Some Spectral Properties of Pseudo-Differential Operators on the Sierpiński Gasket

Marius Ionescu
Joint with Kasso Okoudjou and Luke G. Rogers

U.S. Naval Academy

5th Cornell Conference on Analysis, Probability, and Mathematical Physics on Fractals

The classical Szegö Theorem

Theorem

If P_n is projection onto the span of $\{e^{im\theta}, 0 \leq m \leq n\}$ in $L^2(\mathbb{T})$ and $[f]$ is multiplication by a positive $C^{1+\alpha}$ function for $\alpha > 0$ then

$$\lim_{n\to\infty} \frac{\log \det P_n[f]P_n}{n+1} = \int_0^{2\pi} \log f(\theta)\, d\theta/2\pi.$$

Equivalently, $(n+1)^{-1}\,\mathrm{Tr}\log P_n[f]P_n$ *has the same limit.*

Set up and notation

- X is the Sierpiński gasket.
- μ is the standard measure
- Δ is the Dirichlet Laplacian on X defined by the the symmetric self-similar resistance on X.

Set up and notation

- X is the Sierpiński gasket.
- μ is the standard measure
- Δ is the Dirichlet Laplacian on X defined by the the symmetric self-similar resistance on X.
- For $\lambda \in \mathrm{sp}(-\Delta)$, let E_λ be its eigenspace, $d_\lambda = \dim E_\lambda$, and P_λ the projection onto E_λ.
- For $\Lambda > 0$, let E_Λ be the span of all eigenfunctions corresponding to $\lambda \leq \Lambda$ and let P_Λ be the projection onto E_Λ.

The Szegö Theorem for the Sierpiński Gasket

Theorem (Okoudjou, Rogers, Strichartz, 2010)

Let $f > 0$ be a continuous function on X. Then

$$\lim_{\Lambda \to \infty} \frac{1}{d_\Lambda} \log \det P_\Lambda [f] P_\Lambda = \int_X \log f(x) d\mu(x).$$

Main Theorem

Theorem (I., Okoudjou, Rogers, 2014)

Let $p : X \times (0,\infty) \to \mathbb{R}$ be a bounded measurable function such that $p(\cdot, \lambda_n)$ is continuous for all $n \in \mathbb{N}$. Assume that $\lim_{n\to\infty} p(x, \lambda_n) = q(x)$ is uniform in x. Then, for any continuous function F supported on $[A, B]$, we have that

$$\lim_{\Lambda\to\infty} \frac{1}{d_\Lambda} \operatorname{Tr} F(P_\Lambda p(x, -\Delta) P_\Lambda) = \int_X F(q(x)) d\mu(x).$$

Fact

- *The spectrum decomposes naturally into three sets called the* 2*-series,* 5*-series and* 6*-series eigenvalues.*
- *Each eigenvalue has a generation of birth* j*.*
- 2*-series eigenfunctions have* $j = 1$ *and multiplicity* 1*.*
- *Each* $j \in \mathbb{N}$ *occurs in the* 5*-series and the corresponding eigenspace has multiplicity* $(3^{j-1} + 3)/2$*.*
- *Each* $j \geq 2$ *occurs in the* 6*-series with multiplicity* $(3^j - 3)/2$*.*
- *There are* 5 *and* 6*-series eigenfunctions that are localized.*

Definition

Definition

- If $p : (0, \infty) \to \mathbb{C}$ is measurable then

$$p(-\Delta)u = \sum_n p(\lambda_n)\langle u, \varphi_n\rangle\varphi_n$$

for $u \in D$ gives a densely defined operator on $L^2(\mu)$ called a *constant coefficient pseudo-differential operator.*

Definition

- If $p : (0, \infty) \to \mathbb{C}$ is measurable then

$$p(-\Delta)u = \sum_n p(\lambda_n)\langle u, \varphi_n\rangle\varphi_n$$

for $u \in D$ gives a densely defined operator on $L^2(\mu)$ called a *constant coefficient pseudo-differential operator*.

- If $p : X \times (0, \infty) \to \mathbb{C}$ is measurable we define a *variable coefficient pseudo-differential operator* $p(x, -\Delta)$ via

$$p(x, -\Delta)u(x) = \sum_n \int_X p(x, \lambda_n)P_{\lambda_n}(x, y)u(y)d\mu(y).$$

Assumptions

Fact

We assume that $p : X \times (0, \infty) \to \mathbb{R}$ *is measurable and that*

- $p(\cdot, \lambda)$ *is continuous for all* $\lambda \in \mathrm{sp}(-\Delta)$ *and*
- $\lim_{\lambda \in \mathrm{sp}(-\Delta), \lambda \to \infty} p(x, \lambda) = q(x)$ *uniformly in* x.

Some key lemmas

Theorem

The eigenvalues of $P_\Lambda p(x, -\Delta) P_\Lambda$ *are contained in a bounded interval* $[A, B]$ *for all* $\Lambda > 0$.

Some key lemmas

Theorem

The eigenvalues of $P_\Lambda p(x, -\Delta) P_\Lambda$ *are contained in a bounded interval* $[A, B]$ *for all* $\Lambda > 0$.

Theorem

Let $\Lambda > 0$. *Then the map on* $C[A, B]$ *defined by*

$$F \mapsto \frac{1}{d_\Lambda} \operatorname{Tr} F(P_\Lambda p(x, -\Delta) P_\Lambda)$$

is a continuous non-negative functional.

Single Eigenspace

Fact

If $\lambda \in \mathrm{sp}(-\Delta)$ *then* $\Gamma_\lambda := P_\lambda p(x, -\Delta) P_\lambda$ *is a* $d_\lambda \times d_\lambda$ *matrix with entries*

$$\gamma_\lambda(i,j) = \int p(x,\lambda) u_i(x) u_j(x) d\mu(x).$$

Single Eigenspace

Theorem

Let $\{\lambda_j\}$ be an increasing sequence of 6- or 5-series eigenvalues where λ_j has generation of birth j. Let $N \geq 1$ be fixed, and suppose $f = \sum_{i=1}^{3^N} a_i \chi_{C_i}$ is a simple function. Then for all $k \geq 0$

$$\lim_{j \to \infty} \frac{\mathrm{Tr}(P_j[f]P_j)^k}{d_j} = \int f(x)^k d\mu(x).$$

Sketch of the proof.

The matrix $P_j[f]P_j$ has the following structure with respect to the basis $\{u_m\}_{m=1}^{d_j}$:

$$\begin{bmatrix} R_j & 0 \\ 0 & N_j \end{bmatrix}.$$

Sketch of the Proof

Sketch of the proof.

The matrix $P_j[f]P_j$ has the following structure with respect to the basis $\{u_m\}_{m=1}^{d_j}$:

$$\begin{bmatrix} R_j & 0 \\ 0 & N_j \end{bmatrix}.$$

Moreover

$$\operatorname{Tr}(R_j)^k = \sum_{i=1}^{3^N} m_j^N a_i^k = d_j^N \sum_{i=1}^{3^N} \frac{a_i^k}{3^N} = d_j^N \int f(x)^k \, d\mu(x)$$

and

$$|\operatorname{Tr}(N_j)^k| \leq (\alpha^N)^k \|f\|_\infty^k.$$

□

Theorem

Let $\{\lambda_j\}$ be an increasing sequence of 6- or 5-series eigenvalues where λ_j has generation of birth j. Then

$$\lim_{j\to\infty} \frac{1}{d_j} \operatorname{Tr} F(P_j p(x, -\Delta) P_j) = \int_X F(q(x)) d\mu(x)$$

for any continuous F supported on $[A, B]$.

Sketch of the proof.

It suffices to prove

$$\lim_{j\to\infty} \frac{1}{d_j} \mathrm{Tr}(P_j p(x, -\Delta) P_j)^k = \int_X q(x)^k d\mu(x).$$

Sketch of the proof.

It suffices to prove

$$\lim_{j\to\infty} \frac{1}{d_j} \mathrm{Tr}(P_j p(x,-\Delta)P_j)^k = \int_X q(x)^k d\mu(x).$$

It suffices to assume that $p(x,\lambda) \geq C > 0$ for all (x,λ).

Sketch of the Proof

Sketch of the proof.

It suffices to prove

$$\lim_{j \to \infty} \frac{1}{d_j} \operatorname{Tr}(P_j p(x, -\Delta) P_j)^k = \int_X q(x)^k d\mu(x).$$

It suffices to assume that $p(x, \lambda) \geq C > 0$ for all (x, λ).
Approximate $p(x, \lambda)$ with a simple function f_N such that

$$0 \leq P_j[f_N - \delta]P_j \leq P_j p(x, -\Delta) P_j \leq P_j[f_N + \delta]P_j.$$

Then

$$\left| \frac{1}{d_j} \operatorname{Tr}(P_j p(x, -\Delta) P_j)^k - \frac{1}{d_j} \operatorname{Tr}(P_j [f_N] P_j)^k \right| < \varepsilon.$$

□

Theorem

Let $p : X \times (0, \infty) \to \mathbb{R}$ *be a bounded measurable function such that* $p(\cdot, \lambda_n)$ *is continuous for all* $n \in \mathbb{N}$. *Assume that* $\lim_{n\to\infty} p(x, \lambda_n) = q(x)$ *is uniform in* x. *Then, for any continuous function* F *supported on* $[A, B]$, *we have that*

$$\lim_{\Lambda\to\infty} \frac{1}{d_\Lambda} \operatorname{Tr} F(P_\Lambda p(x, -\Delta)P_\Lambda) = \int_X F(q(x))d\mu(x).$$

Sketch of the Proof

Sketch of the proof.

$$\left|\frac{\operatorname{Tr}(P_\Lambda p(x,-\Delta)P_\Lambda)^k}{d_\Lambda} - \int q(x)^k d\mu(x)\right|$$
$$\leq \frac{\sum_{\lambda\in\tilde{\Gamma}_J(\Lambda)} |\operatorname{Tr}(P_\lambda p(x,-\Delta)P_\lambda)^k - d_\lambda \int q(x)^k d\mu(x)|}{d_\Lambda}$$
$$+ \frac{\sum_{\lambda\in\Gamma_J(\Lambda)} |\operatorname{Tr}(P_\lambda p(x,-\Delta)P_\lambda)^k - d_\lambda \int q(x)^k d\mu(x)|}{d_\Lambda}.$$

□

Examples

Examples

Examples

Examples

1. If $p : (0,\infty) \to \mathbb{R}$ is a bounded measurable map such that $\lim_{j\to\infty} p(\lambda_j) = q$, then for any continuous F supported on $[-\|p(-\Delta)\|, \|p(-\Delta)\|]$ we have

$$\lim_{\Lambda\to\infty} \frac{1}{d_\Lambda} \operatorname{Tr} F(P_\Lambda p(-\Delta) P_\Lambda) = F(q).$$

Examples

1. If $p : (0, \infty) \to \mathbb{R}$ is a bounded measurable map such that $\lim_{j\to\infty} p(\lambda_j) = q$, then for any continuous F supported on $[-\|p(-\Delta)\|, \|p(-\Delta)\|]$ we have

$$\lim_{\Lambda\to\infty} \frac{1}{d_\Lambda} \operatorname{Tr} F(P_\Lambda p(-\Delta) P_\Lambda) = F(q).$$

2. **Riesz and Bessel Potentials:** If $p(\lambda) = 1 + \lambda^{-\beta}$ or $p(\lambda) = 1 + (1+\lambda)^{-\beta}$, $\lambda > 0$, $\beta > 0$, then

$$\lim_{\Lambda\to\infty} \frac{\operatorname{Tr} F(P_\Lambda p(-\Delta) P_\Lambda)}{d_\Lambda} = F(1).$$

Example

Examples

Example

Examples

1. If p is a 0-symbol, then for any continuous F supported on $[-\|p(x,-\Delta)\|, \|p(x,-\Delta)\|]$ we have

$$\lim_{\Lambda\to\infty} \frac{1}{d_\Lambda} \operatorname{Tr} F(P_\Lambda p(x,-\Delta) P_\Lambda) = \int_X F(q(x)) d\mu(x).$$

Example

Examples

1. If p is a 0-symbol, then for any continuous F supported on $[-\|p(x,-\Delta)\|, \|p(x,-\Delta)\|]$ we have

$$\lim_{\Lambda\to\infty} \frac{1}{d_\Lambda} \operatorname{Tr} F(P_\Lambda p(x,-\Delta)P_\Lambda) = \int_X F(q(x))d\mu(x).$$

2. If $p(x,\lambda) = f(x)$, then for any continuous F supported on $[-\|[f]\|, \|[f]\|]$ we have

$$\lim_{\Lambda\to\infty} \frac{1}{d_\Lambda} \operatorname{Tr} F(P_\Lambda [f] P_\Lambda) = \int_X F(f(x))d\mu(x).$$

Example: General Schrödinger operators

Definition

Let $p : (0, \infty) \to \mathbb{R}$ be a measurable function and let χ be a real-valued bounded measurable function on X. We call the operator $H = p(-\Delta) + [\chi]$ a *generalized Schrödinger operator* with potential χ.

Example

Assume that that $\lim_{\lambda\to\infty} p(\lambda) = l$ exists and χ is a continuous function on X. Let F be a continuous function supported on $[-\|H\|, \|H\|]$. Then, if $\{\lambda_j\}_{j\geq 1}$ is an increasing sequence of 6-series or, respectively, 5-series eigenvalues, we have that

$$\lim_{j\to\infty} \frac{\operatorname{Tr} F(P_j H P_j)}{d_j} = \int F(l + \chi(x)) d\mu(x).$$

Example

Assume that that $\lim_{\lambda\to\infty} p(\lambda) = l$ exists and χ is a continuous function on X. Let F be a continuous function supported on $[-\|H\|, \|H\|]$. Then, if $\{\lambda_j\}_{j\geq 1}$ is an increasing sequence of 6-series or, respectively, 5-series eigenvalues, we have that

$$\lim_{j\to\infty} \frac{\operatorname{Tr} F(P_j H P_j)}{d_j} = \int F(l + \chi(x)) d\mu(x).$$

Hence

$$\lim_{\Lambda\to\infty} \frac{\operatorname{Tr} F(P_\Lambda H P_\Lambda)}{d_\Lambda} = \int F(l + \chi(x)) d\mu(x).$$

Non-uniform limit

Theorem

Let $\{\lambda_j\}_{j\in\mathbb{N}}$ *be an increasing sequence of* 6*- or* 5*-series eigenvalues such that* λ_j *has generation of birth* j*, for all* $j \geq 1$*. Assume that*

$$\lim_{j\to\infty} p(x,\lambda_j) = q(x) \quad \textit{for all } x \in X.$$

Suppose that $p(\cdot,\lambda_j) \in \mathrm{Dom}(\Delta)$ *for all* $j \in \mathbb{N}$ *and that both* $p(\cdot,\lambda_j)$ *and* $\Delta_x p(\cdot,\lambda_j)$ *are bounded uniformly in* j*. Then there is a subsequence* $\{\lambda_{k_j}\}$ *of* $\{\lambda_j\}$ *such that*

$$\lim_{j\to\infty} \frac{1}{d_{k_j}} F(P_{k_j} p(x,-\Delta) P_{k_j}) = \int_X F(q(x))\,d\mu(x).$$

Application: Asymptotics of eigenvalue clusters for general Schrödinger operators

- Let $H = p(-\Delta) + [\chi]$ be a Schrödinger operator, where $p : (0, \infty) \to \mathbb{R}$ is a continuous function, such that there is $\overline{\lambda} > 0$ so that p is increasing on $[\overline{\lambda}, \infty)$ and

$$|p(\lambda) - p(\lambda')| \geq c|\lambda - \lambda'|^{\beta}$$

for all $\lambda, \lambda' \geq \overline{\lambda}$, and χ is a continuous function on X.

Application: Asymptotics of eigenvalue clusters for general Schrödinger operators

- Let $H = p(-\Delta) + [\chi]$ be a Schrödinger operator, where $p : (0, \infty) \to \mathbb{R}$ is a continuous function, such that there is $\overline{\lambda} > 0$ so that p is increasing on $[\overline{\lambda}, \infty)$ and

$$|p(\lambda) - p(\lambda')| \geq c|\lambda - \lambda'|^{\beta}$$

for all $\lambda, \lambda' \geq \overline{\lambda}$, and χ is a continuous function on X.

- Let $\{\lambda_j\}$ be a sequence of 6-series eigenvalues of $-\Delta$ such that the separation between λ_j and the next higher and lower eigenvalues of $-\Delta$ grows exponentially in j.

Schrödinger operators (cont'd)

Fact

Schrödinger operators (cont'd)

Fact

- *Let* $\tilde{\Lambda}_j$ *be the portion of the spectrum of* H *lying in* $[p(\lambda_j) + \min \chi, p(\lambda_j) + \max \chi]$.

Fact

- *Let* $\tilde{\Lambda}_j$ *be the portion of the spectrum of* H *lying in* $[p(\lambda_j) + \min \chi, p(\lambda_j) + \max \chi]$.
- *For large* j, $\tilde{\Lambda}_j$ *contains exactly* d_j *eigenvalues* $\{\nu_i^j\}_{i=1}^{d_j}$.

Fact

- *Let* $\tilde{\Lambda}_j$ *be the portion of the spectrum of* H *lying in* $[p(\lambda_j) + \min \chi, p(\lambda_j) + \max \chi]$.
- *For large* j, $\tilde{\Lambda}_j$ *contains exactly* d_j *eigenvalues* $\{\nu_i^j\}_{i=1}^{d_j}$.
- *We call this the* $p(\lambda_j)$ *cluster of the eigenvalues of* H.

Schrödinger operators (cont'd)

Fact

- *Let* $\tilde{\Lambda}_j$ *be the portion of the spectrum of* H *lying in* $[p(\lambda_j) + \min \chi, p(\lambda_j) + \max \chi]$.
- *For large* j, $\tilde{\Lambda}_j$ *contains exactly* d_j *eigenvalues* $\{\nu_i^j\}_{i=1}^{d_j}$.
- *We call this the* $p(\lambda_j)$ *cluster of the eigenvalues of* H.
- *The characteristic measure of the* $p(\lambda_j)$ *cluster of* H *is*

$$\Psi_j(\lambda) = \frac{1}{d_j} \sum_{i=1}^{d_j} \delta(\lambda - (\nu_i^j - p(\lambda_j)).$$

Theorem

The sequence $\{\Psi_j\}_{j\geq 1}$ *converges weakly to the pullback of the measure* μ *under* χ *defined for all continuous functions* f *supported on* $[\min \chi, \max \chi]$ *by*

$$\langle \Psi_0, f \rangle = \int_X f(\chi(x)) d\mu(x).$$

Sketch of the Proof: some lemmas

Theorem

If $p : (0, \infty) \to \mathbb{C}$ *is continuous then* $\operatorname{sp} p(-\Delta) = \overline{p(\operatorname{sp}(-\Delta))}$.

Sketch of the Proof: some lemmas

Theorem

If $p : (0,\infty) \to \mathbb{C}$ *is continuous then* $\operatorname{sp} p(-\Delta) = \overline{p(\operatorname{sp}(-\Delta))}$.

Theorem

Let $p : (0,\infty) \to \mathbb{R}$ *be a continuous function such that there is* $A \in \mathbb{R}$ *with* $p(\lambda) \geq A$ *for all* $\lambda \geq \lambda_1$, *where* λ_1 *is the smallest positive eigenvalue of* $-\Delta$. *For* $i = 1, 2$, *let* χ_i *be real-valued bounded measurable functions on* X. *Let* $H_i = p(-\Delta) + [\chi_i]$ *denote the corresponding generalized Schrödinger operators. For* $n \geq 1$, *the* n*th eigenvalues* ν_n^i *of* H_i, $i = 1, 2$, *satisfy the following inequality:*

$$|\nu_n^1 - \nu_n^2| \leq \|\chi_1 - \chi_2\|_{L^\infty}.$$

Another lemma

Theorem

Assume that $N > 0$ and that $\chi_N = \sum_{i=1}^{N} a_i \chi_{C_i}$ is a simple function. Let $H_N = p(-\Delta) + [\chi_N]$ be the corresponding generalized Schrödinger operator, $\tilde{\Lambda}_j^N$ the $p(\lambda_j)$ cluster of H_N, and let $\overline{P}_j^N$ be the spectral projection for H_N associated with the $p(\lambda_j)$ cluster. Then

$$\lim_{j\to\infty} \frac{\mathrm{Tr}\left(\overline{P}_j^N (p(-\Delta) + [\chi_N] - p(\lambda_j))\overline{P}_j^N\right)^k}{d_j} = \lim_{j\to\infty} \frac{\mathrm{Tr}(P_j[\chi_N]P_j)^k}{d_j}$$

$$= \int_X \chi_N(x)^k \, d\mu(x),$$

for all $k \geq 0$.