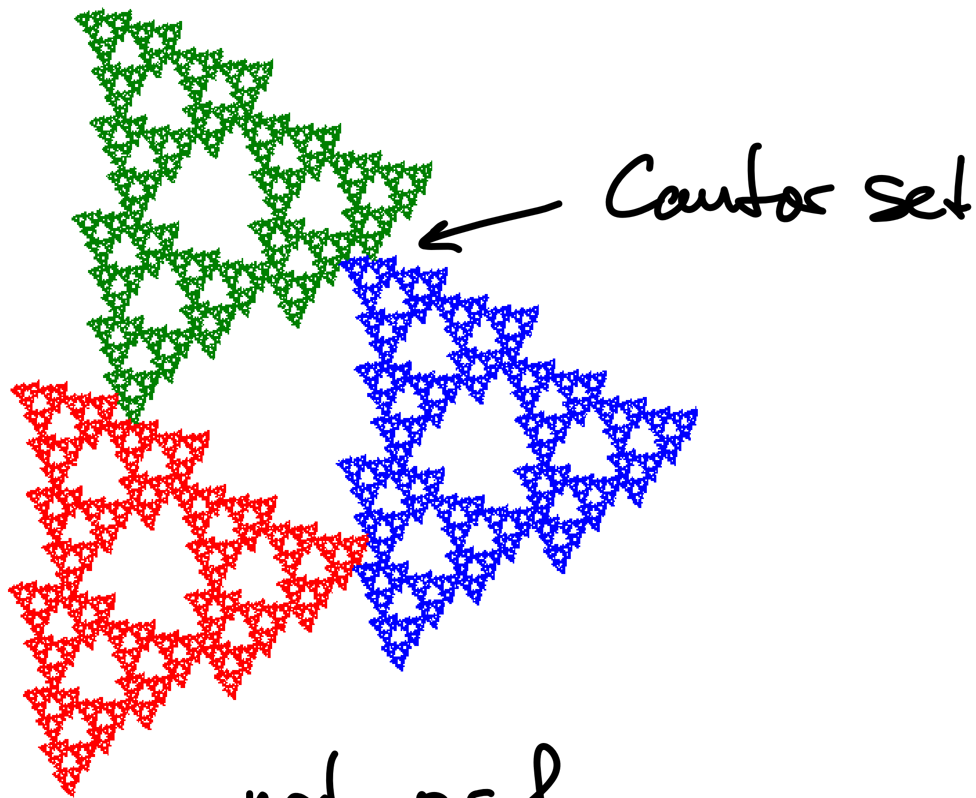
2.1 non-pcf fractals



Boundary sets of pieces are continua – for example Cantor sets, or intervals.

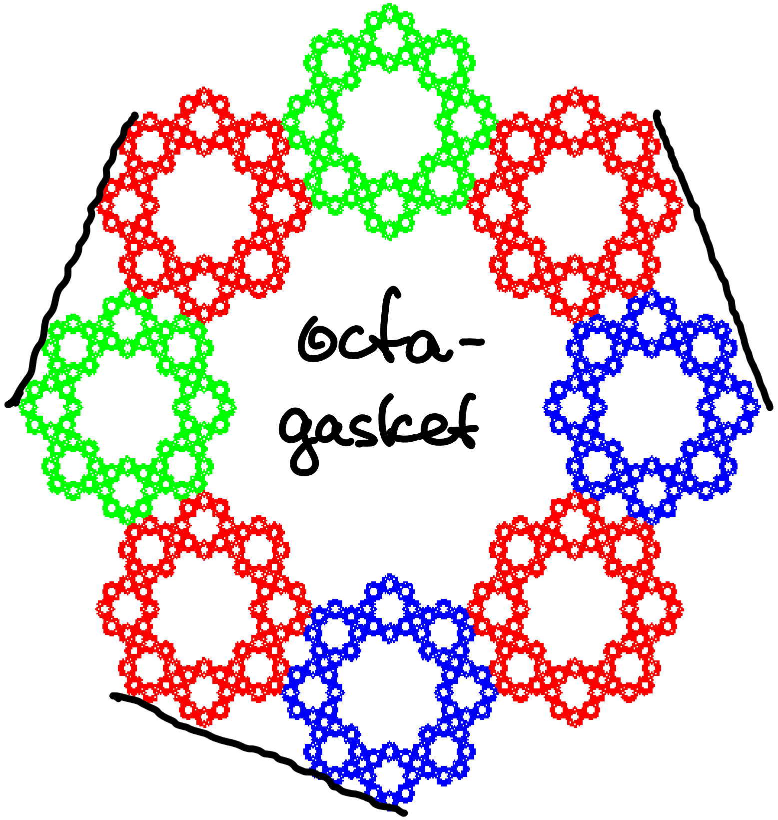


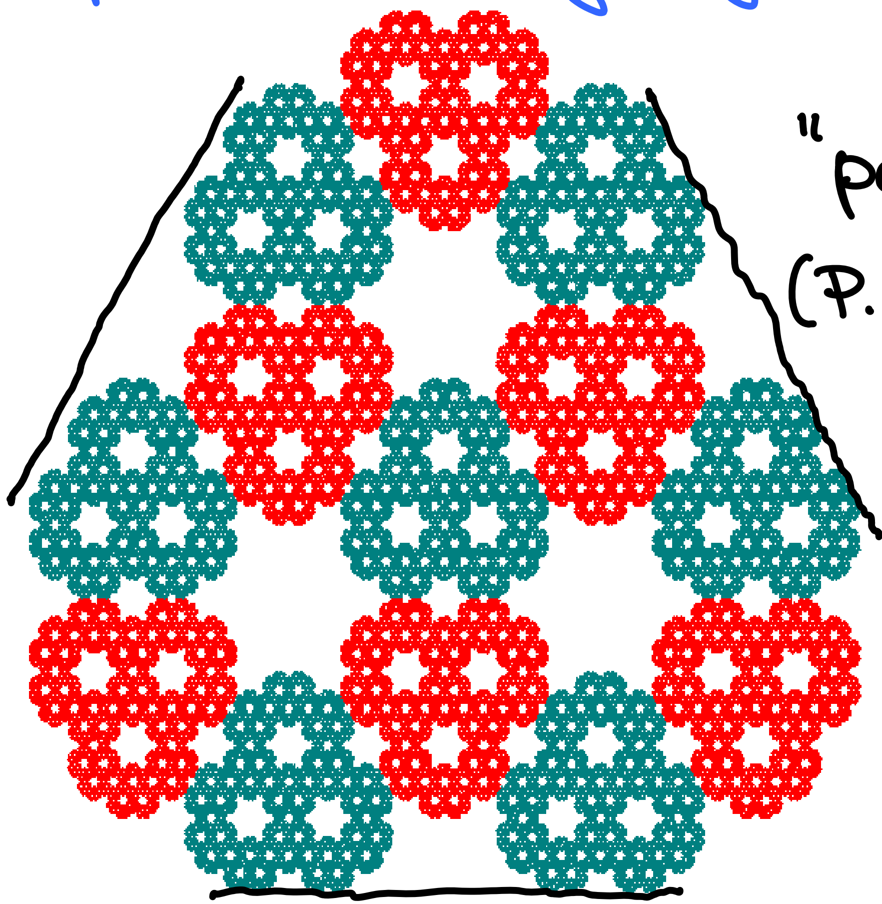
pcf



not pcf

2.2 Two non-pcf examples with high symmetry



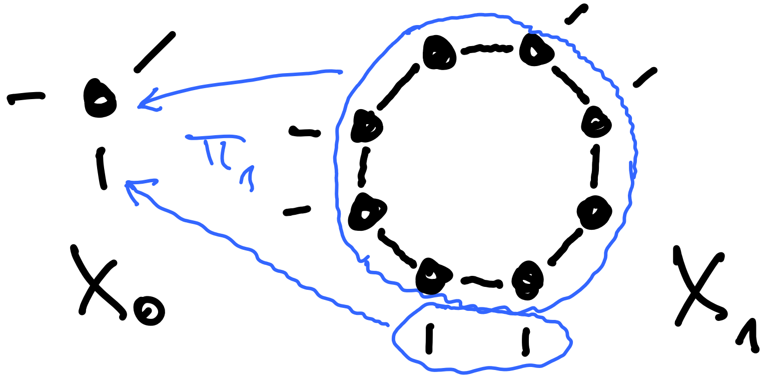


"pansy"
(P. Nestler)

Both have three linear Cantor sets as boundary.
But pieces can have 2 or 3 neighbors.

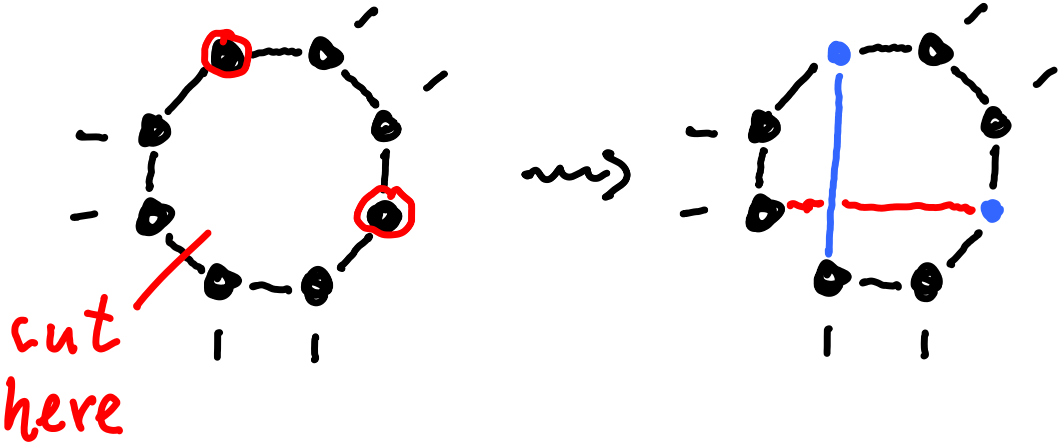
$\rightsquigarrow$ Lack of symmetry, compared with $\triangle$.

Approximation of octagasket must be projective



Two boundary points in X_m have the same image in X_{m-1}.

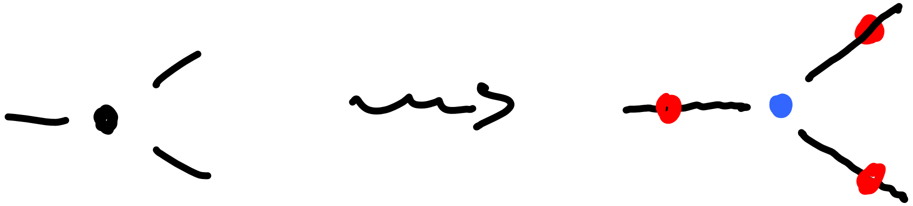
Idea: Modify octagasket so that all pieces have 3 neighbors.



Will this work on all levels?

2.3 The modified octagasket

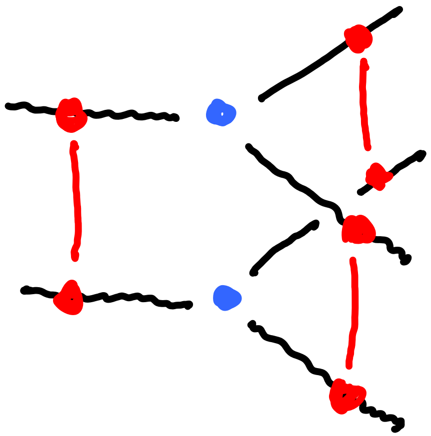
nickname: modoc



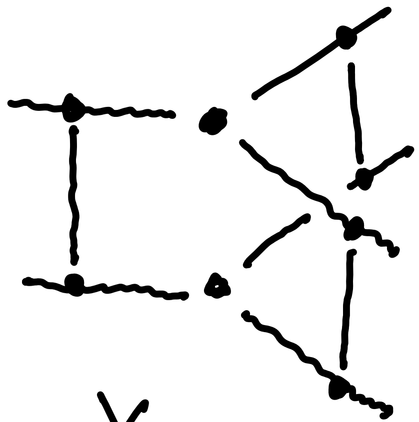
X_0

new vertex on each boundary edge

double the figure, insert new edge between every pair of new vertices

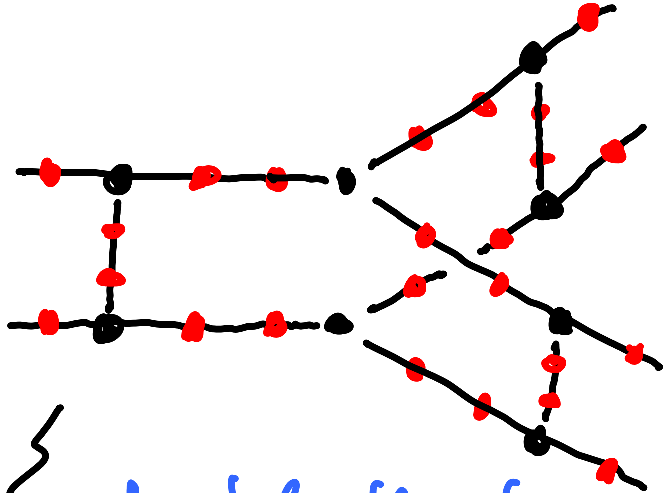


X_1

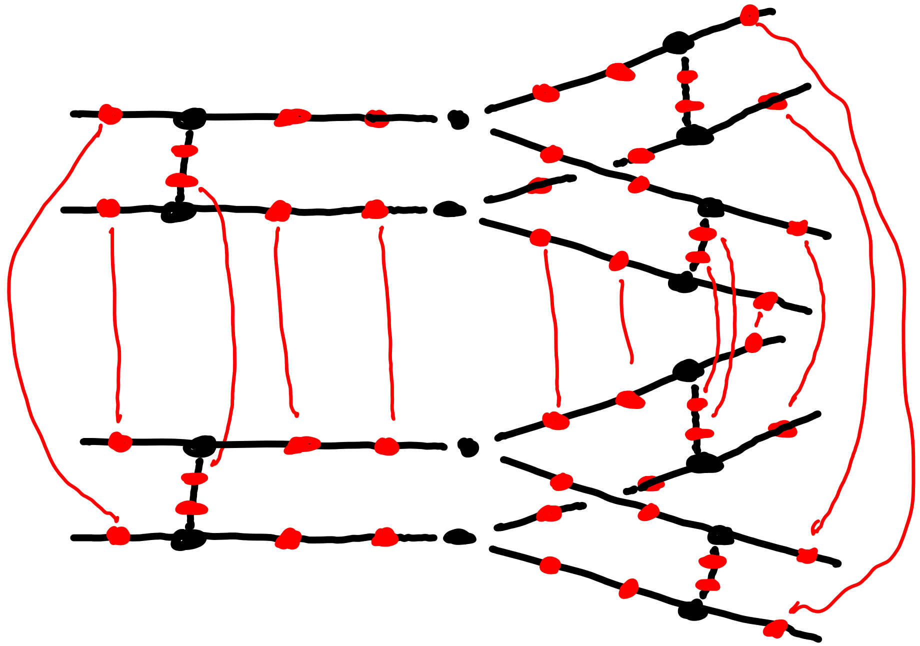


X_1

2 new vertices on each edge, 1 on boundary edge

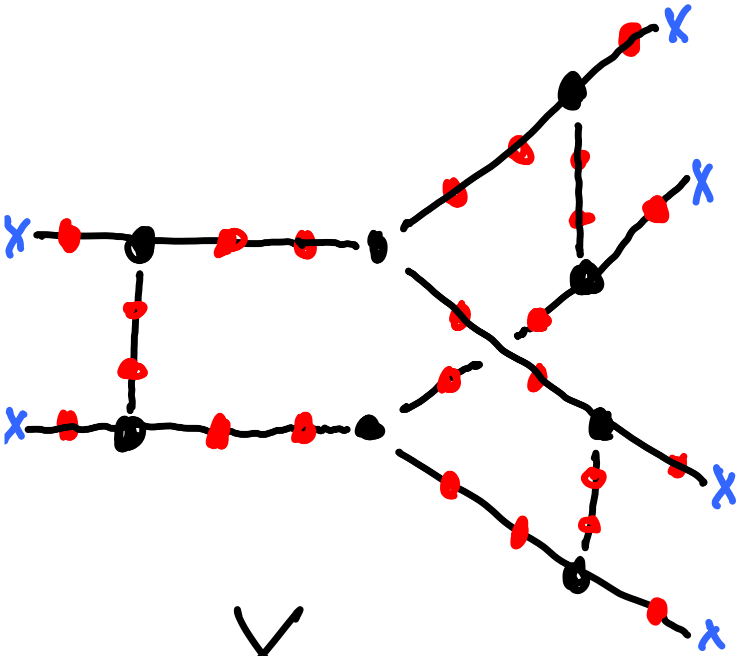


double the figure, insert edges between pairs of new vertices

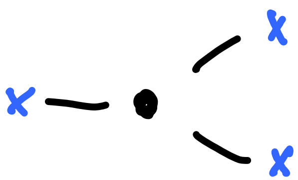


lower copy reflected

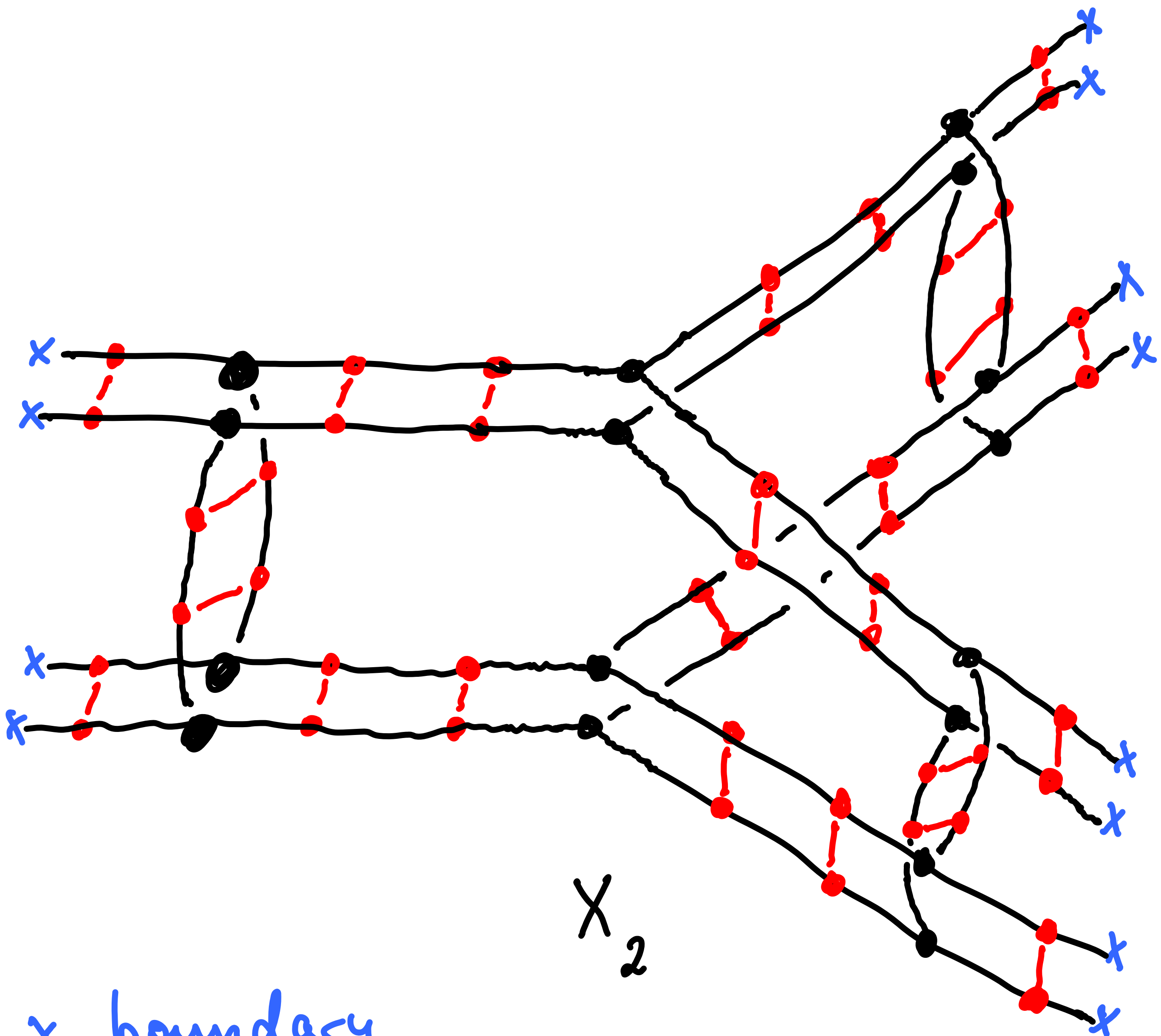
better: arrange copies near to each other



X_1



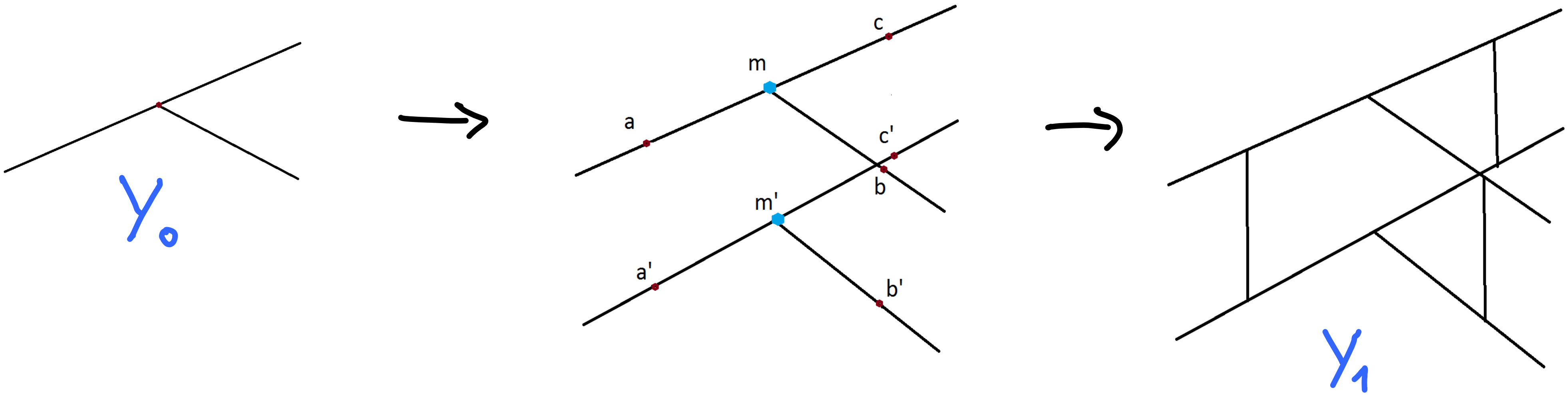
X_0



X_2

x boundary

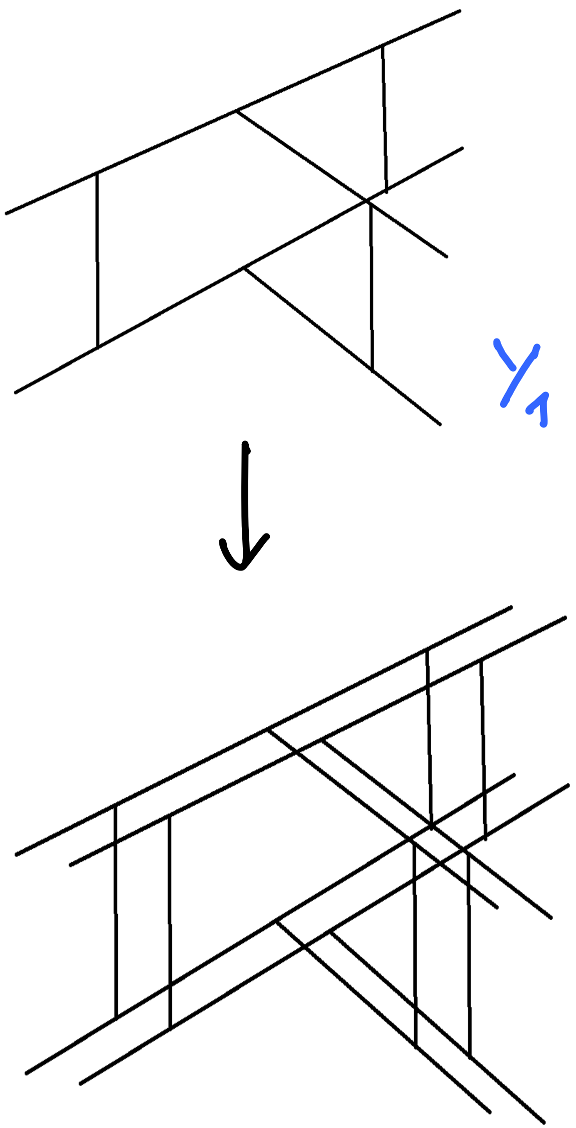
2.4 Geometric construction of modoc in $\mathbb{R}^3$

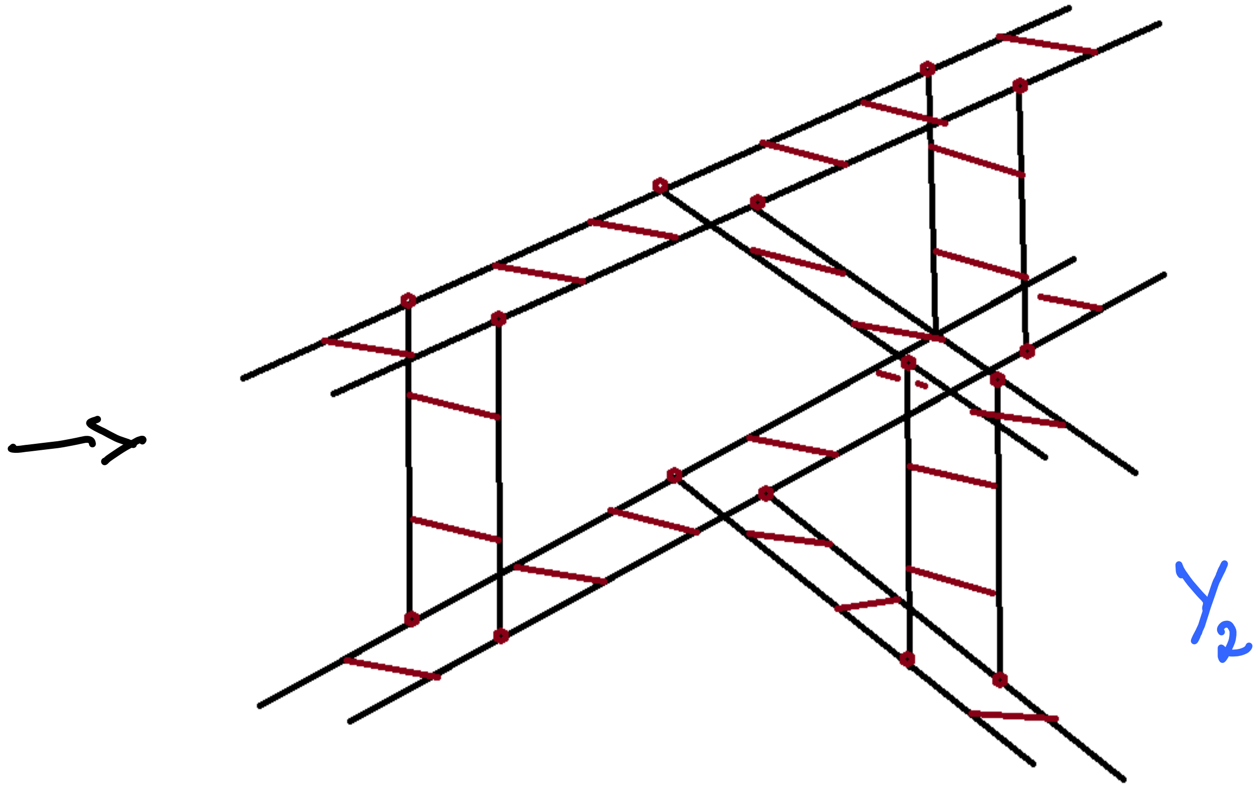


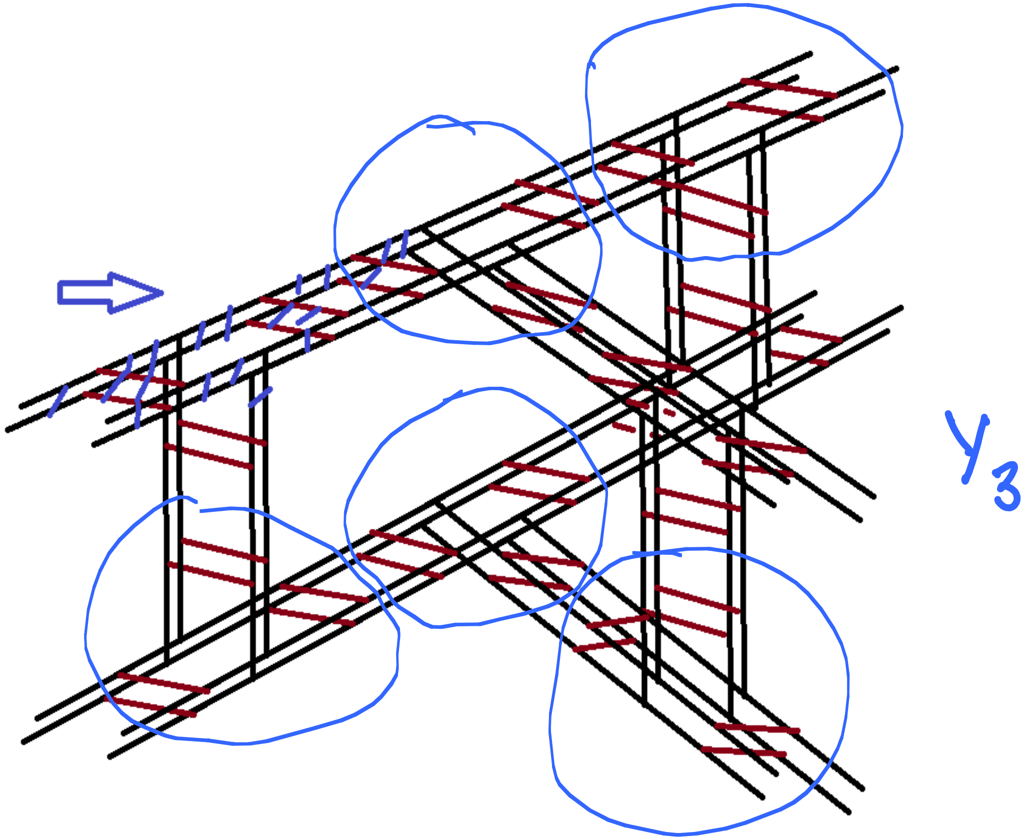
Recursive step:

- Divide each edge into three, mark 2 new vertices (only 1 vertex at boundary)
- Add a translate of the set and connect pairs of new vertices by a segment.

- translation about 3^{-m} at level m
- choose direction so that copies do not intersect







- limit with respect to Hausdorff metric is modoc.

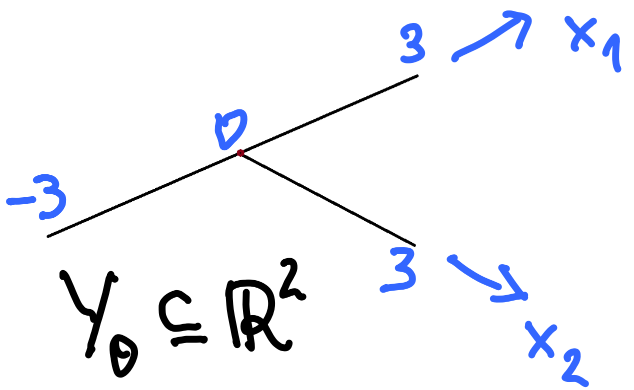
- Can we use the apparent self-similarity ?

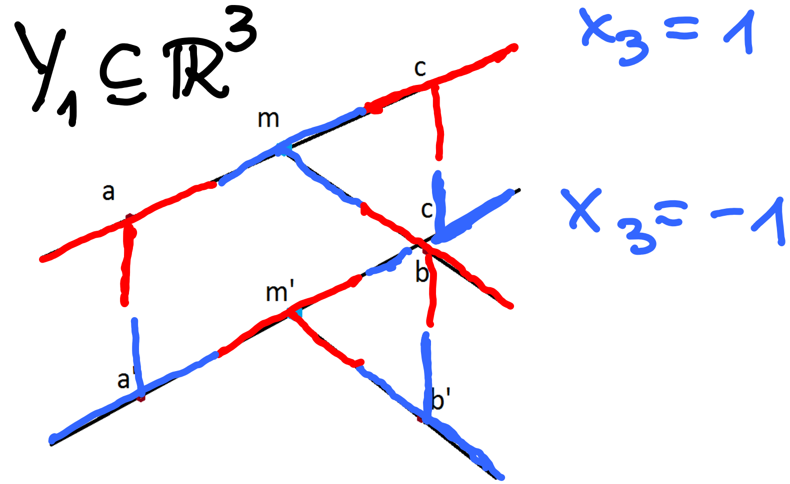
Can modoc be represented as a self-similar or self-affine set in $\mathbb{R}^3$, or $\mathbb{R}^4$?

probably not

2.5 Modoc as a self-similar set in ℓ_2

Let Y_0 consist of 3 segments in the x_1-x_2-plane





Let the first doubling translation be along the x_3-axis so that $\pi_1 : Y_1 \longrightarrow Y_0$ is the projection from $\mathbb{R}^3$ to $\mathbb{R}^2$,

$\pi_1(x_1, x_2, x_3) = (x_1, x_2)$. Y_1 is self-affine with 8 pieces.

For $Y_2 \subseteq \mathbb{R}^4$, $Y_3 \subseteq \mathbb{R}^5, \dots$ we take the translations along the x_4-axis, x_5-axis etc. More precisely,

$$Y_{n+1} = Y_n \times \{-3^{-n}, 3^{-n}\} \cup \bigcup_{y \text{ new vertex in } Y_n} \{y\} \times [-3^{-n}, 3^{-n}]$$

$$\subseteq Y_n \times [-3^{-n}, 3^{-n}] \subseteq \mathbb{R}^{n+3} \qquad \text{for } n \geq 1.$$

<u>Th.</u> Modoc is the inverse limit of

$$Y_0 \xleftarrow{\pi_1} Y_1 \xleftarrow{\pi_2} Y_2 \xleftarrow{\pi_3} Y_3 \dots \quad \text{in } \ell_2$$

where $\pi_i(x_1 \dots x_{i+1}, x_{i+2}) = (x_1, \dots x_{i+1})$.

Th. Modoc is the invariant set of the following eight similitudes on $[-3,3]^2 \times \prod_{n=0}^{\infty} [3^{-n}, 3^{-n}] \subseteq \ell_2$.

$$\varphi_i (x_1, x_2, x_3, \dots) = \frac{1}{3} (z_1, z_2, z_3, z_4, x_4, x_5, \dots)$$

where (z_1, z_2, z_3, z_4) depends on (x_1, x_2) as follows:

i	z_1	z_2	z_3	z_4
m	x_1	x_2	3	x_3
a	$x_1 - 6$	0	$3 - x_2$	x_3
c	$x_1 + 6$	0	$3 - x_2$	x_3
b	0	$x_1 + 6$	$3 - x_2$	x_3

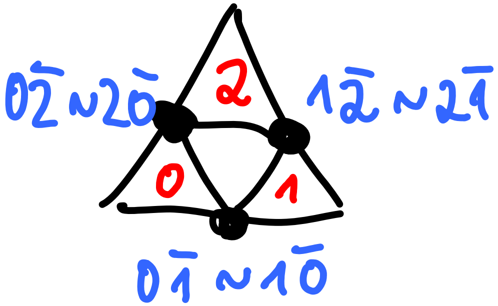
i	z_1	z_2	z_3	z_4
m'	x_1	x_2	-3	$-x_3$
a'	$x_1 - 6$	0	$x_2 - 3$	$-x_3$
c'	$x_1 + 6$	0	$x_2 - 3$	$-x_3$
b'	0	$x_1 + 6$	$x_2 - 3$	$-x_3$

Cor. Modoc has Hausdorff dimension $\frac{\log 8}{\log 3}$ like the Sierpiński carpet.

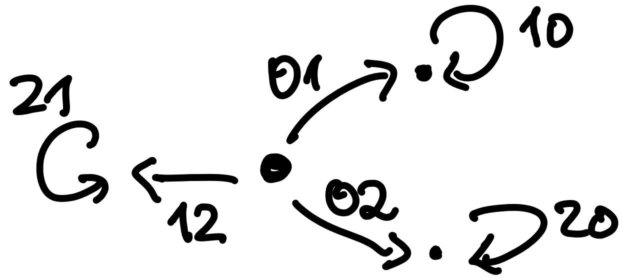
Rem. The interior distance on Modoc can be determined in terms of the ℓ_1-metric of the points $x=(x_1, x_2, \dots)$ and $y=(y_1, y_2, \dots)$

$$d^*(x,y) \geq \|x-y\|_1$$ but not much larger.

2.6 Modoc as a quotient of address space

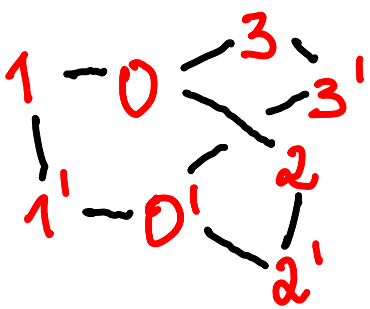


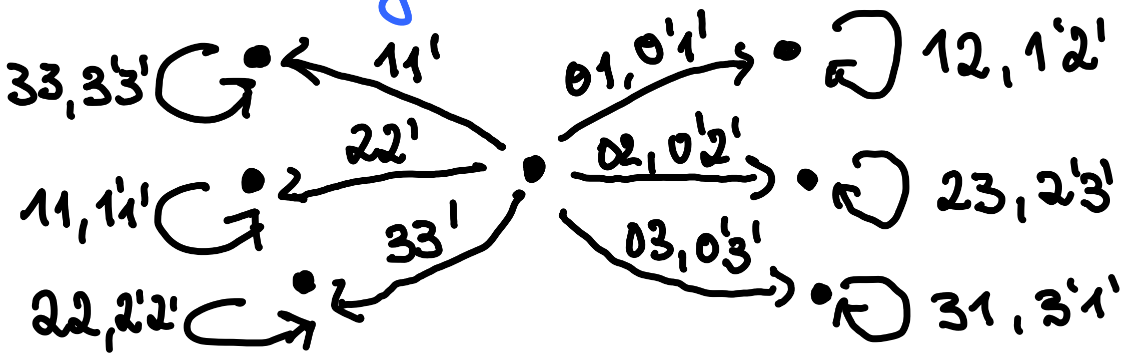
automaton
for address
identification



Def $s_1 s_2 \ldots \sim t_1 t_2 \ldots$ if there is a path in the graph with labels $s_1 t_1, s_2 t_2, \ldots$
(or $t_1 s_1, t_2 s_2, \ldots$)

Modoc automaton (symmetric version)





3.1 Harmonic functions on Modoc

Def. f with averages a_w is harmonic if

$$3a_w = \sum_{w' \sim w} a_{w'}$$ for interior pieces

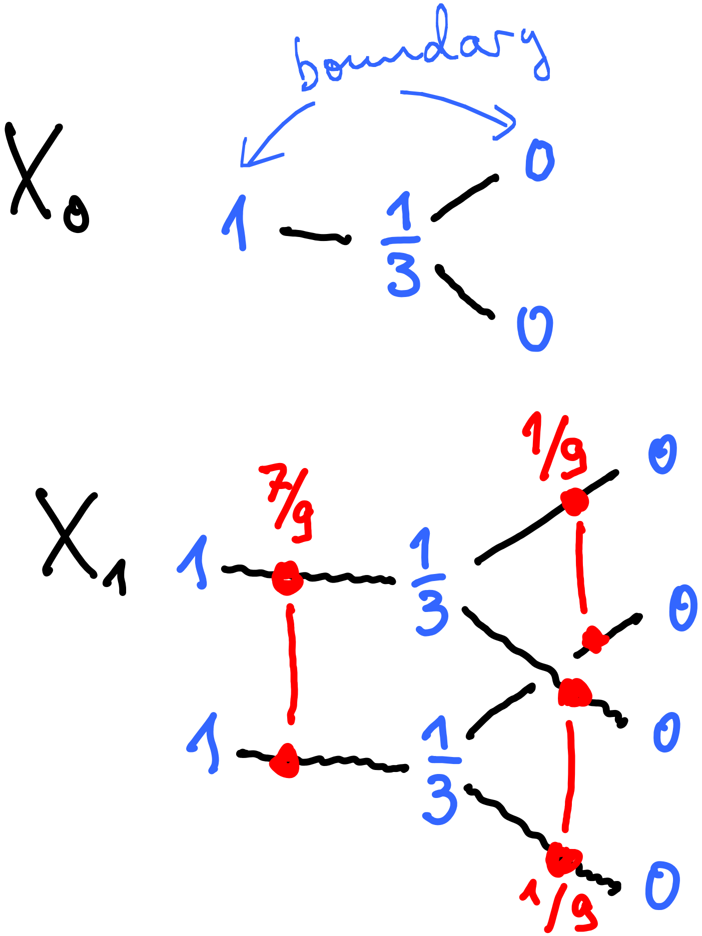
$$4a_w = 2b + a_{w'} + a_{w''}$$

at the boundary

Th. Each harmonic function on X_m (graph of words of length m) extends linearly to Y_m and by projection to $Y_{m+1}, Y_{m+2}, \dots$ and to the limit X.

In other words, if boundary values are given on a finite level m, then the finite graph calculation provides a harmonic function on X.

Ex.



on all new edges, the harmonic function will be constant.

3.2 The Dirichlet problem on Modoc

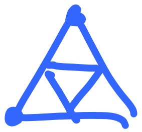
Th. For each continuous function on the boundary of modoc, there is a unique harmonic extension to the whole space X.

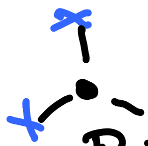
Pf. For finite level functions this was proved. By Stone-Weierstrass, each continuous function on a Cantor set can be approximated by finite level functions.

3.3 Resistance scaling on modoc

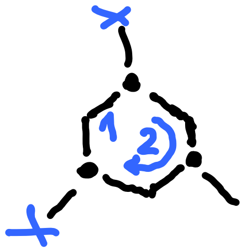
edges of graph considered as resistances, $R=1$

for boundary edges $R=\frac{1}{2}$ $\quad \bullet - \times - \bullet$

for  resistance between vertices in V_m $\frac{5}{3}$ times larger than in V_{m-1}



$R=1$



$$R=\frac{1}{2}+\frac{1}{2}+\frac{1}{1+\frac{1}{2}}=\frac{5}{3}$$

resistance factor $\quad 5/3$

length factor $\quad 2$

resistance exponent $\quad \dfrac{\log 5/3}{\log 2} \approx 0.74 \quad \begin{pmatrix} \text{interval: } 1 \\ \text{square: } 0 \end{pmatrix}$

Th For modoc, the resistance factor is $3/2$.

Pf. In V_m, any path between boundaries has 3 times more edges than in V_{m-1}. But there are twice as many parallel paths, and no flow through new edges of V_m.

length factor 3 $\rightsquigarrow$ resistance exponent $\dfrac{\log 3/2}{\log 3}$

≈ 0.37

The following is work in progress.

4.1 Definition of Laplacian

f continuous function on modoc with averages a_w.

Let

$$b_w = 12^m \sum_{w' \sim_m w} (a_{w'} - a_w)$$

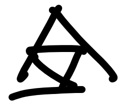
If b_w converges to a continuous function g we say that $f \in \operatorname{dom} \Delta$ and $\Delta f = g$.

Example: harmonic functions f, $b_w \equiv 0$.

Are there other examples?

Yes, eigenfunctions of Δ

4.2 The spectrum of the Laplacian

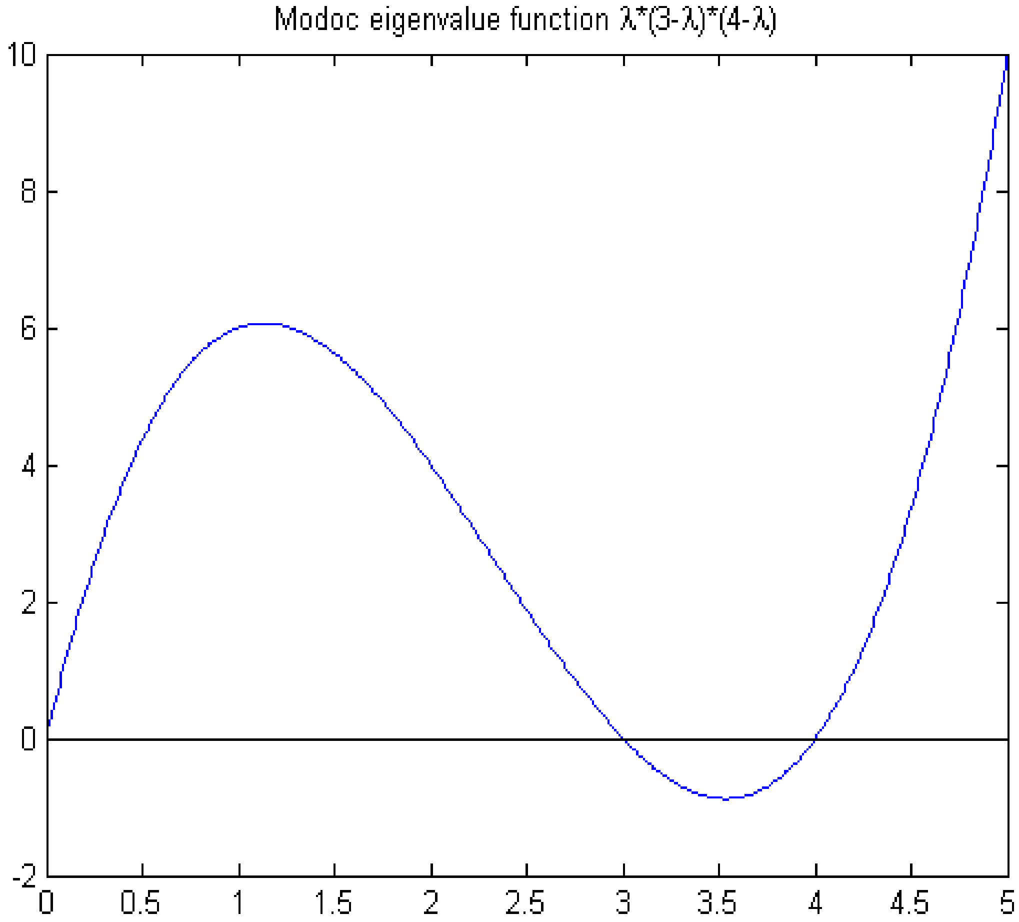
Summary: Decimation method works as for , some details more intricate

Let $-\lambda_{m-1} a_w = b_w = \sum_{w' \sim_{m-1} w} a_{w'} - a_w \quad$ for $w \in X_{m-1}$

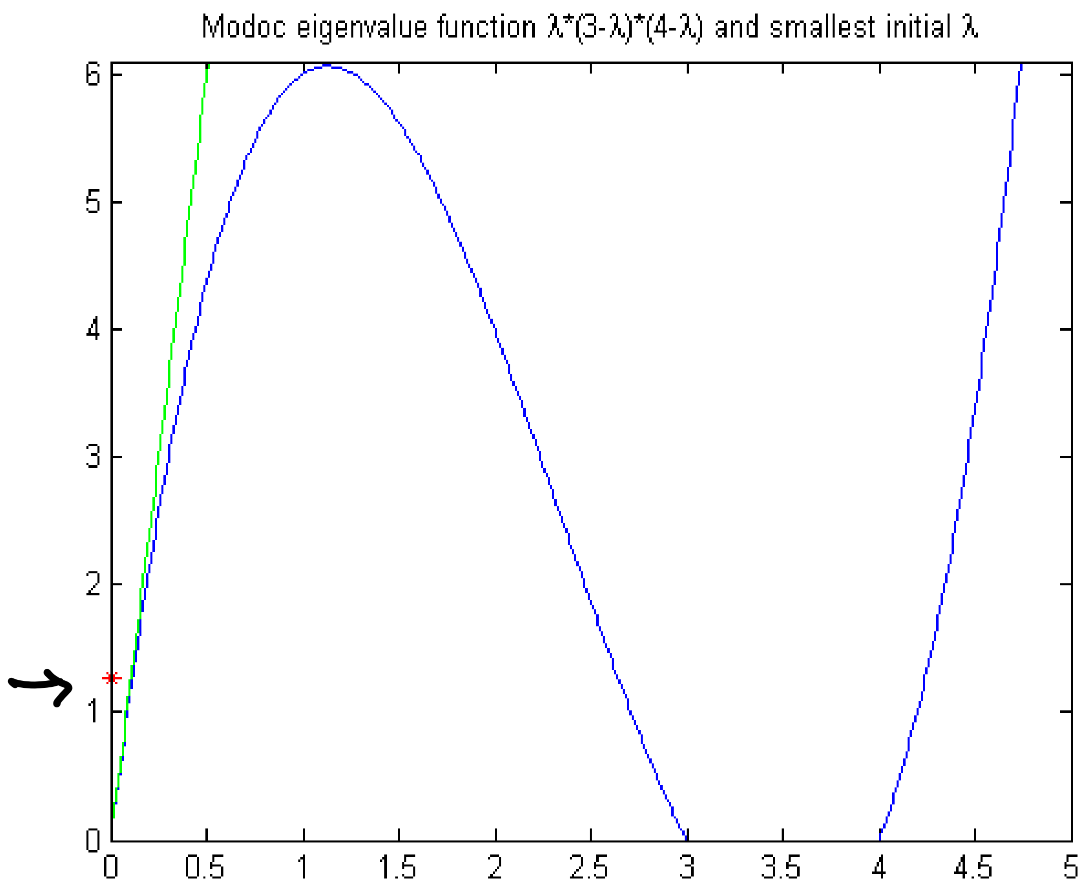
and $-\lambda_m a_v = b_v \quad$ for subpieces $A_v = A_{wi}$ of A_w

then certain equations among a_v, a_w and λ are fulfilled, and

$$\lambda_{m-1} = \lambda_m (3 - \lambda_m)(4 - \lambda_m)$$



- $\lambda_m < 6.1$ for all m
- given λ_{m-1}, there are 3 solutions for λ_m
- for $\lambda_m \to 0$, only the smallest solution can be taken for all $m \geq m_0$



$f'(0) = 12 = 8 \cdot \frac{3}{2}$

(compare $\triangle$ $5 = 3 \cdot \frac{5}{3}$)

- If we take the smallest root λ_n for $n \geq n_0$ then

$$\lambda = \lim_{n \to \infty} 12^n \cdot \lambda_n$$

exists, so b_n converges.

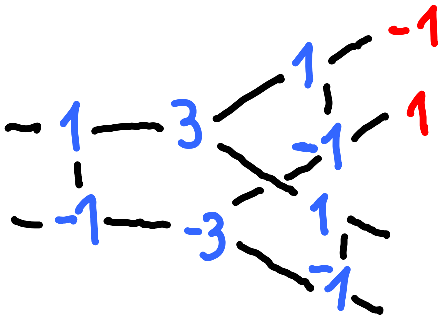
Ex.: smallest $\lambda_1 = 3 - \sqrt{3} \approx 1.27$

gives $12^{15} \cdot \lambda_{15} = 1.364\,077\,901\,915\,411\ldots$

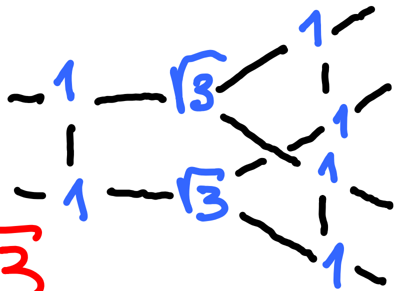
Initial Dirichlet eigenfunctions on X_1

$$(3-\lambda)\, a_w = \sum_{w' \sim_1 w} a_{w'}$$

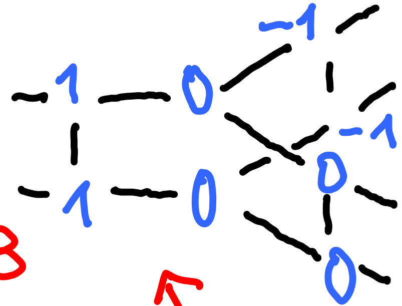
$\lambda = 2$



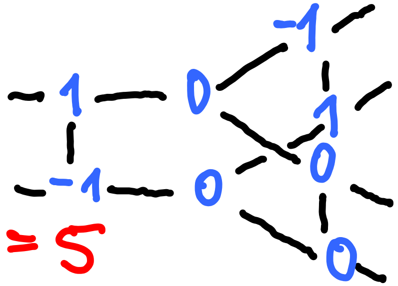
$\lambda = 3 - \sqrt{3}$



$\lambda = 3$

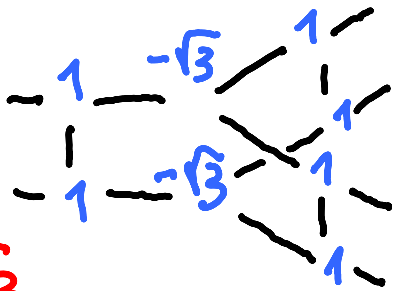


$\lambda = 5$



2 eigenfcts (rotate)

$\lambda = 3 + \sqrt{3}$



$\lambda = 6$