FRACTALS 2014

Cornell University

PALLE JØRGENSEN

Univ. Iowa

CO- AUTHORS (alphabetic)

Daniel Alpay

Ola Bratteli

Dorin Dutkay

Keri Kornelson

Steen Pedersen

Erin Pearse

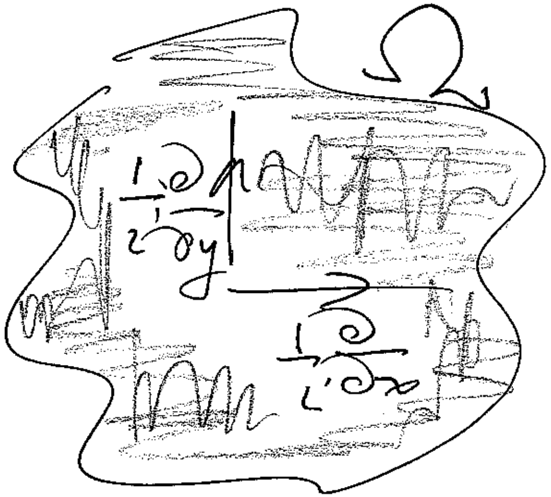
Karen Shuman

Examples

$\mathbb{R}^2 / \mathbb{Z}^2 = \mathbb{T}^2 \qquad d > 1$

$\mathbb{R}^2 / \text{lattice}$

$C_c^\infty(\Omega) \qquad L^2(\Omega)$



Conjecture
Theorems

translation set

$\exists$? discrete set D

st $\Omega \dot{+} D = \mathbb{R}^2$

Induced representation

$d = 1$

$\mathbb{Z} \ni n \longmapsto e^{i2\pi n\theta} = \chi_n(\theta) \in \mathbb{T}^1$

induction

$\mathbb{R} : \ f(x+n) = \chi_n(\theta) f(x) : \int_{\mathbb{R}/\mathbb{Z}} |f|^2 \, dx$

J. Von Neumann, I.E. Segal, B. Fuglede

Eigen-functions : $d > 1$

$$\frac{\partial}{\partial x_k} e^{i\lambda\cdot x} = i\lambda_k e^{i\lambda\cdot x}$$

$$\lambda\cdot x = \lambda_1 x_1 + \cdots + \lambda_d x_d$$

$$\Delta e^{i\lambda\cdot x} = -|\lambda|^2 e^{i\lambda\cdot x}$$

$$\Delta = \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2} \quad , \quad |\lambda|^2 = \sum_k \lambda_k^2$$

e_λ : $e^{i\lambda\cdot x}$, or its restriction to Ω

<u>Theorem</u> If there are commuting selfadjoint extensions of $\left(\frac{1}{i}\frac{\partial}{\partial x_k}, C_c^\infty(\Omega)\right)$ in $L^2(\Omega)$;

$\mathbb{R}^d$

$\lambda_d(\Omega) < \infty$

$C_c^\infty(\Omega) \subset L^2(\Omega)$

then the joint spectrum is pure-point; and the joint eigenspaces $\subset L^2(\Omega)$ are one-dimensional; each spanned by

$e_\lambda(x) = e^{i\lambda\cdot x}$, $x \in \Omega \subset \mathbb{R}^d$, for some discrete subset $\Lambda \subset \mathbb{R}^d$, $\lambda \in \Lambda$.

(4

$d=1$

$\frac{1}{i}\frac{\partial}{\partial x}$ $\quad L^2(0,1)$

PNP $\quad e^{i2\pi(\theta+n)x} \quad C_c^\infty(0,1)$

θ fixed: Spectrum $\quad \{\theta+n \mid n\in\mathbb{Z}\}$ spectrum

$[0,1) \dot{+} \mathbb{Z} = \mathbb{R}$ translation

"

tile

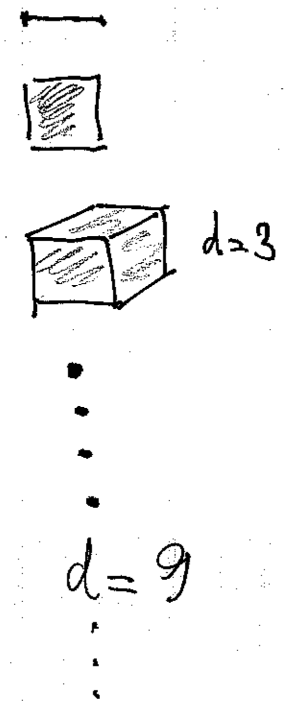
WARNING

$$I^d = \underbrace{[0,1]\times[0,1]\times\cdots\times[0,1]}_{d}$$

is not trivial.

What are the sets $\Lambda \subset \mathbb{R}^d$ such that (I^d, Λ) is a <u>spectral pair</u>?

Known if $d \le 3$, J.-Pedersen, and ...

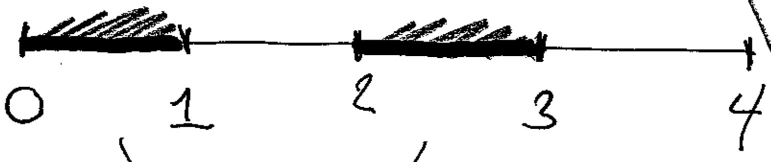


SPECTRAL PAIRS

$$(\Omega, \Lambda)$$

$$L^2(\Omega)$$

$d=1$



$$\Omega_0 \cup \Omega_1 = \Omega$$

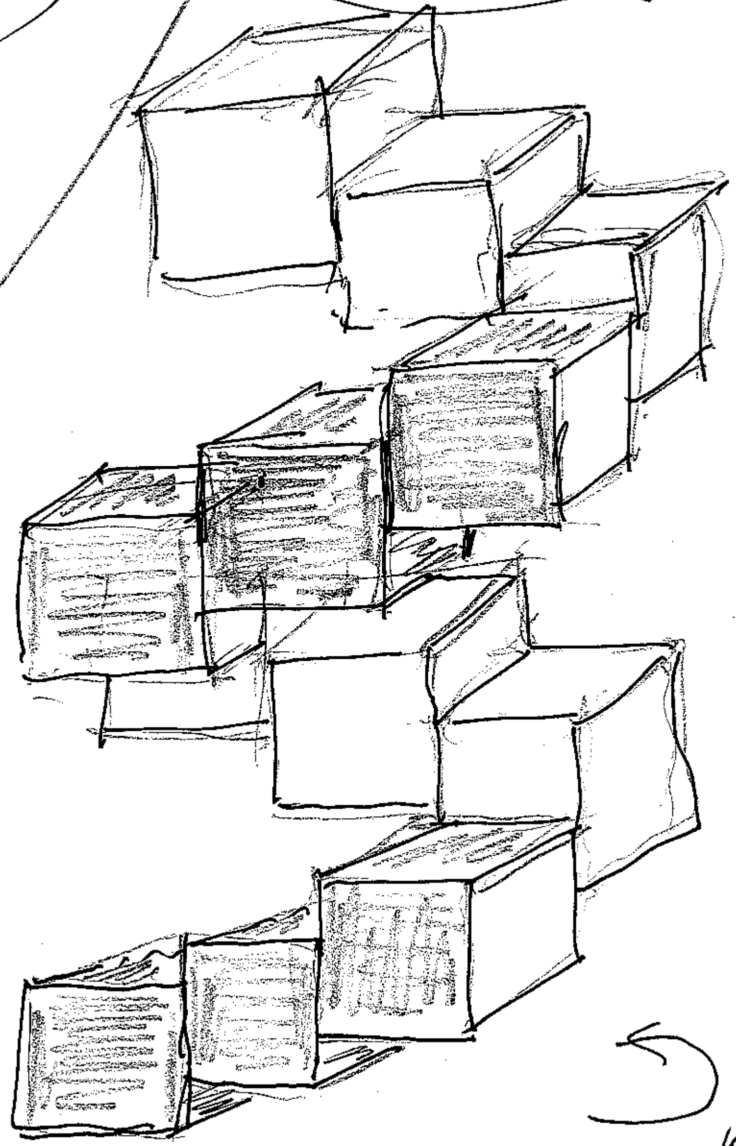
Spectrum

$$\{0,1\}+4\mathbb{Z}$$

$$=4\mathbb{Z}\cup(1+4\mathbb{Z})$$

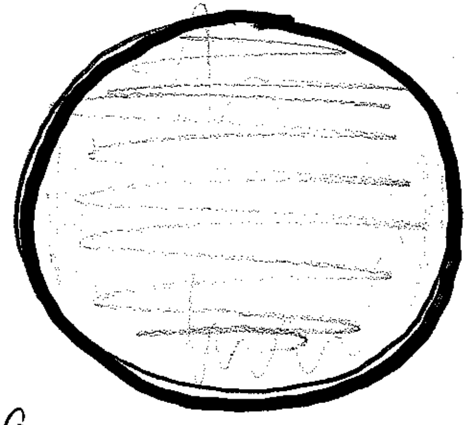
not a lattice

$d=3$

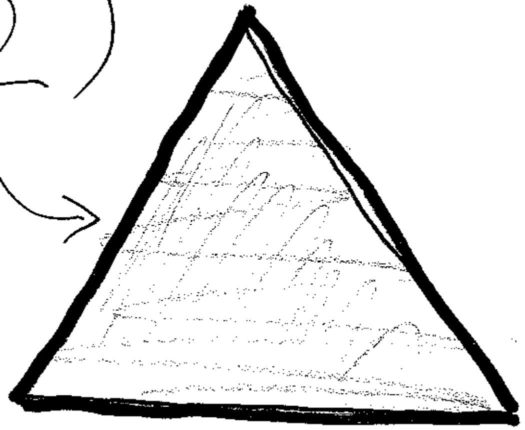


$$(4\mathbb{Z})\times(4\mathbb{Z})\times(\{0,1\}+4\mathbb{Z})$$

$L^2(\Omega)$



no



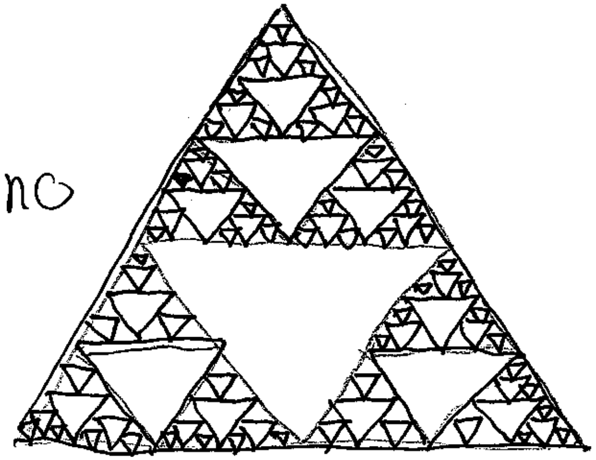
no, but

$\{e_\lambda\}$

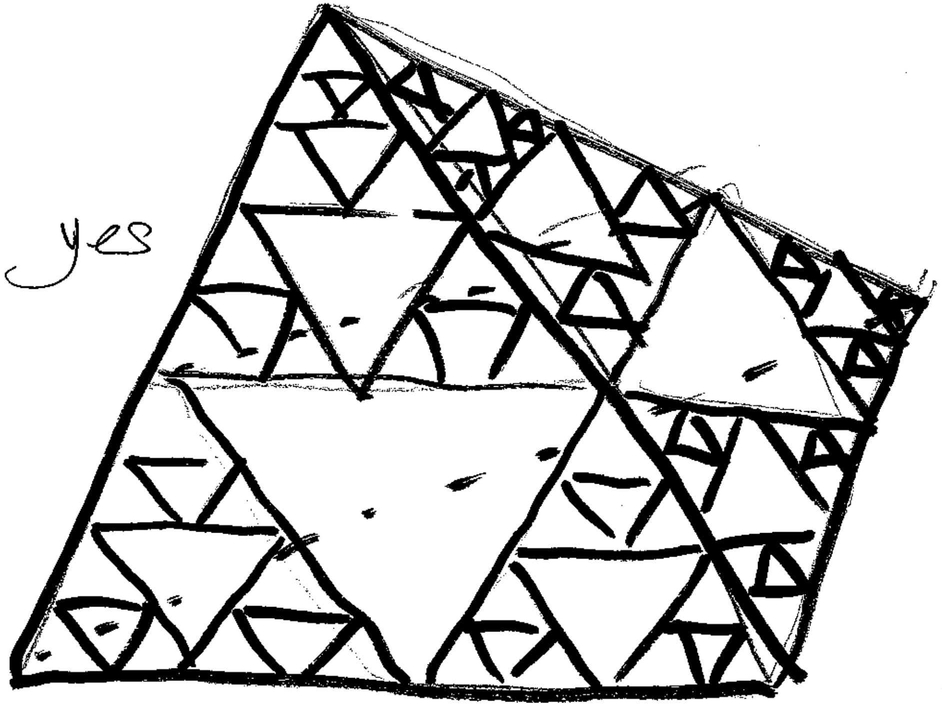
PLANAR: NO

FRACTAL

no



yes



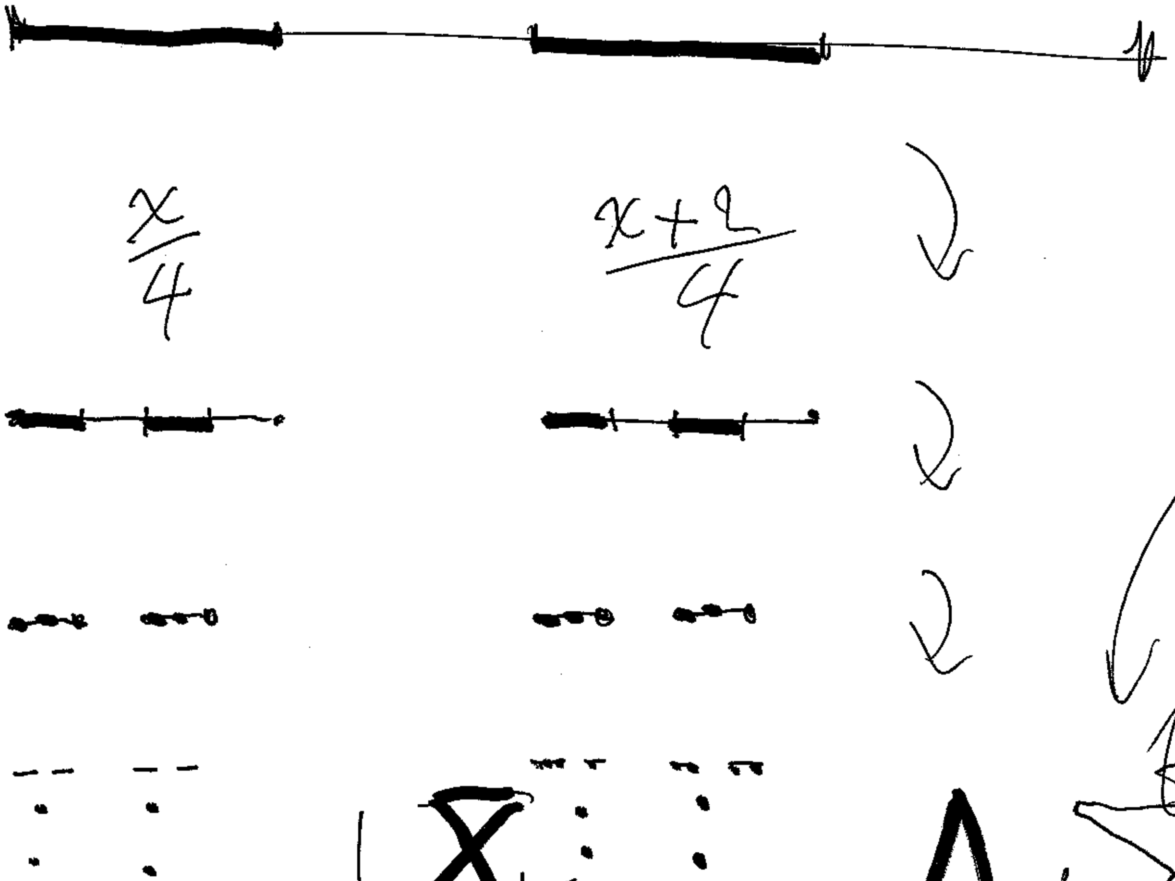
$L^2(\mu)$ not spectral

$\mu :=$ affine max entropy measure on $\overline{X}$

Jo-Pedersen

yes! $\{e_\lambda\}$ ONB $L^2(\mu)$

gaps/similarity



$$X_{\frac{1}{4}} = \left\{ \sum_{k=1}^{\infty} \frac{b_k}{4^k} \;\middle|\; b_k \in \{0,2\} \right\}$$

Spectrum

$\{0,1\} + 4\mathbb{Z}$

$\{0,1\} + \{0,1\}4 + 4^2\mathbb{Z}$

$\{0,1\} + \{0,1\}4 + \{0,1\}4^2 + 4^3\mathbb{Z}$

$\vdots$

$$\Lambda_4 : \sum_{k=0}^{\text{finite}} a_k 4^k = a_0 + a_1 4 + a_2 4^2 + \cdots$$

$$= \{0, 1, 4, 5, 16, 17, 20, 21, \ldots\}$$

Λ_M, $M > 1$

$$\prod_{k=0}^{\infty} \left(1 + x^{M^k}\right) = \sum_{\lambda \in \Lambda_M} x^{\lambda}$$

<u>Ex</u> $M = 4$: $\prod_{k=0}^{\infty} \left(1 + x^{4^k}\right) = \sum_{\lambda \in \Lambda_4} x^{\lambda}$

$$\Lambda_4 = \{0, 1, 4, 5, 16, 17, 21, 64, 65, \ldots\}$$

(10)

Jorgensen / Pedersen

$\frac{1}{4}$ Bernoulli/Cantor meas μ $\quad (\mu_{\frac{1}{4}}, \Lambda_4)$: spectral pair

$$\frac{1}{2}\int \left(f\left(\frac{x}{4}\right) + f\left(\frac{x+2}{4}\right) \right) d\mu(x) = \int f(x)\, d\mu(x) \;;$$

$$\text{support}(\mu_{\frac{1}{4}}) = X_{1/4} \;; \qquad L^2(\mu_{\frac{1}{4}}) = L^2(X_{\frac{1}{4}}, \mu_{\frac{1}{4}})$$

Theorem (Jo-Pe) : $L^2(\mu_{\frac{1}{4}})$ has as an ONB

$$\{ e_\lambda \mid \lambda \in \Lambda_4 \}$$

Contrast to $\mu_{\frac{1}{3}}$ (middle third Cantor!)

(7A)

BUT NOT the middle third

$$\left\{ \sum_{1}^{\infty} \frac{b_k}{3^k} \;\middle|\; b_k \in \{0,2\} \right\}$$

IN FACT $L^2(\mu_{1/3})$ has no more than two orthogonal e_x's.

$$\int f \, d\mu_{1/3} = \frac{1}{2} \int \left(f\left(\frac{x}{3}\right) + f\left(\frac{x+2}{3}\right) \right) d\mu_{1/3}(x)$$

32

Bernoulli-measure μ_λ $\quad \lambda \in (0,1)$

λ fixed

$\mu_\lambda :=$ the distribution of the random POWER SERIES

$$X_\lambda(\cdot) = \sum_{1}^{\infty} \{\pm 1\} \lambda^k \quad \text{on } \Omega = \prod_{\mathbb{N}} \{+1, -1\}$$

fair coin $\left(\frac{1}{2} \frac{1}{2}\right)$

$(\Omega, \mathbb{P})$

$$\mathbb{P} = \prod_{\mathbb{N}} \left\{\tfrac{1}{2}, \tfrac{1}{2}\right\}$$

$$\widehat{\mu_\lambda}(t) = \prod_{k=1}^{\infty} \cos(\pi \lambda^k t)$$

MANY AUTHORS

$\lambda < \frac{1}{2}$, $\lambda \geq \frac{1}{2}$, , , ,

Erdös ... Solomyak

$\lambda < \frac{1}{2}$ When does $L^2(\mu_\lambda)$ have an $\{e_\lambda\}$ ONB, i.e., when is μ_λ spectral?

Answer (many authors DD .. PJ --- Sie ---)

$$\Longleftrightarrow \lambda = \frac{1}{\text{even}}, \quad \lambda \in \{\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{6}, \tfrac{1}{8}, \dots\}.$$

i.e. the Bernoulli measure μ_λ is spectral

$$\Updownarrow$$

$$\lambda = \frac{1}{\text{even}}$$

Questions: Spectral pairs (Λ, μ):

$\mathcal{M}_B :=$ Borel probability measures on $\mathbb{R}^d$,

$\Lambda \subset \mathbb{R}^d$ discrete, fixed.

Def $\mathcal{M}(\Lambda) = \left\{ \mu \in \mathcal{M}_B \;\middle|\; \sum_{\lambda \in \Lambda} |\hat{\mu}(s-\lambda)|^2 \geq 1, \forall s \in \mathbb{R}^d \right\}$

Q

1. When is $\mathcal{M}(\Lambda) \neq \emptyset$?
2. Λ_3
3. If $\neq \emptyset$, then describe $\mathcal{M}(\Lambda)$.

AFFINE FRACTALS

$\mathbb{R}^d$, d fixed, M $d \times d$, $|\lambda| > 1$, over $\mathbb{Z}$

$B \subset \mathbb{R}^d$ finite system of vectors $B = \{b\}$

$\tau_b(x) := M^{-1}(x+b) \rightsquigarrow \mu = \mu_{(M,B)}$

$$\frac{1}{\# B} \int \sum_b f(\tau_b(x))\, d\mu(x) = \int f\, d\mu$$

Could ~~take~~ take any $p_b > 0$, $\sum p_b = 1$, but then not spectral.

Λ is a spectrum $\Longleftrightarrow$ (Jo-Pedersen) $\sum_{\lambda \in \Lambda} |\hat{\mu}(s-\lambda)|^2 \equiv 1$, $\forall s \in \mathbb{R}^d$.

Notation $\quad \hat{\rho}_{\Lambda}(s) := \sum_{\lambda \in \Lambda} \left| \hat{\mu}(s-\lambda) \right|^2,$

$$V_{\mathcal{B}}(s) = \frac{1}{\# \mathcal{B}} \sum_{b \in \mathcal{B}} e_b(M^{T^{-1}} s)$$

$$\hat{\mu}(s) = V_B(s)\, \hat{\mu}(M^{T^{-1}} s)$$

Let $\mathcal{B} = \{\beta\} \subset \mathbb{R}^d$ be a finite set, and set

$$(R_{\mathcal{B}} f)(s) = \sum_{\beta \in \mathcal{B}} \left| V_{\mathcal{B}}(s-\beta) \right|^2 f\left(M^{T^{-1}}(s-\beta)\right);$$

Ruelle operator (transfer-operator).

<u>Thm</u> Suppose $\exists\ \mathcal{B}$ s.t. $\Gamma_{\Lambda}(\cdot)$ satisfies

$$R_{\mathcal{B}}\left(\Gamma_{\Lambda}\right) = \Gamma_{\Lambda}$$

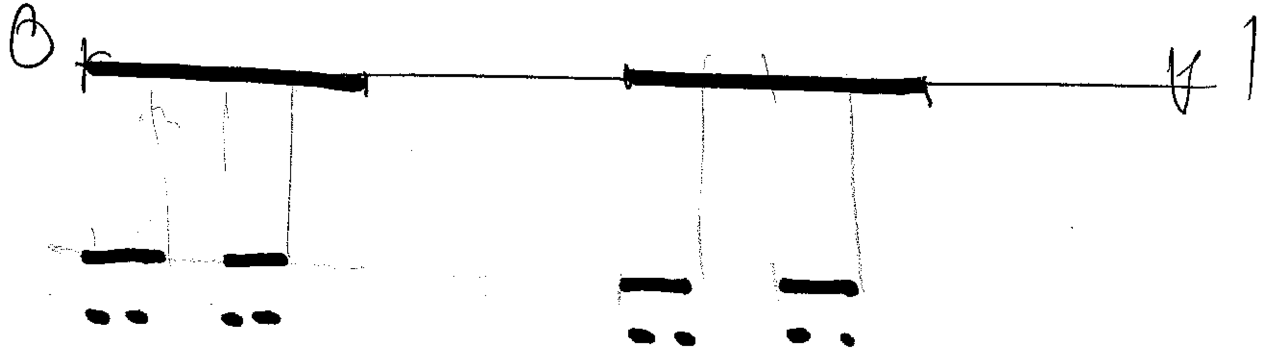
then Λ is a spectrum for $\mu = \mu_{\mathcal{B},M}$

$\Longleftrightarrow$ the constant function $\mathbb{1}$ is, up to a constant multiple, the only continuous eigenfunction of $R_{\mathcal{B}}$.

(Perron - Frobenius)

CASE IN POINT

$d = \frac{1}{4}$



$f(s) := \Gamma_{\Lambda_4}(s)$

$\Lambda_4 = \{0, 1, 4, 5, 16, 17, 20, 21, \dots\}$

$\left(\Sigma_{\frac{1}{4}}, \mu_{\frac{1}{4}} \right)$

$\Lambda = (4\Lambda) \cup \{1 + 4\Lambda\}$

$$(Rf)(s) = \cos^2\left(\frac{\pi s}{2}\right) f\left(\frac{s}{4}\right) + \sin^2\left(\frac{\pi s}{2}\right) f\left(\frac{s-1}{4}\right), \quad s \in \mathbb{R}.$$

Complex version ⤳ RKHS

Λ ⤳ $1, z, z^4, z^5, z^{16}, z^{17}, z^{20}, z^{21}, \ldots$ ↓ ONB

$f(z) = c_0 + c_1 z + c_4 z^4 + c_5 z^5 + c_{16} z^{16} + \cdots$

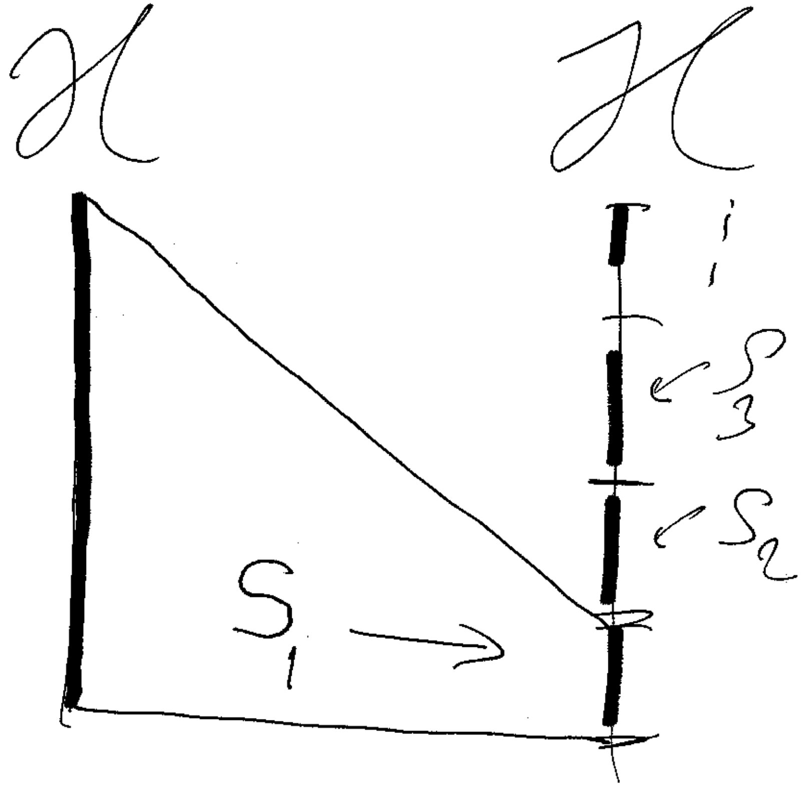
$$\mathcal{H}_{\frac{1}{4}} \ni f(z) = f_0(z^4) + z f_1(z^4) = \cdots$$

$|z| < 1$ closed subspace in the Hardy space

lacunary $\quad f(z) = \sum_{\lambda \in \Lambda} c_\lambda z^\lambda$

$$(S_0 f)(z) = f(z^4), \quad (S_1 f)(z) = z f(z^4)$$

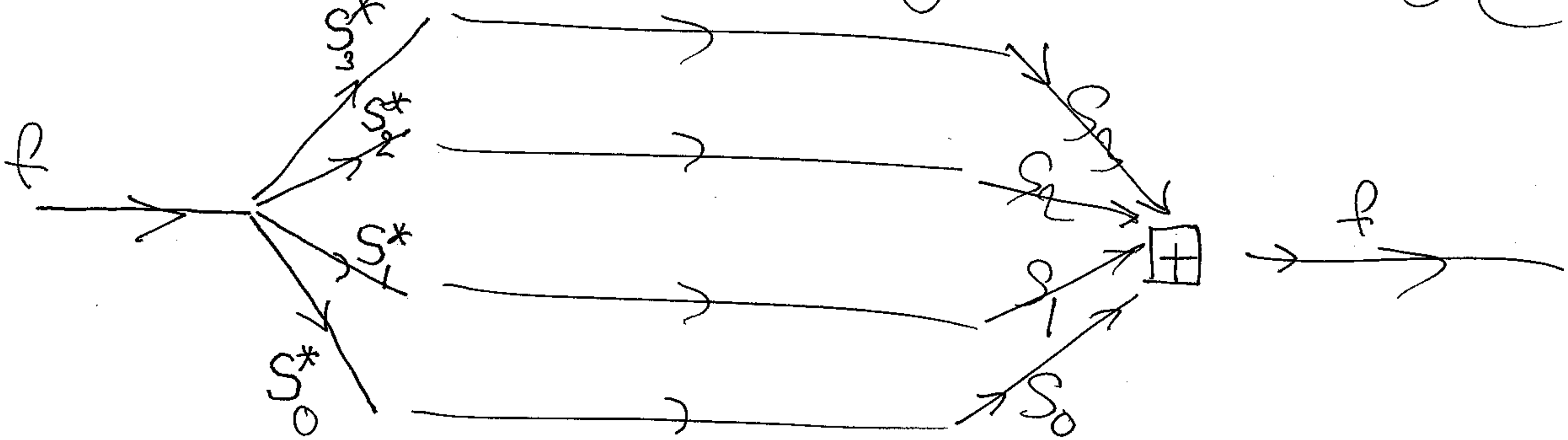
Cuntz relations: $\quad S_0^* S_1 = 0, \quad S_0 S_0^* + S_1 S_1^* = I$



$$(S_j f)(z) = m_j(z)\, f(\varphi(z))$$

$$S_k^* S_j = \delta_{j,k}$$

$$\sum_{j=0}^{3} S_j S_j^* = I_{\mathcal{H}}$$



<u>Thm</u> (D. Dutkay, PJ, S. Pedersen) .. D. Alpay

$L^2(\mu_{\frac{1}{4}}) \ni \{e_\lambda \mid \lambda \in \Lambda\} \longleftrightarrow \{z^\lambda\} \in \mathcal{H}_{\frac{1}{4}}$

define a unitary intertwining isomorphism

$\left\{\frac{x}{4}, \frac{x+2}{4}\right\}$ in $L^2(\mu_{\frac{1}{4}}) \xleftrightarrow[\text{INTERTWINING}]{} \underbrace{\{S_0, S_1\}, \cdots \mathcal{H}_{\frac{1}{4}}}_{\text{RKHS}}$

reproducing kernel

fractional Szegö kernel on $\mathbb{D} = (|z| < 1)$

$$K(z,w) = \prod_{n=0}^{\infty} \left(1 + (z\bar{w})^{4^n}\right), \quad \forall z, w \in \mathbb{D}$$

$$\mathcal{H}(\Lambda) := \operatorname{cl\,span}\{z^\lambda \mid \lambda \in \Lambda\} \subset H_2$$

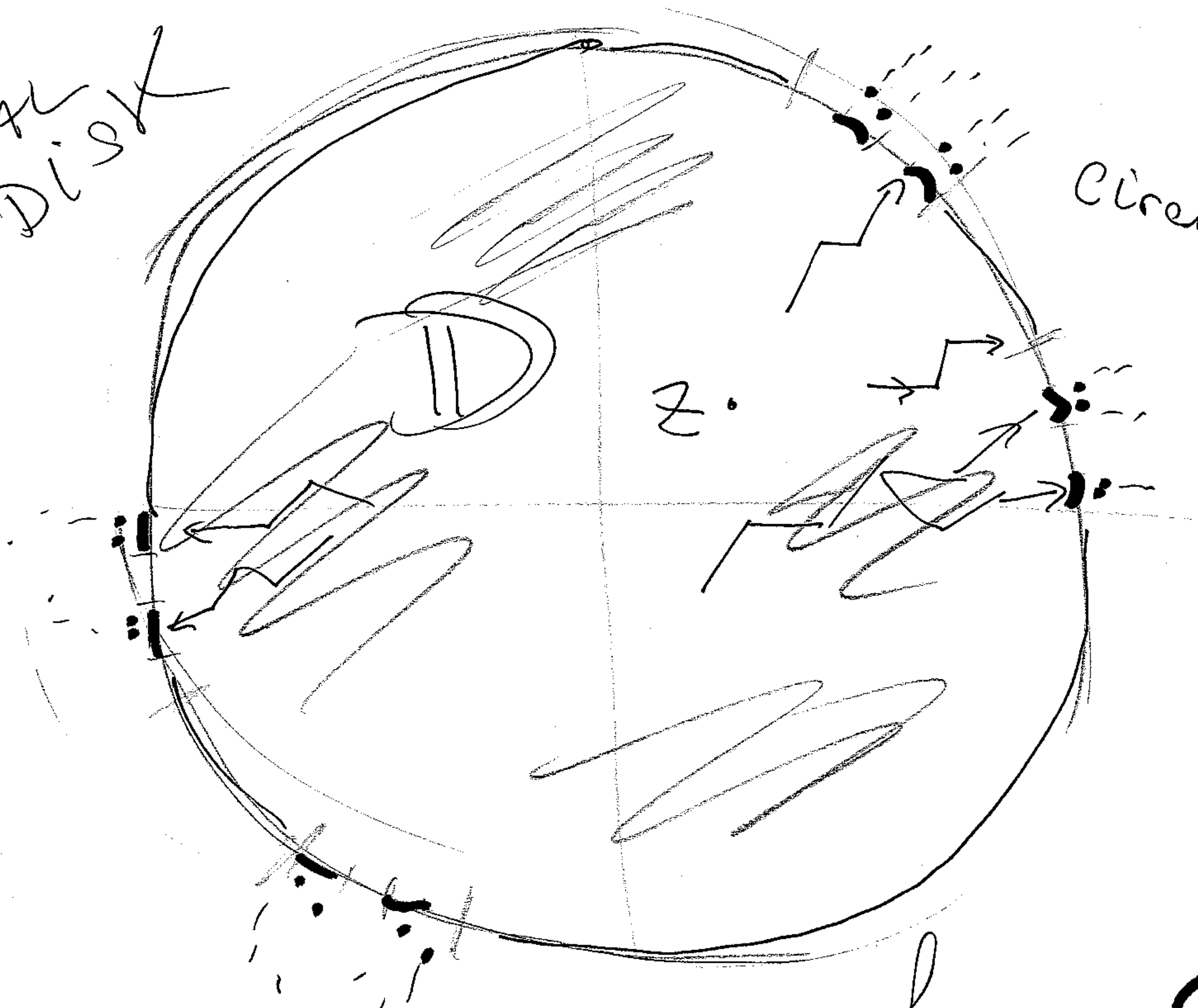
__Coroll__ $\mathcal{H}_{\frac{1}{4}}$ is a (RK) ~~space of analytic~~ functions, a closed subspace of ~~the~~ Hardy ~~space~~

$$\cong \mathcal{L}^2(\mu_{\frac{1}{4}})$$

i.e., $\mathcal{H}(\Lambda_4) = \mathcal{L}^2(\mu_{\frac{1}{4}})$

23/17

FRACTAL DISK



Circle $X_{\frac{1}{4}}$

Lebesgue $= 0$

$\tilde{f} \in L^2(\mu_{\frac{1}{4}})$

Thm

$f \longmapsto \tilde{f}$ is onto!

$$f(z) = \int_{X_{\frac{1}{4}}} K(z,x)\tilde{f}(x)\, d\mu_{\frac{1}{4}}(x)$$

Fix $\Lambda \subset \mathbb{N}_0 = \{0\} \cup \mathbb{N}$

Set $\mathcal{H}(\Lambda) \subset H_2$ closure in the Hardy space

$$\mathcal{H}(\Lambda) = \text{cl.sp.}\{z^\lambda \mid \lambda \in \Lambda\}.$$

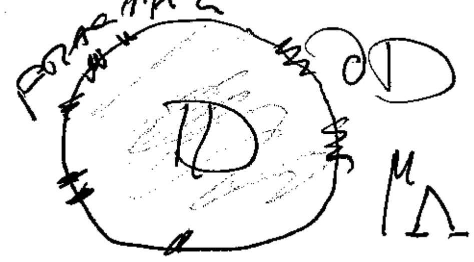
Set:

$$\mathcal{H}(\Lambda) \ni f(z) = \sum_{\lambda \in \Lambda} c_\lambda z^\lambda, \quad (c_\lambda) \in \ell^2(\Lambda)$$

$\leadsto$ RK $\leadsto$ RKHS $\quad K_\Lambda(\cdot,\cdot) \quad \mathbb{D} \times \mathbb{D}$

<u>Theorem</u>

$$K_\Lambda(z,w) = \sum_{\lambda \in \Lambda} (z w^*)^\lambda \leadsto !$$



then $\exists\ \mu_\Lambda \in \mathcal{M}(\partial\mathbb{D}) \simeq \mathcal{M}(\mathbb{T})$

s.t.

$$f(z) = \int_{X_\mu} K(z,x)\, \tilde{f}(x)\, d\mu(x), \quad z \in \mathbb{D}$$

$\tilde{f} \to L^2(\mu_\Lambda)$

$\forall f \in \mathcal{H}_\Lambda$

$\Updownarrow$

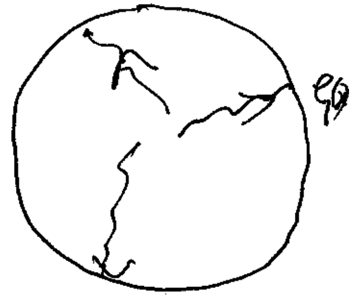
$f \mapsto \tilde{f}$ is onto $L^2(\mu_\Lambda)$

$\left\{ e_\lambda(x) = e^{i2\pi\lambda x} \right\}_{\lambda \in \Lambda}$ is an ONB in $L^2(\mu_\Lambda)$

so (μ_Λ, Λ) is a spectral pair.

Boundary limit

Proposition $\Lambda \subset \mathbb{N}_0$, (Λ, μ) spectral pair; $X \subset [0,1) \; (\simeq \mathbb{T})$, $X := \mathrm{supp}(\mu)$



$\mathrm{supp}(\mu) \subset \mathbb{T}$

Def

Set $\tilde{f}(x) = \sum_{\lambda \in \Lambda} c_\lambda e^{i 2\pi \lambda x}$, $(c_\lambda) \in \ell^2(\Lambda)$

and if $r < 1$, $f(rx) = \sum_{\lambda \in \Lambda} c_\lambda \, (r\, e(x))^\lambda$, $\left(e_\lambda(x) = e^{i2\pi\lambda x}\right)$, $x \in [0, 1)$.

Then $\lim_{r \to 1} \| \tilde{f} - f_r \|_{L^2(\mu)} = 0$.

REMARKS

Existence

Given K, there are probability spaces $(\mathbb{X}, \mu)$ and extensions $\tilde{K}(z, x)$, $x \in \mathbb{X}$ s.t.

$$K(z,w) = \int_{\mathbb{X}} \tilde{K}(z,x) \overline{\tilde{K}(w,x)} \, d\mu(x)$$

but not with $\mathbb{X} \subseteq \mathbb{T} = \partial\mathbb{D}$.

Examples

KERNELS $\quad \mu$?

$$\Lambda_1 = \mathbb{N}_0 = \{0, 1, 2, \ldots\}, \qquad K(z,w) = \frac{1}{1 - zw^*} \qquad X_1 \quad \text{no}$$

Szegö

$$\Lambda_3 = \left\{ \sum_0^{\text{fin}} a_k 3^k \,;\, a_k \in \{0,1\} \right\}, \qquad \prod_{n=0}^{\infty} \left(1 + (zw^*)^{3^n}\right) \qquad X_3 \quad \text{no}$$

$$= \{0, 1, 3, 4, 9, 10, 12, \ldots\}$$

$$\Lambda_4 = \left\{ \sum_0^{\text{fin}} a_k 4^k \,;\, a_k \in \{0,1\} \right\}, \qquad \prod_{n=0}^{\infty} \left(1 + (zw^*)^{4^n}\right) \qquad X_4 \quad \text{yes !}$$

$$= \{0, 1, 4, 5, 16, 17, 20, \ldots\}$$

$$X_{1/4}(\cdot) = \sum_{n=1}^{\infty} (\pm 1) \, 1/4^n \rightsquigarrow \text{distribution } \mu_{1/4}$$

$$\Omega = \prod_{\mathbb{N}} \{\pm 1\}$$

$$\mathbb{P} = \times_{\mathbb{N}} \left\{\tfrac{1}{2}, \tfrac{1}{2}\right\}$$

$$\mu_{1/4} \downarrow \text{CANTOR fractal} \Big\} \text{lacunary} \downarrow \Lambda_4$$

$$L^2(\Omega, \mathbb{P}) \rightsquigarrow L^2(X_{1/4}, \mu_{1/4}) \rightsquigarrow \mathcal{H}(\Lambda_4) \longrightarrow X_{1/4}$$

The kernels $\begin{cases} K_3 = K_{\Lambda_3} \\ K_4 = K_{\Lambda_4} \end{cases}$

<u>differences</u>

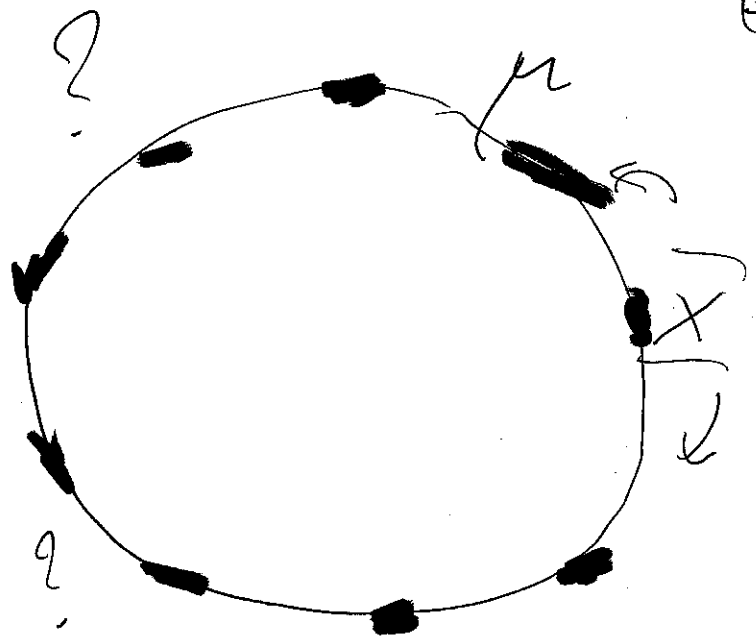
$$K_3(z,w) = \prod_{n=0}^{\infty} \left(1 + (zw^*)^{3^n}\right) = \sum_{\lambda \in \Lambda_3} z^\lambda w^{*\lambda}$$

$$K_4(z,w) = \prod_{n=0}^{\infty} \left(1 + (zw^*)^{4^n}\right) = \sum_{\lambda \in \Lambda_4} z^\lambda w^{*\lambda}$$

$$\int_{\mathbb{T}_{1/4}} K_4(z, e_1(x)) K_4(w, e_1(x))^* \, d\mu_{\frac{1}{4}}(x) = K_4(z,w),$$

$\forall (z,w) \in \mathbb{D} \times \mathbb{D}$

but we do not know measures $\mu \neq$ Leb. on $\partial\mathbb{D}$ s.e.



$$\int_X K_3(z, \xi(x)) K_3(w, \xi(x))^* \, d\mu(x) = K_3(z, w).$$

??

What is $\mathcal{M}(K_3)$?

<u>Definition</u> Given $K : \mathbb{D} \times \mathbb{D} \to$ analytic in z, w^* $\rightsquigarrow \mathcal{H}(K) = $ RKHS

Set

$$\mathcal{M}(K) = \left\{ \mu \text{ on } \partial\mathbb{D} : \int_{\partial\mathbb{D}} K(z,x) K(w,x)^* \, d\mu(x) = K(z,w) \right\}$$

If $K = K_{\wedge}$ then Leb $\in \mathcal{M}(K_{\wedge})$

$\mu_{\frac{1}{4}} \in \mathcal{M}(K_4)$, $\mu_{\frac{1}{3}} \notin \mathcal{M}(K_3)$.

<u>Question</u>: What is $\mathcal{M}(K_3)$?

Question! Given $\Lambda \subset \mathbb{R}^{d}$,
Given $\mu \in \mathcal{M}(K_{\Lambda})$; what
is $\overline{\text{span}}^{L^2(\mu)}\{e_\lambda(x) \mid \lambda \in \Lambda\} \; (\subset L^2(\mu))$?

Question! Is $\mathcal{M}(K_{\Lambda_3}) = \{Leb\}$ (singleton)?

Moral $\mathcal{H}(K_{\Lambda_3})$ is "too small" on the boundary!

FACT Given $K : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ p.d. analytic in z, w^*

then $\mathfrak{M}(K)$ is convex and w^*-compact.

Questions 1. What is $\mathrm{Ext}(\mathfrak{M}(K))$,

i.e., its extreme points?

2. Is $\mu_{\frac{1}{4}} \in \mathrm{Ext}(\mathfrak{M}(K_{\wedge 4}))$?

$$5\Lambda_4 = \{0, 5, 20, 25, 80, 85, 100, 105, \ldots\}$$

Question

$$p\Lambda_4 + q\Lambda_4 \quad ?$$

yes!

$$\mathcal{H}(5\Lambda_4) \quad \text{ONB} \quad \text{yes!}$$

<u>Theorem</u>

$(U_5 f)(x) = f((5x) \bmod X_{1/4})$ is unitary in $L^2(\mu_{1/4})$.

Q. What is its spectrum? $\rightsquigarrow$ fractal---.

The property of the infinite product formula

$$K(z,w)=\prod_{n=0}^{\infty}\left(1+(zw^*)^{4^n}\right), \quad (z,w)\in\mathbb{D}\times\mathbb{D}$$

is a special case of the following transformation rules: Suppose

$$(S_k f)(z)=m_k(z)\, f(\varphi(z)), \text{ and}$$

$$\sum_k S_k S_k^* = I_{\mathcal{H}}, \quad \mathcal{H}=\mathcal{H}(K) \text{ some}$$

RKHS ; $\hookrightarrow \Omega\times\Omega$ then

$$K(z,w) = \left(\sum_j m_j(z)\, m_j(w)^*\right) K(\varphi(z), \varphi(w)),$$

$$(z,w) \in \Omega \times \Omega .$$

Application: Julia sets : $\varphi(z) = c + z^2$,

φ = pol, $\quad \varphi(z) = \dfrac{P(z)}{Q(z)}$ ...

This includes many (K, Ω) st. with

$$\varphi^{\circ n}(z) = \varphi(\varphi(\cdots \varphi(z)) \cdots) \quad n \text{ fold substitution}$$

$$\varphi^{\circ n+1}(z) = \varphi^{\circ n}(\varphi(z))$$

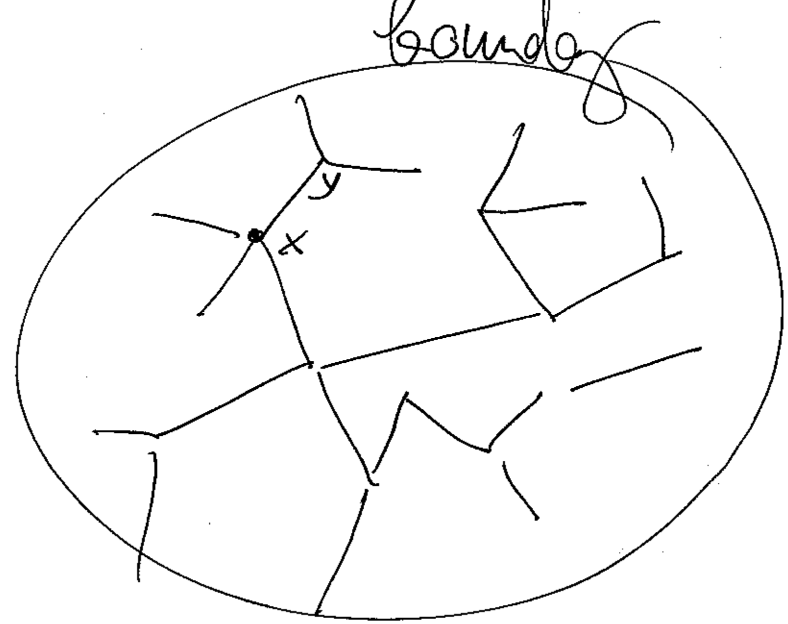
$$K(z,w) = \prod_{n=0}^{\infty} \left(\sum_j m_j \left(\varphi^{\circ n}(z), \varphi^{\circ n}(w) \right) \right) K(\ell, \ell)$$

G

V: Vertices x, y, z

E: Edges $e = (x, y)$

Notation $x \sim y : (x,y) \in E$



$$(\Delta f)(x) = \sum_{y \sim x} c_{xy} \left(f(x) - f(y) \right)$$

$\{c_{xy}\}_{(xy) \in E}$ fixed conductance nos.