Simple random walk on the two-dimensional uniform spanning tree and its scaling limits

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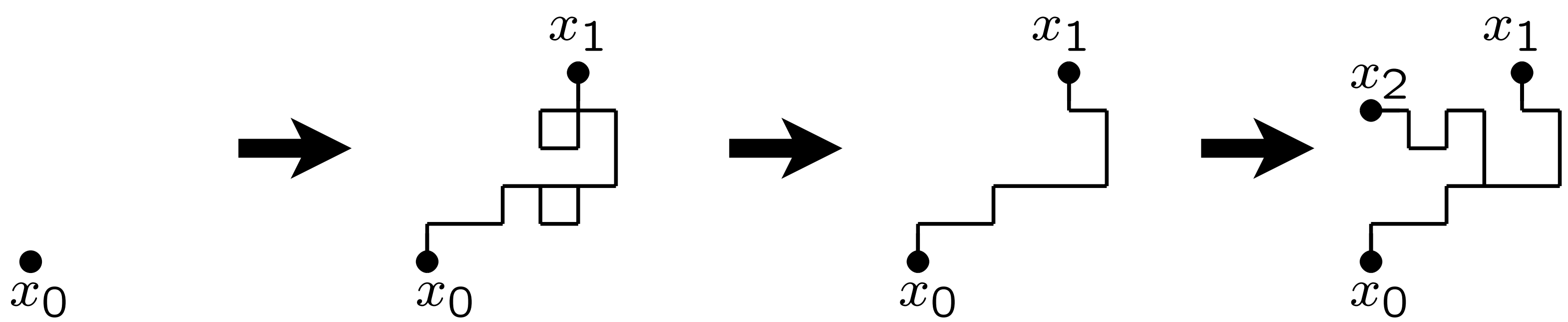
UNIFORM SPANNING TREE IN TWO DIMENSIONS

Let $\Lambda_n := [-n, n]^2 \cap \mathbb{Z}^2$, $\mathcal{U}^{(n)}$: spanning tree over Λ_n (i.e. vertices Λ_n, no cycle) - selected uniformly at random from all possibilities

$\mathcal{U}$: UST on $\mathbb{Z}^2$, which is the local limit of $\mathcal{U}^{(n)}$.
NB. Wired/free boundary conditions unimportant.
Almost-surely, $\mathcal{U}$ is a spanning tree of $\mathbb{Z}^2$.
[Aldous, Benjamini, Broder, Häggström, Lyons, Pemantle, Peres, Schramm. . .]

WILSON'S ALGORITHM ON $\mathbb{Z}^2$

$\mathbb{Z}^2 = \{x_0 = 0, x_1, x_2, \dots\}$, $\mathcal{U}(0) = \{x_0\}$.
Given $\mathcal{U}(k-1)$, $k \geq 1$, define $\mathcal{U}(k)$ as a union of $\mathcal{U}(k-1)$ and the loop-erased random walk (LERW) path run from x_k to $\mathcal{U}(k-1)$.
UST $\mathcal{U}$ is the a local limit of $\mathcal{U}(k)$.



LERW $\to$ SLE(2), UST Peano curve $\to$ SLE(8)
[Schramm '00, Lawler-Schramm-Werner '04].
$M_n = |LERW(0, B_E(0,n))|$: length of a LERW from 0 to $B_E(0,n)^c$.

$\Rightarrow \lim_{n\to\infty} \frac{\log E^0 M_n}{log n} = 5/4$ [Kenyon '00], $E^0 M_n \asymp n^{5/4}$ [Lawler '13]

RW on random graphs: General theory.

Let $\mathcal{G}(\omega)$ be a random graph on $(\Omega, \mathbb{P})$. Assume $\exists 0 \in \mathcal{G}(\omega)$.

Let $D \geq 1$. For $\lambda \geq 1$, we sat that $B(0,R)$ in $\mathcal{G}(\omega)$ is λ-**good** if

$$\lambda^{-1} R^D \leq |B(0,R)| \leq \lambda R^D,$$
$$\lambda^{-1} R \leq R_{\text{eff}}(0, B(0,R)^c) \leq R + 1.$$

λ-good is a nice control of the volume and resistance for $B(0,R)$.

Theorem. [Barlow/Jarai/K/Slade 2008, K/Misumi 2008]

Suppose $\exists p > 0$ such that

$$\mathbb{P}(\{\omega : B(0,R) \text{ is } \lambda\text{-good.}\}) \geq 1 - \lambda^{-p} \qquad \forall R \geq R_0, \forall \lambda \geq \lambda_0.$$

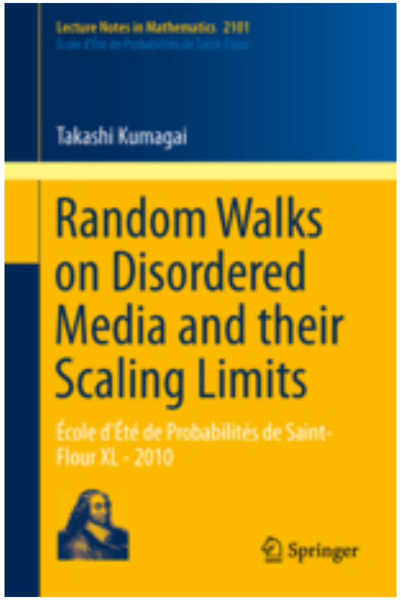
Then $\exists \alpha_1, \alpha_2 > 0$ and $N(\omega), R(\omega) \in \mathbb{N}$ s.t. the following holds for $\mathbb{P}$-a.e. ω:

$$(\log n)^{-\alpha_1} n^{-\frac{D}{D+1}} \leq p^{\omega}_{2n}(0,0) \leq (\log n)^{\alpha_1} n^{-\frac{D}{D+1}}, \qquad \forall n \geq N(\omega),$$

$$(\log R)^{-\alpha_2} R^{D+1} \leq E^0_{\omega}\tau_{B(0,R)} \leq (\log R)^{\alpha_2} R^{D+1}, \qquad \forall R \geq R(\omega).$$

In particular,

$$d_s(G) := \lim_{n\to\infty} \frac{\log p^{\omega}_{2n}(0,0)}{\log n} = \frac{2D}{D+1}$$



(Advertisement)

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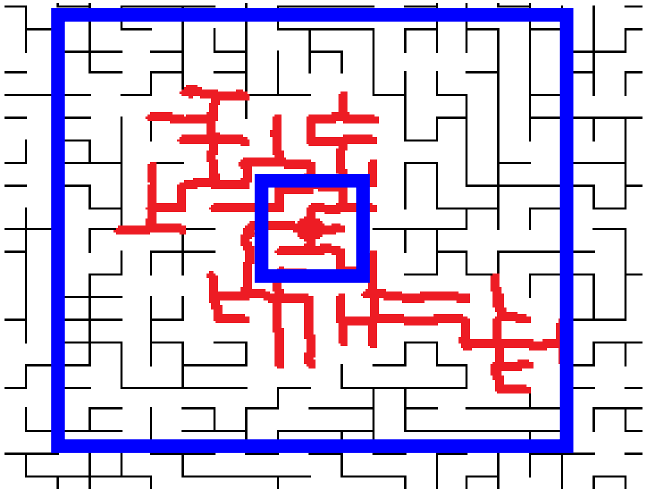
Random Walks on Disordered Media and their Scaling Limits.

Lecture Notes in Mathematics, Vol. 2101, École d'Été de Probabilités de Saint-Flour XL–2010. Springer, New York (2014).

VOLUME AND RESISTANCE ESTIMATES [BARLOW/MASSON 2010,2011]



With high probability,

$$B_E(x, \lambda^{-1}R) \subseteq B_{\mathcal{U}}(x, R^{5/4}) \subseteq B_E(x, \lambda R),$$

as $R \to \infty$ then $\lambda \to \infty$.

It follows that with high probability,

$$\mu_{\mathcal{U}}(B_{\mathcal{U}}(x, R)) \asymp R^{2/(5/4)} = R^{8/5}.$$

Also with high probability,

$$\text{Resistance}(x, B_{\mathcal{U}}(x, R)^c) \asymp R.$$

$\Rightarrow$ Exit time for intrinsic ball radius R is $R^{13/5}$,
HK bounds $p^{\mathcal{U}}_{2n}(0,0) \asymp n^{-8/13}$. ($D = 8/5, d_s = 16/13$)

(Q) How about scaling limit for UST?

UST SCALING [SCHRAMM 2000]

$\mathfrak{U} = \left\{(a, b, \pi_{ab}) : a, b \in \mathbb{Z}^2\right\}$: π_{ab} is a path from a to b

Scaling limit $\mathfrak{T}$ satisfies the following a.s.:

- Each pair $a, b \in \dot{\mathbb{R}}^2$ (cpt) are connected by the following path.
- $a \neq b \rightarrow$ simple path, $a = b \rightarrow$ one point or a simple loop.
- Trunk, $\cup_{\mathfrak{T}} \pi_{ab} \backslash \{a, b\}$ is a dense tree, the degree is at most 3.
  cf. [Aizenman-Burchard-Newman-Wilson '99].

Problem : No info. on intrinsic distance, volume, resistance.

We thus consider the (generalized) Gromov-Hausdorff topology.

Let $\mathbb{T}$ be the collection of measured, rooted, spatial trees, i.e.

$$(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}}),$$

- $(\mathcal{T}, d_{\mathcal{T}})$: loc. cpt real tree • $\mu_{\mathcal{T}}$: Borel measure on $(\mathcal{T}, d_{\mathcal{T}})$
- $\phi_{\mathcal{T}}$: $(\mathcal{T}, d_{\mathcal{T}}) \to \mathbb{R}^2$ cont. map • $\rho_{\mathcal{T}}$: distinguished vertex on $\mathcal{T}$

Δ_c: distance on $\mathbb{T}_c$ (compact trees only) defined as follows

$$\inf_{\substack{Z,\psi,\psi',\mathcal{C},\\ (\rho_{\mathcal{T}},\rho'_{\mathcal{T}})\in\mathcal{C}}} \left\{ d_P^Z(\mu_{\mathcal{T}} \circ \psi^{-1}, \mu'_{\mathcal{T}} \circ \psi'^{-1}) + \sup_{(x,x')\in\mathcal{C}} \left(d_Z(\psi(x), \psi'(x')) + \left| \phi_{\mathcal{T}}(x) - \phi'_{\mathcal{T}}(x') \right| \right) \right\}.$$

Theorem. [Tightness] $\mathbf{P}_\delta$: law of the following spatial tree

$$\left(\mathcal{U}, \delta^{5/4} d_{\mathcal{U}}, \delta^2 \mu_{\mathcal{U}}(\cdot), \delta\phi_{\mathcal{U}}, 0\right)$$

Then, under $\mathbf{P}$, $(\mathbf{P}_\delta)_{\delta\in(0,1)}$ is tight on $\mathcal{M}_1(\mathbb{T})$.

(Proof) Strengthening estimates of Barlow-Masson.

Comparison of Euclidean and intrinsic distance along paths.

UST LIMIT PROPERTIES

If $\tilde{\mathbf{P}}$ is a subsequential limit of $(\mathbf{P}_\delta)_{\delta\in(0,1)}$, then for $\tilde{\mathbf{P}}$-a.e. $(\mathcal{T}, d_\mathcal{T}, \mu_\mathcal{T}, \phi_\mathcal{T}, \rho_\mathcal{T})$ it holds that:

(i) $\mu_\mathcal{T}$ is non-atomic, supported on the leaves of $\mathcal{T}$,
i.e. $\mu_\mathcal{T}(\mathcal{T}^o) = 0$, where $\mathcal{T}^o := \mathcal{T} \backslash \{x \in \mathcal{T} : \deg_\mathcal{T}(x) = 1\}$;

(ii) for any $R > 0$,

$$\liminf_{r\to 0} \frac{\inf_{x\in B_\mathcal{T}(\rho_\mathcal{T},R)} \mu_\mathcal{T}\left(B_\mathcal{T}(x,r)\right)}{r^{8/5}(\log r^{-1})^{-c}} > 0,$$

(iii) $\phi_\mathcal{T}$ is a homeo. between $\mathcal{T}^o$ and $\phi_\mathcal{T}(\mathcal{T}^o)$ (dense in $\mathbb{R}^2$);

(iv) $\max_{x\in\mathcal{T}} \deg_\mathcal{T}(x) = 3$;

(v) $\mu_\mathcal{T} = \mathcal{L} \circ \phi_\mathcal{T}$.

To prove this, we need 'uniform controls' in a ball that requires more detailed estimates than those of Barlow/Masson.

$\Rightarrow$ As a by-product of the detailed estimates, we can sharpen some HK estimates.

Proposition. For each $q > 0$, there exist $c_q, C_q > 0$ such that the following holds

$$c_q n^{5q/13} \leq \mathbb{E}\left(d_{\mathcal{U}}(0, X_n)^q\right) \leq C_q n^{5q/13} \quad \forall n \geq 1.$$

NB. Marlow/Masson's estimates include $(\log n)^{\pm c}$.

LIMITING PROCESS FOR SRW ON UST

Suppose $(\mathbf{P}_{\delta_i})_{i\geq 1}$, the laws of

$$\left(\mathcal{U}, \delta_i^{5/4} d_{\mathcal{U}}, \delta_i^2 \mu_{\mathcal{U}}, \delta_i \phi_{\mathcal{U}}, 0\right),$$

form a convergent sequence with limit $\tilde{\mathbf{P}}$.

Let $(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}}) \sim \tilde{\mathbf{P}}$.

It is then the case that $\mathbb{P}_{\delta_i}$, the annealed laws of

$$\left(\delta_i X^{\mathcal{U}}_{\delta_i^{-13/4} t}\right)_{t\geq 0},$$

converge to $\tilde{\mathbb{P}}$, the annealed law of

$$\left(\phi_{\mathcal{T}}(X^{\mathcal{T}}_t)\right)_{t\geq 0},$$

as probability measures on $C(\mathbb{R}_+, \mathbb{R}^2)$.

HEAT KERNEL ESTIMATES FOR SRW LIMIT

Let $R > 0$. For $\tilde{\mathbf{P}}$-a.e. realisation of $(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}})$, there exist random constants $c_1, c_2, c_3, c_4, t_0 \in (0, \infty)$ and deterministic constants $\theta_1, \theta_2, \theta_3, \theta_4 \in (0, \infty)$ such that the heat kernel associated with the process $X^{\mathcal{T}}$ satisfies:

$$p_t^{\mathcal{T}}(x,y) \leq c_1 t^{-8/13} \ell(t^{-1})^{\theta_1} \exp\left\{ -c_2 \left(\frac{d_{\mathcal{T}}(x,y)^{13/5}}{t} \right)^{5/8} \ell(d_{\mathcal{T}}(x,y)/t)^{-\theta_2} \right\},$$

$$p_t^{\mathcal{T}}(x,y) \geq c_3 t^{-8/13} \ell(t^{-1})^{-\theta_3} \exp\left\{ -c_4 \left(\frac{d_{\mathcal{T}}(x,y)^{13/5}}{t} \right)^{5/8} \ell(d_{\mathcal{T}}(x,y)/t)^{\theta_4} \right\},$$

for all $x, y \in B_{\mathcal{T}}(\rho_{\mathcal{T}}, R)$, $t \in (0, t_0)$, where $\ell(x) := 1 \vee \log x$.